The 22 periods of a quartic surface

Pierre Lairez

MATHEXP, Université Paris–Saