```
> restart:
```

Échauffement : dimension 1

Quel est le nombre moyen de retour à l'origine d'une marche aléatoire dans ℤ avec probabilité de transition (1+a)/2 à gauche et (1-a)/2 à droite ?

Tourver `a` tel que ce nombre soit égal à 2.

```
> E := sum(binomial(2*n, n)*((1-a^2)/4)^n, n=0..infinity) assuming
  a > 0 and a < 1;
```

$$E := \frac{1}{a} \tag{1.1}$$

```
> solve(E = 2, a);
```

$$\frac{1}{2} \tag{1.2}$$

Dimension 2

Soit a ∈ [0,1].
On définit une marche dans $\mathbb{Z}^2$ est une suite d'éléments de $\mathbb{Z}^2$, dont chacun est voisin du suivant et partant de l'origine.

On considère une marche aléatoire : on part de l'origine et chaque pas est choisit aléatoirement, vers le haut et le bas avec probabilité 1/4, vers la gauche avec probabilité (1+a)/4 et vers la droite avec probabilité (1-a)/4.

Trouver `a` tel que le nombre moyen de retour à l'origine de la marche soit égal à 2.

```
> E := sum((1/16)^n*binomial(2*n, n)*sum(binomial(n,k)^2*(1-a^2)^
  (k), k=0..n), n=0..infinity);
```

$$E := \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n} \left(a^2\right)^n \mathrm{LegendreP}\left(n, \frac{-a^2+2}{a^2}\right) \tag{2.1}$$

```
> a0 := fsolve(E=2, a);
```

$$a0 := -0.2476558179 \tag{2.2}$$

Méthode 1

Pour approcher `a`, on approxime E(a) par un polynôme dont on calculera les racines.

On somme les premiers N termes de E.

```
> N := 50:
> Eapprox := expand(add((1/16)^n*binomial(2*n, n)*add(binomial(n,k)
  ^2*(1-a^2)^(k), k=0..n), n=0..N));
```

Eapprox := **(3.1)**

58157541499435631090316731510517646027268559531441269212729/
2510840694154672305534315769283066566440942177785613805158\
4 + 1261141806819552416685156215 7/
2008672555323737844427452615426453253152753742228491044126\
72 a^{100} K 161050356492082741696384721787 75/
1004336277661868922213726307713226626576376871114245522063\
36 a^{98} + 103052015806434248004068917905810 75/
1004336277661868922213726307713226626576376871114245522063\
36 a^{96}
K 1833567183228029664540108674210697 45/
6277101735386680763835789423207666416102355444464034512896
a^{94}
+ 23488907949238440253255485144412982 5305/
5021681388309344611068631538566133132881884355571227610316\
8 a^{92}
K 1202388760195615524150364760383714240 2345/
2510840694154672305534315769283066566440942177785613805158\
4 a^{90}
+ 42657602638455748179719697288848731621 8075/
1255420347077336152767157884641533283220471088892806902579\
2 a^{88}
K 55429538859851656502442727136321393292 58275/
3138550867693340381917894711603833208051177722232017256448
a^{86} + 70310440019493961836334412434281393551044 97895/
1004336277661868922213726307713226626576376871114245522063\
36 a^{84}
K 10957588502498297114275042353135129253388869 5705/
5021681388309344611068631538566133132881884355571227610316\
8 a^{82}
+ 27505633134064975481246238104426727345514970 54495/
5021681388309344611068631538566133132881884355571227610316\

$8\,a^{80}$

K 7085051262173059627515126754548405883499451901075/
62771017353866807638357894232076664161023554444640345128961

a^{78}

$+$ 9739745801570243412692818368859180591586877552303751/
5021681388309344611068631538566133132881884355571227610316\

$8\,a^{76}$

K 7069257545376624738730847013508536142913922655007083/
2510840694154672305534315769283066566440942177785613805158\

$4\,a^{74}$

$+$ 27387657006691040876820844175729317671389257305050271/
7846377169233350954794736779009583020127944305580043141121

a^{72}

K 585333799958321591704147353234522473161752956365473931
/
15692754338466701909589473558019166040255888611160086282241

a^{70}

$+$ 1738955353435264064237159149431728340371644358898287\
685/
5021681388309344611068631538566133132881884355571227610316\

$8\,a^{68}$

K 7060836250187154320992581766418003993200468344199100\
255/
2510840694154672305534315769283066566440942177785613805158\

$4\,a^{66}$

$+$ 5044990049804946086766487488497967820536285687123533\
5439/
2510840694154672305534315769283066566440942177785613805158\

$4\,a^{64}$

K 9961449191661449175039163828680351209856406360396428\
991/
7846377169233350954794736779009583020127944305580043141121

a^{62}

$+$ 4472394378093443433628075930169905203601342845109792\
50351/
6277101735386680763835789423207666416102355444464034512896

a^{60}

K 11189332915088197760380714265134873921574228744378162\
720295/
3138550867693340381917894711603833208051177722232017256448

a^{58}

$+$ 25041470488328572405427198681615284974292909389658899\
804235/
1569275433846670190958947355801916604025588861116008628224

a^{56}

K 25136216029306072910218499668903289114959065866554770\
798187/
392318858461667547739736838950479151006397215279002157056

a^{54}

$+$ 29040508814935341657072254391443535097324039450332485407\
36947/
1255420347077336152767157884641533283220471088892806902579\
$2\,a^{52}$

K 47233984895745137936304718515124147045041722139083485541\
88693/
6277101735386680763835789423207666416102355444464034512896

a^{50}

$+$ 13866795133131151351198329956602402233829504837746618433\
754775/
6277101735386680763835789423207666416102355444464034512896

a^{48}

K 45984959054207226146825402284448667294113783502922092631\
15725/
784637716923335095479473677900958302012794430558004314112

a^{46}

$+$ 88281230748348895719740780840016866661938695726531917883\
305225/
6277101735386680763835789423207666416102355444464034512896

a^{44}

K 95878131559633601509663376853561348559337313057116706050\
010525/
3138550867693340381917894711603833208051177722232017256448

a^{42}

$+$ 47141875730346989570229138030190055590347286707832110270\
991185/
784637716923335095479473677900958302012794430558004314112
a^{40}

K 26234814635556743003311315468214220871223731424451169462\
14975/
245199286538542217337335524344049469378998259549376348l6
a^{38}

$+$ 17321942565256970901597865553159131458944146704689960576\
055558075/
10043362776618689222137263077132266265763768711142455220 63\
36 a^{36}

K 12629185240942339166390999352424666465963398988111013067\
976256625/
50216813883093446110686315385661331328818843555712276103 16\
8 a^{34}

$+$ 16641221065034642824567573982810978064832501252123111036\
613044125/
50216813883093446110686315385661331328818843555712276103 16\
8 a^{32}

K 12366631024692357748398622957085476763207192765000959778\
15700405/
313855086769334038191789471160383320805117772223201725644 8
a^{30}

$+$ 10594593601180567624556000833464876427203062000525535357\
857725725/
25108406941546723055343157692830665664409421777856138051 58\
4 a^{28}

K 50969987598925336968007029249074744136764771675132676361\
25637725/
12555420347077336152767157884641533283220471088892806902 579\
2 a^{26}

$+$ 21966981647139900762098833768771363493729745841150713021\
78866395/
627710173538668076383578942320766641610235544446403451289 6

a^{24}
K 4225087068643148676768347183720509407501373275384591424\
1202915/
1569275433846670190958947355801916604025588861116008628224
a^{22}
$+$ 9243030698501480502539960456963310049557169671584004585\
76111015/
5021681388309344611068631538566133132881884355571227610316\
8 a^{20}
K 2792278727591308970713274992053577643773765300526358112\
89238825/
2510840694154672305534315769283066566440942177785613805158\
4 a^{18}
$+$ 1481167281147853234192545138876244924249276009604452783\
72533975/
2510840694154672305534315769283066566440942177785613805158\
4 a^{16}
K 8552936730763916768351932993726600621584415910393936582\
244655/
3138550867693340381917894711603833208051177722232017256448
a^{14}
$+$ 2725520109624335018161297093642504227984132058017734306\
8867795/
2510840694154672305534315769283066566440942177785613805158\
4 a^{12}
K 4623985093552905952136490652238109606528950184859741577\
465655/
1255420347077336152767157884641533283220471088892806902579\
2 a^{10}
$+$ 1646216937353317014183729539051790588287820016231020643\
67575/
1569275433846670190958947355801916604025588861116008628224
a^{8}
$+$ 2378831552074549530319016496807979952281049519797994854\
01835/
5021681388309344611068631538566133132881884355571227610316\

$8\,a^{4}$

K 19356944143991777382139683668753077006835535157772965726\
8275/
78463771692333509547947367790095830201279443055800431411 2
a^{6}

K 19553621705156276865212401285166517199346342789467272378\
7465/
25108406941546723055343157692830665664409421777856138051 58\
$4\,a^{2}$

```
> Digits := 10:
> candidates := CodeTools[Usage]([fsolve(Eapprox=2, a)]);
```

memory used=1.26MiB, alloc change=0 bytes, cpu time=104.00ms, real time=104.00ms, gc time=0ns

candidates := [K 42.00546126, K 18.31060011, K 11.69241428, K 8.593888770, K 6.800000877, K 5.631435642, K 4.810581842, K 4.202909139, K 3.735332057, K 3.364736036, K 3.064049289, K 2.815419910, K 2.606601186, K 2.428913244, K 2.276056429, K 0.2336185007, 0.2336185007, 2.276056429, 2.428913244, 2.606601186, 2.815419910, 3.064049289, 3.364736036, 3.735332057, 4.202909139, 4.810581842, 5.631435642, 6.800000877, 8.593888770, 11.69241428, 18.31060011, 42.00546126] **(3.2)**

```
> a1 := select(`>`, candidates, 0)[1];
```

$$a1 := 0.2336185007 \qquad \textbf{(3.3)}$$

Méthode 2

Grâce à une relation de récurrence, on peut évaluer numeriquement E(a) efficacement.

Comme E est monotone sur [0,1], on peut résoudre E(a)=2, par dichotomie.

On peut trouver la récurrence par reconstruction, à partir des premiers termes. Pour la démontrer, on peut utiliser l'algorithme de Zeilberger.

```
> L := [seq(expand((1/16)^n*binomial(2*n, n)*add(binomial(n,k)^2*(1-a^2)^(k), k=0..n)), n=0..15)];
```

$$L := \left[1, \mathrm{K}\ \frac{a^{2}}{8} + \frac{1}{4}, \frac{9}{64}\ \mathrm{K}\ \frac{9}{64}\ a^{2} + \frac{3}{128}\ a^{4}, \frac{25}{256}\ \mathrm{K}\ \frac{75}{512}\ a^{2} + \frac{15}{256}\ a^{4}\right. \qquad \textbf{(4.1)}$$

$$\mathrm{K}\ \frac{5}{1024}\, a^{6}, \frac{1225}{16384}\ \mathrm{K}\ \frac{1225}{8192}\, a^{2} + \frac{1575}{16384}\, a^{4}\mathrm{K}\ \frac{175}{8192}\, a^{6} + \frac{35}{32768}\, a^{8},$$

$$\frac{3969}{65536}\ \mathrm{K}\ \frac{19845}{131072}\, a^{2} + \frac{2205}{16384}\, a^{4}\mathrm{K}\ \frac{6615}{131072}\, a^{6} + \frac{945}{131072}\, a^{8}$$

$$\mathrm{K}\ \frac{63}{262144}\, a^{10}, \frac{53361}{1048576}\ \mathrm{K}\ \frac{160083}{1048576}\, a^{2} + \frac{363825}{2097152}\, a^{4}\mathrm{K}\ \frac{24255}{262144}\, a^{6}$$

$$+ \frac{24255}{1048576}\, a^{8}\mathrm{K}\ \frac{4851}{2097152}\, a^{10} + \frac{231}{4194304}\, a^{12}, \frac{184041}{4194304}$$

$$\mathrm{K}\ \frac{429}{33554432}\, a^{14} + \frac{3003}{4194304}\, a^{12}\mathrm{K}\ \frac{81081}{8388608}\, a^{10} + \frac{225225}{4194304}\, a^{8}$$

$$+ \frac{891891}{4194304}\, a^{4}\mathrm{K}\ \frac{2477475}{16777216}\, a^{6}\mathrm{K}\ \frac{1288287}{8388608}\, a^{2}, \frac{41409225}{1073741824}$$

$$+ \frac{6435}{2147483648}\, a^{16}\mathrm{K}\ \frac{57915}{268435456}\, a^{14} + \frac{2027025}{536870912}\, a^{12}$$

$$\mathrm{K}\ \frac{7432425}{268435456}\, a^{10} + \frac{111486375}{1073741824}\, a^{8} + \frac{135270135}{536870912}\, a^{4}\mathrm{K}\ \frac{57972915}{268435456}\, a^{6}$$

$$\mathrm{K}\ \frac{41409225}{268435456}\, a^{2}, \frac{147744025}{4294967296}\ \mathrm{K}\ \frac{12155}{17179869184}\, a^{18}$$

$$+ \frac{546975}{8589934592}\, a^{16}\mathrm{K}\ \frac{6016725}{4294967296}\, a^{14} + \frac{14039025}{1073741824}\, a^{12}$$

$$\mathrm{K}\ \frac{547521975}{8589934592}\, a^{10} + \frac{766530765}{4294967296}\, a^{8} + \frac{78217425}{268435456}\, a^{4}$$

$$\mathrm{K}\ \frac{1277551275}{4294967296}\, a^{6}\mathrm{K}\ \frac{1329696225}{8589934592}\, a^{2}, \frac{2133423721}{68719476736}$$

$$+ \frac{46189}{274877906944}\, a^{20}\mathrm{K}\ \frac{2540395}{137438953472}\, a^{18} + \frac{68590665}{137438953472}\, a^{16}$$

$$\mathrm{K}\ \frac{99075405}{17179869184}\, a^{14} + \frac{4854694845}{137438953472}\, a^{12}\mathrm{K}\ \frac{8738450721}{68719476736}\, a^{10}$$

$$+ \frac{4854694845}{17179869184}\, a^{8} + \frac{45475610895}{137438953472}\, a^{4}\mathrm{K}\ \frac{1684281885}{4294967296}\, a^{6}$$

$$\mathrm{K}\ \frac{10667118605}{68719476736}\, a^{2}, \frac{7775536041}{274877906944}\ \mathrm{K}\ \frac{88179}{2199023255552}\, a^{22}$$

$$+ \frac{2909907}{549755813888}\, a^{20}\mathrm{K}\ \frac{189143955}{1099511627776}\, a^{18} + \frac{1324007685}{549755813888}\, a^{16}$$

$$\mathrm{K}\ \frac{19860115275}{1099511627776}\, a^{14} + \frac{5560832277}{68719476736}\, a^{12}\mathrm{K}\ \frac{31511382903}{137438953472}\, a^{10}$$

$$+ \frac{28939025115}{68719476736}\, a^{8} + \frac{101822495775}{274877906944}\, a^{4}\mathrm{K}\ \frac{549841477185}{1099511627776}\, a^{6}$$

$$\mathrm{K}\ \frac{85530896451}{549755813888}\, a^{2}, \frac{457028729521}{17592186044416} + \frac{676039}{70368744177664}\, a^{24}$$

$$\mathrm{K}\ \frac{26365521}{17592186044416}\ a^{22} + \frac{2030145117}{35184372088832}\ a^{20}$$
$$\mathrm{K}\ \frac{16917875975}{17592186044416}\ a^{18} + \frac{152260883775}{17592186044416}\ a^{16}$$
$$\mathrm{K}\ \frac{103537400967}{2199023255552}\ a^{14} + \frac{724761806769}{4398046511104}\ a^{12}\ \mathrm{K}\ \frac{843090265017}{2199023255552}\ a^{10}$$
$$+ \frac{21077256625425}{35184372088832}\ a^{8} + \frac{7213105600701}{17592186044416}\ a^{4}\ \mathrm{K}\ \frac{5464473939925}{8796093022208}\ a^{6}$$
$$\mathrm{K}\ \frac{1371086188563}{8796093022208}\ a^{2},\ \frac{1690195005625}{70368744177664}\ \mathrm{K}\ \frac{1300075}{562949953421312}\ a^{26}$$
$$+ \frac{118306825}{281474976710656}\ a^{24}\ \mathrm{K}\ \frac{5323807125}{281474976710656}\ a^{22}$$
$$+ \frac{6506875375}{17592186044416}\ a^{20}\ \mathrm{K}\ \frac{553084406875}{140737488355328}\ a^{18}$$
$$+ \frac{1791993478275}{70368744177664}\ a^{16}\ \mathrm{K}\ \frac{3783097343025}{35184372088832}\ a^{14}$$
$$+ \frac{2702212387875}{8796093022208}\ a^{12}\ \mathrm{K}\ \frac{170239380436125}{281474976710656}\ a^{10}$$
$$+ \frac{115594641036875}{140737488355328}\ a^{8} + \frac{7910112626325}{17592186044416}\ a^{4}$$
$$\mathrm{K}\ \frac{106347069753925}{140737488355328}\ a^{6}\ \mathrm{K}\ \frac{21972535073125}{140737488355328}\ a^{2},\ \frac{25145962430625}{1125899906842624}$$
$$+ \frac{5014575}{9007199254740992}\ a^{28}\ \mathrm{K}\ \frac{526530375}{4503599627370496}\ a^{26}$$
$$+ \frac{6844894875}{1125899906842624}\ a^{24}\ \mathrm{K}\ \frac{38787737625}{281474976710656}\ a^{22}$$
$$+ \frac{3839986024875}{2251799813685248}\ a^{20}\ \mathrm{K}\ \frac{14591946894525}{1125899906842624}\ a^{18}$$
$$+ \frac{72959734472625}{1125899906842624}\ a^{16}\ \mathrm{K}\ \frac{31268457631125}{140737488355328}\ a^{14}$$
$$+ \frac{2407671237596625}{4503599627370496}\ a^{12}\ \mathrm{K}\ \frac{2050979202397125}{2251799813685248}\ a^{10}$$
$$+ \frac{1230587521438275}{1125899906842624}\ a^{8} + \frac{1101765687238125}{2251799813685248}\ a^{4}$$
$$\mathrm{K}\ \frac{254253620131875}{281474976710656}\ a^{6}\ \mathrm{K}\ \frac{176021737014375}{1125899906842624}\ a^{2},$$
$$\frac{93990019574025}{4503599627370496}\ \mathrm{K}\ \frac{9694845}{72057594037927936}\ a^{30}$$
$$+ \frac{145422675}{4503599627370496}\ a^{28}\ \mathrm{K}\ \frac{17305298325}{9007199254740992}\ a^{26}$$

$$
+\frac{224968878225}{4503599627370496}a^{24}-\frac{12823226058825}{18014398509481984}a^{22}
+\frac{28211097329415}{4503599627370496}a^{20}-\frac{329129468843175}{9007199254740992}a^{18}
+\frac{664975865621925}{4503599627370496}a^{16}-\frac{1529444909304275}{36028797018963968}a^{14}
+\frac{3965226457967775}{4503599627370496}a^{12}-\frac{11895679373903325}{9007199254740992}a^{10}
+\frac{6390240985981125}{4503599627370496}a^{8}+\frac{2382160840927875}{4503599627370496}a^{4}
-\frac{19170722957943375}{18014398509481984}a^{6}-\frac{1409850293610375}{9007199254740992}a^{2}\Big]
$$

```
> rec := gfun[listtorec](L, u(n))[1];
```

$$
rec := \Big\{\left(4\,n^2+8\,n+3\right)a^4\,u(n)+\left(32\,a^2\,n^2+96\,a^2\,n+72\,a^2-64\,n^2-192\,n-144\right)u(n+1)+\left(64\,n^2+256\,n+256\right)u(n+2),\ u(0)=1,\ u(1)=-\frac{a^2}{8}+\frac{1}{4}\Big\} \quad (4.2)
$$

```
> SumTools[Hypergeometric][Zeilberger]((1/16)^n*binomial(2*n, n)*
  binomial(n,k)^2*(1-a^2)^(k), n, k, En);
```

$$
\Big[\left(64\,n^2+256\,n+256\right)En^2+\left(32\,a^2\,n^2+96\,a^2\,n+72\,a^2-64\,n^2-192\,n-144\right)En+4\,a^4\,n^2+8\,a^4\,n+3\,a^4,\ \frac{1}{64\,(-n+k-1)^2\,(-n+k-2)^2}\Bigg(\bigg(-\frac{a^2\left(4\,n^2+8\,n+3\right)k^2}{n^2+4\,n+4}+\frac{2\left(a^2\,n+2\,a^2+n+1\right)\left(4\,n^2+8\,n+3\right)k}{n^2+4\,n+4}-\frac{4\,a^2\,n^3+16\,a^2\,n^2+19\,a^2\,n+12\,n^3+6\,a^2+36\,n^2+33\,n+9}{n+2}\bigg)k^2\left(\frac{1}{16}\right)^n\binom{2\,n}{n}\binom{n}{k}^2\left(-a^2+1\right)^k\left(64\,n^2+256\,n+256\right)\Bigg)\Big] \quad (4.3)
$$

La récurrence permet de calculer les termes de E.

```
> simplify(isolate(simplify(subs(n=n-2, rec[1])), u(n)));
```

$$u(n) = -\frac{\left(n - \frac{1}{2}\right)\left(8\left(a^2 - 2\right)\left(n - \frac{1}{2}\right)u(n-1) + a^4\,u(n-2)\left(n - \frac{3}{2}\right)\right)}{16\,n^2} \qquad \textbf{(4.4)}$$

```
maxn := 0:
numeval := 0:

# calcule E()
evalE := proc(a)
     local tot, un, unm1, unm2, n:
     global maxn, numeval:
     numeval := numeval + 1:
     n := 1:
     un := -a^2/8 + 1/4:
     unm1 := 1:
     tot := unm1 + un:
     while unm1 > 10^(-Digits) do
          n := n + 1:
          un, unm1 := -((n - 1/2)*(8*(a^2 - 2)*(n - 1/2)*un + a^4*unm1*(n - 3/2)))/(16*n^2), un
:         tot := un + tot:
     end do:
     maxn := max(n, maxn):
     return tot:
end:

# Résout l'équation E()=val, par dichotomie
solveE := proc(val)
     local up, down, mid, ev:
     up := 1.0:
     down := 0.0:
     while up - down > 10^(-Digits + 1) do
          mid := (up+down)/2:
          ev := evalE(mid):
          if ev < 2 then
               up := mid:
          else
               down := mid:
          fi:
```

> Digits:=15;

$$Digits := 15 \qquad \textbf{(4.5)}$$

> a2 := CodeTools[Usage](solveE(2));

memory used=133.49MiB, alloc change=32.00MiB, cpu time=727.00ms, real time=539.00ms, gc time=318.62ms

$$a2 := 0.247655817895944 \qquad \textbf{(4.6)}$$

Vérification

> evalE(a2);

$$1.99999999999998 \qquad \textbf{(4.7)}$$

```
> numeval;
```

$$48 \tag{4.8}$$

```
> maxn;
```

$$3223 \tag{4.9}$$

Notre première approximation n'était vraiment pas très bonne

```
> evalE(a0);
Error, (in evalE) cannot determine if this expression is true or false: 0 < -1/8*a0^2+2499999999999999/10000000000000000
```

Méthode 3

On peut faire encore mieux et obtenir une description de E(a) comme une fonction hypergéométrique.

Pour cela, on commence par calculer une représentation intégrale.

```
> with(BinomSums):
```

```
> E := Sum((1/16)^n*t^n*Binomial(2*n, n)*Sum(Binomial(n,k)^2*(1-a^2)^(k), k=0..n), n=0..infinity);
```

$$E := \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n t^n \mathit{Binomial}(2\,n, n) \left(\sum_{k=0}^{n} \mathit{Binomial}(n, k)^2 \left(-a^2+1\right)^k\right) \tag{5.1}$$

```
> R, ord := sumtores(E, u);
```

$$R, \mathit{ord} := \frac{16}{(1+u_1)^2\,(1+u_2)\,a^2\,t - (1+u_2)^2\,(1+u_1)^2\,t + 16\,u_2\,u_1}, [t, u_1, u_2] \tag{5.2}$$

```
> R1 := subs(t=1, R);
```

$$R1 := \frac{16}{(1+u_1)^2\,(1+u_2)\,a^2 - (1+u_2)^2\,(1+u_1)^2 + 16\,u_2\,u_1} \tag{5.3}$$

```
> N := 4;
```

$$N := 4 \tag{5.4}$$

```
> p[0]*R1 + p[1]*Diff(R1, a) + Diff(R1, a, a) + Diff( add(add(b[i,j]*u[2]^i, i=0..N)*u[1]^j, j=0..N)/denom(R1)^2, u[2]) + Diff( add(add(c[i,j]*u[2]^i, i=0..N)*u[1]^j, j=0..N)/denom(R1)^2, u[1]);
```

$$\frac{16\,p_0}{(1+u_1)^2\,(1+u_2)\,a^2 - (1+u_2)^2\,(1+u_1)^2 + 16\,u_2\,u_1} + p_1\left(\frac{\partial}{\partial a}\left(\frac{16}{(1+u_1)^2\,(1+u_2)\,a^2 - (1+u_2)^2\,(1+u_1)^2 + 16\,u_2\,u_1}\right)\right) + \frac{\partial^2}{\partial a^2} \tag{5.5}$$

$$\left(\frac{16}{\left(1+u_1\right)^2\left(1+u_2\right)a^2 \mathrm{K}\ \left(1+u_2\right)^2\left(1+u_1\right)^2+16\,u_2\,u_1}\right)+\frac{\partial}{\partial u_2}\ \Big(\big(b_{4,\,0}$$
$$u_2^4+b_{3,\,0}\,u_2^3+b_{2,\,0}\,u_2^2+b_{1,\,0}\,u_2+b_{0,\,0}+\big(b_{4,\,1}\,u_2^4+b_{3,\,1}\,u_2^3+b_{2,\,1}\,u_2^2+b_{1,\,1}\,u_2$$
$$+b_{0,\,1}\big)\,u_1+\big(b_{4,\,2}\,u_2^4+b_{3,\,2}\,u_2^3+b_{2,\,2}\,u_2^2+b_{1,\,2}\,u_2+b_{0,\,2}\big)\,u_1^2+\big(b_{4,\,3}\,u_2^4+b_{3,\,3}$$
$$u_2^3+b_{2,\,3}\,u_2^2+b_{1,\,3}\,u_2+b_{0,\,3}\big)\,u_1^3+\big(b_{4,\,4}\,u_2^4+b_{3,\,4}\,u_2^3+b_{2,\,4}\,u_2^2+b_{1,\,4}\,u_2+b_{0,\,4}\big)$$
$$u_1^4\big)\Big/\Big(a^2\,u_1^2\,u_2+a^2\,u_1^2+2\,a^2\,u_1\,u_2 \mathrm{K}\ u_1^2\,u_2^2+2\,a^2\,u_1+a^2\,u_2 \mathrm{K}\ 2\,u_1^2\,u_2$$
$$\mathrm{K}\ 2\,u_1\,u_2^2+a^2 \mathrm{K}\ u_1^2+12\,u_2\,u_1 \mathrm{K}\ u_2^2 \mathrm{K}\ 2\,u_1 \mathrm{K}\ 2\,u_2 \mathrm{K}\ 1\Big)^2\Big)+\frac{\partial}{\partial u_1}\ \Big(\big(c_{4,\,0}\,u_2^4$$
$$+c_{3,\,0}\,u_2^3+c_{2,\,0}\,u_2^2+c_{1,\,0}\,u_2+c_{0,\,0}+\big(c_{4,\,1}\,u_2^4+c_{3,\,1}\,u_2^3+c_{2,\,1}\,u_2^2+c_{1,\,1}\,u_2$$
$$+c_{0,\,1}\big)\,u_1+\big(c_{4,\,2}\,u_2^4+c_{3,\,2}\,u_2^3+c_{2,\,2}\,u_2^2+c_{1,\,2}\,u_2+c_{0,\,2}\big)\,u_1^2+\big(c_{4,\,3}\,u_2^4+c_{3,\,3}\,u_2^3$$
$$+c_{2,\,3}\,u_2^2+c_{1,\,3}\,u_2+c_{0,\,3}\big)\,u_1^3+\big(c_{4,\,4}\,u_2^4+c_{3,\,4}\,u_2^3+c_{2,\,4}\,u_2^2+c_{1,\,4}\,u_2+c_{0,\,4}\big)\,u_1^4\big)$$
$$\Big/\Big(a^2\,u_1^2\,u_2+a^2\,u_1^2+2\,a^2\,u_1\,u_2 \mathrm{K}\ u_1^2\,u_2^2+2\,a^2\,u_1+a^2\,u_2 \mathrm{K}\ 2\,u_1^2\,u_2 \mathrm{K}\ 2\,u_1\,u_2^2$$
$$+a^2 \mathrm{K}\ u_1^2+12\,u_2\,u_1 \mathrm{K}\ u_2^2 \mathrm{K}\ 2\,u_1 \mathrm{K}\ 2\,u_2 \mathrm{K}\ 1\Big)^2\Big)$$

```
> eqs := [coeffs(collect(numer(normal(value(%))), {u[1], u[2]},
  distributed), {u[1], u[2]})];
```

$$eqs := \big[16\,a^4\,p_0 \mathrm{K}\ 32\,a^3\,p_1 \mathrm{K}\ 2\,a^2\,b_{0,\,0}+a^2\,b_{1,\,0} \mathrm{K}\ 4\,a^2\,c_{0,\,0}+a^2\,c_{0,\,1} \mathrm{K}\ 32\,a^2\,p_0 \qquad \textbf{(5.6)}$$
$$+96\,a^2+32\,p_1\,a+4\,b_{0,\,0} \mathrm{K}\ b_{1,\,0}+4\,c_{0,\,0} \mathrm{K}\ c_{0,\,1}+16\,p_0+32, \mathrm{K}\ c_{4,\,1}+4\,c_{4,\,0},$$
$$\mathrm{K}\ 2\,a^2\,b_{0,\,4}+a^2\,b_{1,\,4}+4\,b_{0,\,4} \mathrm{K}\ b_{1,\,4}, 2\,a^2\,b_{4,\,4}+b_{3,\,4} \mathrm{K}\ 4\,b_{4,\,4}, \mathrm{K}\ 4\,c_{4,\,4}+c_{4,\,3},$$
$$\mathrm{K}\ 4\,c_{4,\,4} \mathrm{K}\ 2\,c_{4,\,3}+2\,c_{4,\,2}, 4\,c_{4,\,0} \mathrm{K}\ 2\,c_{4,\,2}+2\,c_{4,\,1}, \mathrm{K}\ 3\,c_{4,\,3}+3\,c_{4,\,1}, \mathrm{K}\ a^2\,b_{1,\,4}$$
$$+2\,a^2\,b_{2,\,4}+4\,b_{0,\,4}+2\,b_{1,\,4} \mathrm{K}\ 2\,b_{2,\,4}, a^2\,b_{3,\,4}+4\,a^2\,b_{4,\,4}+2\,b_{2,\,4} \mathrm{K}\ 2\,b_{3,\,4}$$
$$\mathrm{K}\ 4\,b_{4,\,4}, 3\,a^2\,b_{3,\,4}+3\,b_{1,\,4} \mathrm{K}\ 3\,b_{3,\,4}, \mathrm{K}\ 3\,a^2\,c_{4,\,1}+3\,a^2\,c_{4,\,3}+3\,c_{3,\,1} \mathrm{K}\ 3\,c_{3,\,3}$$
$$+6\,c_{4,\,1} \mathrm{K}\ 6\,c_{4,\,3}, \mathrm{K}\ 4\,a^2\,c_{4,\,0} \mathrm{K}\ 2\,a^2\,c_{4,\,1}+2\,a^2\,c_{4,\,2}+4\,c_{3,\,0}+2\,c_{3,\,1} \mathrm{K}\ 2\,c_{3,\,2}$$
$$+8\,c_{4,\,0} \mathrm{K}\ 12\,c_{4,\,1} \mathrm{K}\ 4\,c_{4,\,2}, a^2\,b_{3,\,3}+2\,a^2\,b_{3,\,4}+4\,a^2\,b_{4,\,3}+8\,a^2\,b_{4,\,4}+2\,b_{2,\,3}$$
$$+4\,b_{2,\,4} \mathrm{K}\ 2\,b_{3,\,3}+12\,b_{3,\,4} \mathrm{K}\ 4\,b_{4,\,3} \mathrm{K}\ 8\,b_{4,\,4}, \mathrm{K}\ 4\,a^2\,c_{4,\,0}+a^2\,c_{4,\,1}+4\,c_{3,\,0}$$
$$\mathrm{K}\ c_{3,\,1} \mathrm{K}\ 24\,c_{4,\,0} \mathrm{K}\ 2\,c_{4,\,1}, \mathrm{K}\ 2\,a^2\,b_{0,\,3} \mathrm{K}\ 4\,a^2\,b_{0,\,4}+a^2\,b_{1,\,3}+2\,a^2\,b_{1,\,4}+4\,b_{0,\,3}$$
$$\mathrm{K}\ 24\,b_{0,\,4} \mathrm{K}\ b_{1,\,3} \mathrm{K}\ 2\,b_{1,\,4}, 2\,a^2\,b_{4,\,0} \mathrm{K}\ 4\,a^2\,c_{3,\,0}+a^2\,c_{3,\,1} \mathrm{K}\ 4\,a^2\,c_{4,\,0}+a^2\,c_{4,\,1}$$
$$+b_{3,\,0} \mathrm{K}\ 4\,b_{4,\,0}+4\,c_{2,\,0} \mathrm{K}\ c_{2,\,1} \mathrm{K}\ 24\,c_{3,\,0} \mathrm{K}\ 2\,c_{3,\,1}+4\,c_{4,\,0} \mathrm{K}\ c_{4,\,1}+16\,p_0,$$
$$16\,a^4\,p_0 \mathrm{K}\ 32\,a^3\,p_1 \mathrm{K}\ 2\,a^2\,b_{0,\,2} \mathrm{K}\ 4\,a^2\,b_{0,\,3} \mathrm{K}\ 2\,a^2\,b_{0,\,4}+a^2\,b_{1,\,2}+2\,a^2\,b_{1,\,3}$$

$$+ a^2 b_{1,4} \mathrm{K}\ a^2 c_{0,3} + 4 a^2 c_{0,4} \mathrm{K}\ 32 a^2 p_0 + 96 a^2 + 32 p_1 a + 4 b_{0,2} \mathrm{K}\ 24 b_{0,3}$$
$$+ 4 b_{0,4} \mathrm{K}\ b_{1,2} \mathrm{K}\ 2 b_{1,3} \mathrm{K}\ b_{1,4} + c_{0,3} \mathrm{K}\ 4 c_{0,4} + 16 p_0 + 32,\ a^2 b_{3,2} + 2 a^2 b_{3,3}$$
$$+ a^2 b_{3,4} + 4 a^2 b_{4,2} + 8 a^2 b_{4,3} + 4 a^2 b_{4,4} \mathrm{K}\ a^2 c_{2,3} + 4 a^2 c_{2,4} \mathrm{K}\ a^2 c_{3,3}$$
$$+ 4 a^2 c_{3,4} \mathrm{K}\ 32 a^2 p_0 + 32 p_1 a + 2 b_{2,2} + 4 b_{2,3} + 2 b_{2,4} \mathrm{K}\ 2 b_{3,2} + 12 b_{3,3}$$
$$\mathrm{K}\ 2 b_{3,4} \mathrm{K}\ 4 b_{4,2} \mathrm{K}\ 8 b_{4,3} \mathrm{K}\ 4 b_{4,4} + c_{1,3} \mathrm{K}\ 4 c_{1,4} + 2 c_{2,3} + 24 c_{2,4} + c_{3,3}$$
$$\mathrm{K}\ 4 c_{3,4} + 64 p_0 + 32,\ 16 a^4 p_0 \mathrm{K}\ 32 a^3 p_1 + 3 a^2 b_{3,2} + 6 a^2 b_{3,3} + 3 a^2 b_{3,4}$$
$$\mathrm{K}\ a^2 c_{1,3} + 4 a^2 c_{1,4} \mathrm{K}\ a^2 c_{2,3} + 4 a^2 c_{2,4} \mathrm{K}\ 96 a^2 p_0 + 96 a^2 + 96 p_1 a + 3 b_{1,2}$$
$$+ 6 b_{1,3} + 3 b_{1,4} \mathrm{K}\ 3 b_{3,2} \mathrm{K}\ 6 b_{3,3} \mathrm{K}\ 3 b_{3,4} + c_{0,3} \mathrm{K}\ 4 c_{0,4} + 2 c_{1,3} + 24 c_{1,4}$$
$$+ c_{2,3} \mathrm{K}\ 4 c_{2,4} + 96 p_0 + 96,\ a^2 b_{3,1} + 2 a^2 b_{3,2} + a^2 b_{3,3} + 4 a^2 b_{4,1}$$
$$+ 8 a^2 b_{4,2} + 4 a^2 b_{4,3} \mathrm{K}\ 2 a^2 c_{2,2} + 2 a^2 c_{2,3} + 4 a^2 c_{2,4} \mathrm{K}\ 2 a^2 c_{3,2} + 2 a^2 c_{3,3}$$
$$+ 4 a^2 c_{3,4} \mathrm{K}\ 128 a^2 p_0 + 128 p_1 a + 2 b_{2,1} + 4 b_{2,2} + 2 b_{2,3} \mathrm{K}\ 2 b_{3,1} + 12 b_{3,2}$$
$$\mathrm{K}\ 2 b_{3,3} \mathrm{K}\ 4 b_{4,1} \mathrm{K}\ 8 b_{4,2} \mathrm{K}\ 4 b_{4,3} + 2 c_{1,2} \mathrm{K}\ 2 c_{1,3} \mathrm{K}\ 4 c_{1,4} + 4 c_{2,2} + 12 c_{2,3}$$
$$\mathrm{K}\ 8 c_{2,4} + 2 c_{3,2} \mathrm{K}\ 2 c_{3,3} \mathrm{K}\ 4 c_{3,4} \mathrm{K}\ 256 p_0 + 128,\ 32 a^4 p_0 \mathrm{K}\ 64 a^3 p_1 \mathrm{K}\ a^2 b_{1,2}$$
$$\mathrm{K}\ 2 a^2 b_{1,3} \mathrm{K}\ a^2 b_{1,4} + 2 a^2 b_{2,2} + 4 a^2 b_{2,3} + 2 a^2 b_{2,4} \mathrm{K}\ a^2 c_{0,3} + 4 a^2 c_{0,4}$$
$$\mathrm{K}\ a^2 c_{1,3} + 4 a^2 c_{1,4} \mathrm{K}\ 96 a^2 p_0 + 192 a^2 + 96 p_1 a + 4 b_{0,2} + 8 b_{0,3} + 4 b_{0,4}$$
$$+ 2 b_{1,2} \mathrm{K}\ 12 b_{1,3} + 2 b_{1,4} \mathrm{K}\ 2 b_{2,2} \mathrm{K}\ 4 b_{2,3} \mathrm{K}\ 2 b_{2,4} + 2 c_{0,3} + 24 c_{0,4} + c_{1,3}$$
$$\mathrm{K}\ 4 c_{1,4} + 64 p_0 + 96,\ 64 a^4 p_0 \mathrm{K}\ 128 a^3 p_1 + 3 a^2 b_{3,1} + 6 a^2 b_{3,2} + 3 a^2 b_{3,3}$$
$$\mathrm{K}\ 2 a^2 c_{1,2} + 2 a^2 c_{1,3} + 4 a^2 c_{1,4} \mathrm{K}\ 2 a^2 c_{2,2} + 2 a^2 c_{2,3} + 4 a^2 c_{2,4} + 128 a^2 p_0$$
$$+ 384 a^2 \mathrm{K}\ 128 p_1 a + 3 b_{1,1} + 6 b_{1,2} + 3 b_{1,3} \mathrm{K}\ 3 b_{3,1} \mathrm{K}\ 6 b_{3,2} \mathrm{K}\ 3 b_{3,3}$$
$$+ 2 c_{0,2} \mathrm{K}\ 2 c_{0,3} \mathrm{K}\ 4 c_{0,4} + 4 c_{1,2} + 12 c_{1,3} \mathrm{K}\ 8 c_{1,4} + 2 c_{2,2} \mathrm{K}\ 2 c_{2,3} \mathrm{K}\ 4 c_{2,4}$$
$$\mathrm{K}\ 640 p_0 \mathrm{K}\ 128,\ a^2 b_{3,0} + 2 a^2 b_{3,1} + a^2 b_{3,2} + 4 a^2 b_{4,0} + 8 a^2 b_{4,1} + 4 a^2 b_{4,2}$$
$$\mathrm{K}\ 3 a^2 c_{2,1} + 3 a^2 c_{2,3} \mathrm{K}\ 3 a^2 c_{3,1} + 3 a^2 c_{3,3} \mathrm{K}\ 192 a^2 p_0 + 192 p_1 a + 2 b_{2,0}$$
$$+ 4 b_{2,1} + 2 b_{2,2} \mathrm{K}\ 2 b_{3,0} + 12 b_{3,1} \mathrm{K}\ 2 b_{3,2} \mathrm{K}\ 4 b_{4,0} \mathrm{K}\ 8 b_{4,1} \mathrm{K}\ 4 b_{4,2} + 3 c_{1,1}$$
$$\mathrm{K}\ 3 c_{1,3} + 6 c_{2,1} \mathrm{K}\ 6 c_{2,3} + 3 c_{3,1} \mathrm{K}\ 3 c_{3,3} \mathrm{K}\ 640 p_0 + 192,\ 128 a^4 p_0$$
$$\mathrm{K}\ 256 a^3 p_1 \mathrm{K}\ a^2 b_{1,1} \mathrm{K}\ 2 a^2 b_{1,2} \mathrm{K}\ a^2 b_{1,3} + 2 a^2 b_{2,1} + 4 a^2 b_{2,2} + 2 a^2 b_{2,3}$$
$$\mathrm{K}\ 2 a^2 c_{0,2} + 2 a^2 c_{0,3} + 4 a^2 c_{0,4} \mathrm{K}\ 2 a^2 c_{1,2} + 2 a^2 c_{1,3} + 4 a^2 c_{1,4} + 128 a^2 p_0$$
$$+ 768 a^2 \mathrm{K}\ 128 p_1 a + 4 b_{0,1} + 8 b_{0,2} + 4 b_{0,3} + 2 b_{1,1} \mathrm{K}\ 12 b_{1,2} + 2 b_{1,3}$$

$\mathrm{K}\ 2\,b_{2,\,1}\mathrm{K}\ 4\,b_{2,\,2}\mathrm{K}\ 2\,b_{2,\,3}+4\,c_{0,\,2}+12\,c_{0,\,3}\mathrm{K}\ 8\,c_{0,\,4}+2\,c_{1,\,2}\mathrm{K}\ 2\,c_{1,\,3}\mathrm{K}\ 4\,c_{1,\,4}$

$\mathrm{K}\ 256\,p_0\mathrm{K}\ 128,\ 2\,a^2\,b_{3,\,0}+a^2\,b_{3,\,1}+8\,a^2\,b_{4,\,0}+4\,a^2\,b_{4,\,1}\mathrm{K}\ 4\,a^2\,c_{2,\,0}$

$\mathrm{K}\ 2\,a^2\,c_{2,\,1}+2\,a^2\,c_{2,\,2}\mathrm{K}\ 4\,a^2\,c_{3,\,0}\mathrm{K}\ 2\,a^2\,c_{3,\,1}+2\,a^2\,c_{3,\,2}\mathrm{K}\ 128\,a^2\,p_0+128\,p_1\,a$

$+\,4\,b_{2,\,0}+2\,b_{2,\,1}+12\,b_{3,\,0}\mathrm{K}\ 2\,b_{3,\,1}\mathrm{K}\ 8\,b_{4,\,0}\mathrm{K}\ 4\,b_{4,\,1}+4\,c_{1,\,0}+2\,c_{1,\,1}\mathrm{K}\ 2\,c_{1,\,2}$

$+\,8\,c_{2,\,0}\mathrm{K}\ 12\,c_{2,\,1}\mathrm{K}\ 4\,c_{2,\,2}+4\,c_{3,\,0}+2\,c_{3,\,1}\mathrm{K}\ 2\,c_{3,\,2}\mathrm{K}\ 256\,p_0+128,\ 96\,a^4\,p_0$

$\mathrm{K}\ 192\,a^3\,p_1+3\,a^2\,b_{3,\,0}+6\,a^2\,b_{3,\,1}+3\,a^2\,b_{3,\,2}\mathrm{K}\ 3\,a^2\,c_{1,\,1}+3\,a^2\,c_{1,\,3}\mathrm{K}\ 3\,a^2\,c_{2,\,1}$

$+\,3\,a^2\,c_{2,\,3}+448\,a^2\,p_0+576\,a^2\mathrm{K}\ 448\,p_1\,a+3\,b_{1,\,0}+6\,b_{1,\,1}+3\,b_{1,\,2}\mathrm{K}\ 3\,b_{3,\,0}$

$\mathrm{K}\ 6\,b_{3,\,1}\mathrm{K}\ 3\,b_{3,\,2}+3\,c_{0,\,1}\mathrm{K}\ 3\,c_{0,\,3}+6\,c_{1,\,1}\mathrm{K}\ 6\,c_{1,\,3}+3\,c_{2,\,1}\mathrm{K}\ 3\,c_{2,\,3}+2624\,p_0$

$\mathrm{K}\ 448,\ 192\,a^4\,p_0\mathrm{K}\ 384\,a^3\,p_1\mathrm{K}\ a^2\,b_{1,\,0}\mathrm{K}\ 2\,a^2\,b_{1,\,1}\mathrm{K}\ a^2\,b_{1,\,2}+2\,a^2\,b_{2,\,0}$

$+\,4\,a^2\,b_{2,\,1}+2\,a^2\,b_{2,\,2}\mathrm{K}\ 3\,a^2\,c_{0,\,1}+3\,a^2\,c_{0,\,3}\mathrm{K}\ 3\,a^2\,c_{1,\,1}+3\,a^2\,c_{1,\,3}+448\,a^2\,p_0$

$+\,1152\,a^2\mathrm{K}\ 448\,p_1\,a+4\,b_{0,\,0}+8\,b_{0,\,1}+4\,b_{0,\,2}+2\,b_{1,\,0}\mathrm{K}\ 12\,b_{1,\,1}+2\,b_{1,\,2}$

$\mathrm{K}\ 2\,b_{2,\,0}\mathrm{K}\ 4\,b_{2,\,1}\mathrm{K}\ 2\,b_{2,\,2}+6\,c_{0,\,1}\mathrm{K}\ 6\,c_{0,\,3}+3\,c_{1,\,1}\mathrm{K}\ 3\,c_{1,\,3}\mathrm{K}\ 640\,p_0\mathrm{K}\ 448,$

$64\,a^4\,p_0\mathrm{K}\ 128\,a^3\,p_1+6\,a^2\,b_{3,\,0}+3\,a^2\,b_{3,\,1}\mathrm{K}\ 4\,a^2\,c_{1,\,0}\mathrm{K}\ 2\,a^2\,c_{1,\,1}+2\,a^2\,c_{1,\,2}$

$\mathrm{K}\ 4\,a^2\,c_{2,\,0}\mathrm{K}\ 2\,a^2\,c_{2,\,1}+2\,a^2\,c_{2,\,2}+128\,a^2\,p_0+384\,a^2\mathrm{K}\ 128\,p_1\,a+6\,b_{1,\,0}$

$+\,3\,b_{1,\,1}\mathrm{K}\ 6\,b_{3,\,0}\mathrm{K}\ 3\,b_{3,\,1}+4\,c_{0,\,0}+2\,c_{0,\,1}\mathrm{K}\ 2\,c_{0,\,2}+8\,c_{1,\,0}\mathrm{K}\ 12\,c_{1,\,1}\mathrm{K}\ 4\,c_{1,\,2}$

$+\,4\,c_{2,\,0}+2\,c_{2,\,1}\mathrm{K}\ 2\,c_{2,\,2}\mathrm{K}\ 640\,p_0\mathrm{K}\ 128,\ \mathrm{K}\,a^2\,c_{4,\,3}+4\,a^2\,c_{4,\,4}+c_{3,\,3}\mathrm{K}\ 4\,c_{3,\,4}$

$+\,2\,c_{4,\,3}+24\,c_{4,\,4},\ \mathrm{K}\,2\,a^2\,c_{4,\,2}+2\,a^2\,c_{4,\,3}+4\,a^2\,c_{4,\,4}+2\,c_{3,\,2}\mathrm{K}\ 2\,c_{3,\,3}\mathrm{K}\ 4\,c_{3,\,4}$

$+\,4\,c_{4,\,2}+12\,c_{4,\,3}\mathrm{K}\ 8\,c_{4,\,4},\ 2\,a^2\,b_{4,\,3}+4\,a^2\,b_{4,\,4}+b_{3,\,3}+2\,b_{3,\,4}\mathrm{K}\ 4\,b_{4,\,3}$

$+\,24\,b_{4,\,4},\ 3\,a^2\,b_{3,\,3}+6\,a^2\,b_{3,\,4}+3\,b_{1,\,3}+6\,b_{1,\,4}\mathrm{K}\ 3\,b_{3,\,3}\mathrm{K}\ 6\,b_{3,\,4},\ \mathrm{K}\,a^2\,b_{1,\,3}$

$\mathrm{K}\ 2\,a^2\,b_{1,\,4}+2\,a^2\,b_{2,\,3}+4\,a^2\,b_{2,\,4}+4\,b_{0,\,3}+8\,b_{0,\,4}+2\,b_{1,\,3}\mathrm{K}\ 12\,b_{1,\,4}\mathrm{K}\ 2\,b_{2,\,3}$

$\mathrm{K}\ 4\,b_{2,\,4},\ 2\,a^2\,b_{4,\,2}+4\,a^2\,b_{4,\,3}+2\,a^2\,b_{4,\,4}\mathrm{K}\ a^2\,c_{3,\,3}+4\,a^2\,c_{3,\,4}\mathrm{K}\ a^2\,c_{4,\,3}$

$+\,4\,a^2\,c_{4,\,4}+b_{3,\,2}+2\,b_{3,\,3}+b_{3,\,4}\mathrm{K}\ 4\,b_{4,\,2}+24\,b_{4,\,3}\mathrm{K}\ 4\,b_{4,\,4}+c_{2,\,3}\mathrm{K}\ 4\,c_{2,\,4}$

$+\,2\,c_{3,\,3}+24\,c_{3,\,4}+c_{4,\,3}\mathrm{K}\ 4\,c_{4,\,4}+16\,p_0,\ 2\,a^2\,b_{4,\,1}+4\,a^2\,b_{4,\,2}+2\,a^2\,b_{4,\,3}$

$\mathrm{K}\ 2\,a^2\,c_{3,\,2}+2\,a^2\,c_{3,\,3}+4\,a^2\,c_{3,\,4}\mathrm{K}\ 2\,a^2\,c_{4,\,2}+2\,a^2\,c_{4,\,3}+4\,a^2\,c_{4,\,4}+b_{3,\,1}$

$+\,2\,b_{3,\,2}+b_{3,\,3}\mathrm{K}\ 4\,b_{4,\,1}+24\,b_{4,\,2}\mathrm{K}\ 4\,b_{4,\,3}+2\,c_{2,\,2}\mathrm{K}\ 2\,c_{2,\,3}\mathrm{K}\ 4\,c_{2,\,4}+4\,c_{3,\,2}$

$+\,12\,c_{3,\,3}\mathrm{K}\ 8\,c_{3,\,4}+2\,c_{4,\,2}\mathrm{K}\ 2\,c_{4,\,3}\mathrm{K}\ 4\,c_{4,\,4}+64\,p_0,\ 96\,a^4\,p_0\mathrm{K}\ 192\,a^3\,p_1$

$\mathrm{K}\ 2\,a^2\,b_{0,\,0}\mathrm{K}\ 4\,a^2\,b_{0,\,1}\mathrm{K}\ 2\,a^2\,b_{0,\,2}+a^2\,b_{1,\,0}+2\,a^2\,b_{1,\,1}+a^2\,b_{1,\,2}\mathrm{K}\ 3\,a^2\,c_{0,\,1}$

$+\,3\,a^2\,c_{0,\,3}\mathrm{K}\;\,192\,a^2\,p_0+576\,a^2+192\,p_1\,a+4\,b_{0,\,0}\mathrm{K}\;\,24\,b_{0,\,1}+4\,b_{0,\,2}\mathrm{K}\;\,b_{1,\,0}$

$\mathrm{K}\;\,2\,b_{1,\,1}\mathrm{K}\;\,b_{1,\,2}+3\,c_{0,\,1}\mathrm{K}\;\,3\,c_{0,\,3}+96\,p_0+192,\,64\,a^4\,p_0\mathrm{K}\;\,128\,a^3\,p_1$

$\mathrm{K}\;\,2\,a^2\,b_{0,\,1}\mathrm{K}\;\,4\,a^2\,b_{0,\,2}\mathrm{K}\;\,2\,a^2\,b_{0,\,3}+a^2\,b_{1,\,1}+2\,a^2\,b_{1,\,2}+a^2\,b_{1,\,3}\mathrm{K}\;\,2\,a^2\,c_{0,\,2}$

$+\,2\,a^2\,c_{0,\,3}+4\,a^2\,c_{0,\,4}\mathrm{K}\;\,128\,a^2\,p_0+384\,a^2+128\,p_1\,a+4\,b_{0,\,1}\mathrm{K}\;\,24\,b_{0,\,2}$

$+\,4\,b_{0,\,3}\mathrm{K}\;\,b_{1,\,1}\mathrm{K}\;\,2\,b_{1,\,2}\mathrm{K}\;\,b_{1,\,3}+2\,c_{0,\,2}\mathrm{K}\;\,2\,c_{0,\,3}\mathrm{K}\;\,4\,c_{0,\,4}+64\,p_0+128,$

$16\,a^4\,p_0\mathrm{K}\;\,32\,a^3\,p_1+3\,a^2\,b_{3,\,0}\mathrm{K}\;\,4\,a^2\,c_{1,\,0}+a^2\,c_{1,\,1}\mathrm{K}\;\,4\,a^2\,c_{2,\,0}+a^2\,c_{2,\,1}$

$\mathrm{K}\;\,96\,a^2\,p_0+96\,a^2+96\,p_1\,a+3\,b_{1,\,0}\mathrm{K}\;\,3\,b_{3,\,0}+4\,c_{0,\,0}\mathrm{K}\;\,c_{0,\,1}\mathrm{K}\;\,24\,c_{1,\,0}\mathrm{K}\;\,2\,c_{1,\,1}$

$+\,4\,c_{2,\,0}\mathrm{K}\;\,c_{2,\,1}+96\,p_0+96,\,a^2\,b_{3,\,0}+4\,a^2\,b_{4,\,0}\mathrm{K}\;\,4\,a^2\,c_{2,\,0}+a^2\,c_{2,\,1}\mathrm{K}\;\,4\,a^2\,c_{3,\,0}$

$+\,a^2\,c_{3,\,1}\mathrm{K}\;\,32\,a^2\,p_0+32\,p_1\,a+2\,b_{2,\,0}\mathrm{K}\;\,2\,b_{3,\,0}\mathrm{K}\;\,4\,b_{4,\,0}+4\,c_{1,\,0}\mathrm{K}\;\,c_{1,\,1}$

$\mathrm{K}\;\,24\,c_{2,\,0}\mathrm{K}\;\,2\,c_{2,\,1}+4\,c_{3,\,0}\mathrm{K}\;\,c_{3,\,1}+64\,p_0+32,\,128\,a^4\,p_0\mathrm{K}\;\,256\,a^3\,p_1$

$\mathrm{K}\;\,2\,a^2\,b_{1,\,0}\mathrm{K}\;\,a^2\,b_{1,\,1}+4\,a^2\,b_{2,\,0}+2\,a^2\,b_{2,\,1}\mathrm{K}\;\,4\,a^2\,c_{0,\,0}\mathrm{K}\;\,2\,a^2\,c_{0,\,1}+2\,a^2\,c_{0,\,2}$

$\mathrm{K}\;\,4\,a^2\,c_{1,\,0}\mathrm{K}\;\,2\,a^2\,c_{1,\,1}+2\,a^2\,c_{1,\,2}+128\,a^2\,p_0+768\,a^2\mathrm{K}\;\,128\,p_1\,a+8\,b_{0,\,0}$

$+\,4\,b_{0,\,1}\mathrm{K}\;\,12\,b_{1,\,0}+2\,b_{1,\,1}\mathrm{K}\;\,4\,b_{2,\,0}\mathrm{K}\;\,2\,b_{2,\,1}+8\,c_{0,\,0}\mathrm{K}\;\,12\,c_{0,\,1}\mathrm{K}\;\,4\,c_{0,\,2}+4\,c_{1,\,0}$

$+\,2\,c_{1,\,1}\mathrm{K}\;\,2\,c_{1,\,2}\mathrm{K}\;\,256\,p_0\mathrm{K}\;\,128,\,2\,a^2\,b_{4,\,0}+4\,a^2\,b_{4,\,1}+2\,a^2\,b_{4,\,2}\mathrm{K}\;\,3\,a^2\,c_{3,\,1}$

$+\,3\,a^2\,c_{3,\,3}\mathrm{K}\;\,3\,a^2\,c_{4,\,1}+3\,a^2\,c_{4,\,3}+b_{3,\,0}+2\,b_{3,\,1}+b_{3,\,2}\mathrm{K}\;\,4\,b_{4,\,0}+24\,b_{4,\,1}$

$\mathrm{K}\;\,4\,b_{4,\,2}+3\,c_{2,\,1}\mathrm{K}\;\,3\,c_{2,\,3}+6\,c_{3,\,1}\mathrm{K}\;\,6\,c_{3,\,3}+3\,c_{4,\,1}\mathrm{K}\;\,3\,c_{4,\,3}+96\,p_0,\,4\,a^2\,b_{4,\,0}$

$+\,2\,a^2\,b_{4,\,1}\mathrm{K}\;\,4\,a^2\,c_{3,\,0}\mathrm{K}\;\,2\,a^2\,c_{3,\,1}+2\,a^2\,c_{3,\,2}\mathrm{K}\;\,4\,a^2\,c_{4,\,0}\mathrm{K}\;\,2\,a^2\,c_{4,\,1}+2\,a^2\,c_{4,\,2}$

$+\,2\,b_{3,\,0}+b_{3,\,1}+24\,b_{4,\,0}\mathrm{K}\;\,4\,b_{4,\,1}+4\,c_{2,\,0}+2\,c_{2,\,1}\mathrm{K}\;\,2\,c_{2,\,2}+8\,c_{3,\,0}\mathrm{K}\;\,12\,c_{3,\,1}$

$\mathrm{K}\;\,4\,c_{3,\,2}+4\,c_{4,\,0}+2\,c_{4,\,1}\mathrm{K}\;\,2\,c_{4,\,2}+64\,p_0,\,32\,a^4\,p_0\mathrm{K}\;\,64\,a^3\,p_1\mathrm{K}\;\,a^2\,b_{1,\,0}$

$+\,2\,a^2\,b_{2,\,0}\mathrm{K}\;\,4\,a^2\,c_{0,\,0}+a^2\,c_{0,\,1}\mathrm{K}\;\,4\,a^2\,c_{1,\,0}+a^2\,c_{1,\,1}\mathrm{K}\;\,96\,a^2\,p_0+192\,a^2$

$+\,96\,p_1\,a+4\,b_{0,\,0}+2\,b_{1,\,0}\mathrm{K}\;\,2\,b_{2,\,0}\mathrm{K}\;\,24\,c_{0,\,0}\mathrm{K}\;\,2\,c_{0,\,1}+4\,c_{1,\,0}\mathrm{K}\;\,c_{1,\,1}+64\,p_0$

$+\,96,\,64\,a^4\,p_0\mathrm{K}\;\,128\,a^3\,p_1\mathrm{K}\;\,4\,a^2\,b_{0,\,0}\mathrm{K}\;\,2\,a^2\,b_{0,\,1}+2\,a^2\,b_{1,\,0}+a^2\,b_{1,\,1}$

$\mathrm{K}\;\,4\,a^2\,c_{0,\,0}\mathrm{K}\;\,2\,a^2\,c_{0,\,1}+2\,a^2\,c_{0,\,2}\mathrm{K}\;\,128\,a^2\,p_0+384\,a^2+128\,p_1\,a\mathrm{K}\;\,24\,b_{0,\,0}$

$+\,4\,b_{0,\,1}\mathrm{K}\;\,2\,b_{1,\,0}\mathrm{K}\;\,b_{1,\,1}+4\,c_{0,\,0}+2\,c_{0,\,1}\mathrm{K}\;\,2\,c_{0,\,2}+64\,p_0+128\big]$

```
> nops(eqs);
```

45 **(5.7)**

```
> nops(indets(eqs));
```

53 **(5.8)**

```
> sol := solve(eqs, indets(eqs) minus {a}):
```

```
> deq := simplify(numer(normal(subs(sol, p[0]*y(a)+p[1]*diff(y(a),
  a)+diff(y(a), a, a)))));
```

$$deq := \left(a^7 + 3\,a^5 \mathrm{K}\ 36\,a^3 + 32\,a\right) \left(\frac{d^2}{da^2}\,y(a)\right) + \left(3\,a^6 \mathrm{K}\ 13\,a^4 \mathrm{K}\ 76\,a^2 + 32\right) \left(\frac{d}{da}\,y(a)\right) + y(a)\,a\,\left(a^4 \mathrm{K}\ 12\,a^2 \mathrm{K}\ 16\right) \tag{5.9}$$

```
> ysol := rhs(dsolve(deq));
```

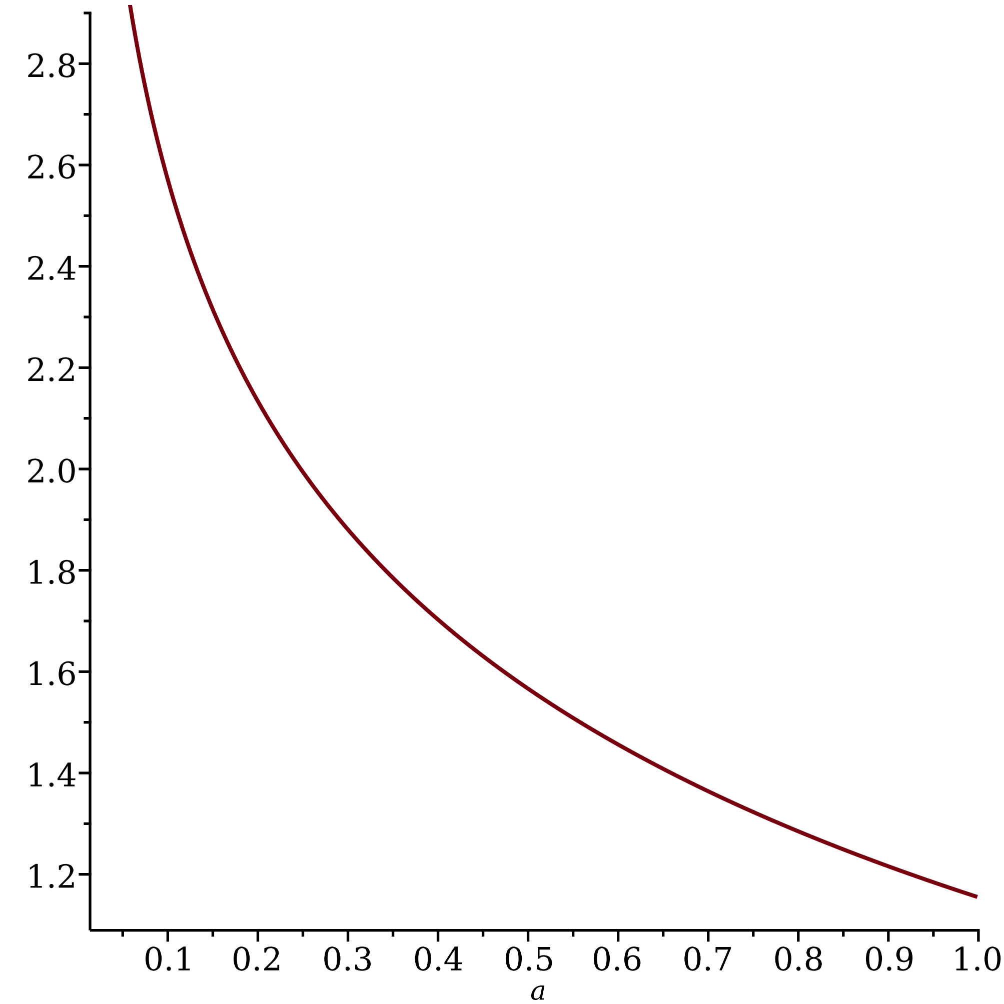
$$ysol := \frac{\_C1\ \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{\mathrm{K}\ 4\,a^2 + 4}{\left(a^2+2\right)^2}\right)}{\sqrt{a^2+2}} + \frac{\_C2\ \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{a^2\left(a^2+8\right)}{\left(a^2+2\right)^2}\right)}{\sqrt{a^2+2}} \tag{5.10}$$

En regardant la valeur E(1) = 2 sqrt(3)/3, on obtient les constantes

```
> ysol := subs(_C1=2, _C2=0, ysol);
```

$$ysol := \frac{2\ \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{\mathrm{K}\ 4\,a^2 + 4}{\left(a^2+2\right)^2}\right)}{\sqrt{a^2+2}} \tag{5.11}$$

```
> plot(ysol, a=0..1);
```



https://dlmf.nist.gov/15.8#E13

```
> zofa := solve({z > 0, (z/(2-z))^2 = (-4*a^2 + 4)/(a^2 + 2)^2}, z)
  assuming a > 0 and a < 1;
```

$$zofa := \left\{z = \frac{4\left(2\,a^2 + \sqrt{\mathrm{K}\,a^6 \mathrm{K}\;3\,a^4 + 4}\;\mathrm{K}\;2\right)}{a^2\,\left(a^2 + 8\right)}\right\} \qquad \textbf{(5.12)}$$

```
> ysolfast := simplify(subs(zofa, 2*hypergeom([1/2, 1/2], [1], z)*
  sqrt(1-z/2)/sqrt(a^2 + 2))) assuming a < 1 and a > 0;
```

$$ysolfast := \qquad \textbf{(5.13)}$$

$$\frac{1}{\pi\,a\,\sqrt{a^2+8}}\left(4\,\mathrm{EllipticK}\left(\frac{2\,\sqrt{\sqrt{\mathrm{K}\,a^2+1}\;a^2 + 2\,a^2 + 2\,\sqrt{\mathrm{K}\,a^2+1}\;\mathrm{K}\;2}}{a\,\sqrt{a^2+8}}\right)\right.$$
$$\left.\sqrt{a^2 \mathrm{K}\;2\,\sqrt{\mathrm{K}\,a^2+1}+2}\right)$$

```
> Digits := 2000:
```

```
> CodeTools[Usage](evalf(eval(ysol, a=1/3)));
memory used=0.56GiB, alloc change=0 bytes, cpu time=1.98s, real
time=1.44s, gc time=838.12ms

1.815274437706456910586455632851310382469250521335047134032O\                (5.14)
    17265858917360048676933252854606921820148119680347784287 5\
    490661341215318366783465824875799630987863320919039921438\
    782220553391628477576156539206053790157698469170746821995\
    036986490371559617769339733795330946210455965421035600323\
    816626696003017662093694694999380191762676952021801394985\
    635969718044899193138741169898639457377591964312914895812\
    090624277999136589312829024993034663072330788249618551149\
    229919144870019630670888718290775199954100193040429677435\
    140329861841348936631647821501704199218119446585037661101\
    728157551135211827102243538566366916979952231043553857929\
    034592206314682125644499989416419284422293659742923977717\
    824322185019521396482476006840345105960317189504487296310\
    279008688566760087450456831274212657678743751941052385632\
    491491509360573252085347043739312596971664603450475117779\
    291166230955034586146300421553037046562852515969548163782\
    218309245296982744046436255494327926408591679065465631966\
    948375041628434357271542565895731835362484735552026991427\
    457960057985228208073452091921342266163589881320878766188\
    506503482611038662547591976366153704646691529088244761787\
    768339948340311759767784700177986474939404057544660347243\
    146324535724397310116139923813795489770274966212220798791\
    429735045669705559030174112290647849872266309066458515598\
    771093341937444295978644021340046862309764720280456832700\
    805932199130587797734433927471420290713910603760122613264\
    726974637553241169672898866666766764463753087012443905996\
    006051091712522193808331634025065470811338396799447476184\
    301998963492250588207190323041594064759254145998797378534\
    696850202231132325707337905331766687281225394732713747517\
    233085958935528528949936979020233762587905697957980892431\
    565077625047270263078176000790729915481099582925249527872\
    116904235248741060663020604400965678017866432037007791056\
    836997477793632916790419057409052102274430801434170491670\
    510660116652461393184851628985775455257820540334121185818\
    771676691834097718743668801427933235771120683546001545196\
```

```
    801
> CodeTools[Usage](evalf(eval(ysolfast, a=1/3)));
memory used=1.23MiB, alloc change=0 bytes, cpu time=6.00ms,
real time=6.00ms, gc time=0ns

1.815274437706456910586455632851310382469250521335047134032O\   (5.15)
    17265858917360048676933252854606921820148119680347784287 5\
    490661341215318366783465824875799630987863320919039921438\
    782220553391628477576156539206053790157698469170746821995\
    036986490371559617769339733795330946210455965421035600323\
    816626696003017662093694694999380191762676952021801394985\
    635969718044899193138741169898639457377591964312914895812\
    090624277999136589312829024993034663072330788249618551149\
    229919144870019630670888718290775199954100193040429677435\
    140329861841348936631647821501704199218119446585037661101\
    728157551135211827102243538566366916979952231043553857929\
    034592206314682125644499989416419284422293659742923977717\
    824322185019521396482476006840345105960317189504487296310\
    279008688566760087450456831274212657678743751941052385632\
    491491509360573252085347043739312596971664603450475117779\
    291166230955034586146300421553037046562852515969548163782\
    218309245296982744046436255494327926408591679065465631966\
    948375041628434357271542565895731835362484735552026991427\
    457960057985228208073452091921342266163589881320878766188\
    506503482611038662547591976366153704646691529088244761787\
    768339948340311759767784700177986474939404057544660347243\
    146324535724397310116139923813795489770274966212220798791\
    429735045669705559030174112290647849872266309066458515598\
    771093341937444295978644021340046862309764720280456832700\
    805932199130587797734433927471420290713910603760122613264\
    726974637553241169672898866666766764463753087012443905996\
    006051091712522193808331634025065470811338396799447476184\
    301998963492250588207190323041594064759254145998797378534\
    696850202231132325707337905331766687281225394732713747517\
    233085958935528528949936979020233762587905697957980892431\
    565077625047270263078176000790729915481099582925249527872\
    116904235248741060663020604400965678017866432037007791056\
    836997477793632916790419057409052102274430801434170491670\
    510660116652461393184851628985775455257820540334121185818\
```

771676691834097718743668801427933235771120683546001545196\
789

```
nbiter := 0:

solveEnewt := proc(val)
     local a, delta, cur, E, Ediff;
     global nbiter;

     cur := 1/4.;
     E := 4*EllipticK(2*sqrt(sqrt(-a^2 + 1)*a^2 + 2*a^2 + 2*sqrt(-a^2 + 1) - 2)/(a*sqrt(a^2 + 8)))*sqrt(a^2 - 2*sqrt(-a^2 + 1) + 2)/(Pi*a*sqrt(a^2 + 8)):
     Ediff := simplify(diff(E, a)):

     while true do
          nbiter := nbiter + 1;
          delta := eval((E-val)/Ediff, a=cur):
          if abs(delta) < 10^(-Digits+3) then
               break
          else
               cur := cur - delta:
          end if:
     end do:

     return cur:
end:
```

> Digits := 1000;

$$Digits := 1000 \tag{5.16}$$

> a3 := CodeTools[Usage](solveEnewt(2));

memory used=45.47MiB, alloc change=32.00MiB, cpu time=
353.00ms, real time=261.00ms, gc time=142.42ms

$a3 :=$ **(5.17)**

0.247655817895963771392700865892848707998555099993448304 2\
458690354397240411903846338362842258300835144548671104865\
841419081408663047914495568307077060840216984891745844240\
879324346658004462057792046762175342832933220894234098837\
398471470354631728104770129192737102159329226381535867961\
188015168058011733581145516620537136526952782061883264163\
195041917798650106956162819428363670545814570110350470007\
747276265741706605216430771035857175178294355948012549477\
159797992375125163819995863068963304017141576742722757823\
083975198859659456308193047705323530355639445734537582581\
311076650082072175694730918891928340618084073697433295659\
127606781650928877566206711193788973968567342359833590607\

```
3568099991989061495436057595990489374732053566788887964555\
4569960847944986429766092459377882348730535207409277305 47\
4560567685165344640146232634984750417963827717111923658 34\
4628196109437071271084271003004160932910508645606072365 28\
7853389729223542173279192690782749387473348944549576839 05\
54260157104633622909536332698924 4
```

```
> nbiter;
```

10 **(5.18)**

```
> evalf(eval(ysolfast, a=a3));
```

1.9999999999999999999999999999999999999999999999999999999999\ **(5.19)**

```
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
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999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
999999999999999999999999999999999999999999999999999999999\
99999999999999999999999999920
```

```
> evalf(eval(ysolfast, a=a2));
```

2.0000000000000498521930193770018463073092039047463966162828\ **(5.20)**

```
185241159719206608587498450497037558864730310019152180283\
854031725332364858104536756797893549671314624750355366721\
198239967013090299573099324300467226681014365546086284519\
676019302280543004642952746284066751921401552005987151272\
620112793514972660384397186958520682206656772091515099773\
291875194634316131670072162781217166198803870212758349243\
507735570627413915633454164290060067751631781095561926510\
```

```
0760744941133136852464830665958835383224098256475674844 12\
3794855155959812813262221256439992802032295257060844259 36\
1778180202964245581857433191029823092992457701378011454 63\
3893308547630664674045971569025607575833175258258454414 05\
1598091285272176484677747465879836730559110516226656389 43\
5678688449737047530243796351182176666423941716540274135 09\
6342109496966360335917635921911240420474537061211530974 55\
9680374957755464868936987608372917111613417736444576321 63\
4275978057727571284522048329707572538211101443839581144 86\
901302634595741260020853612 54
```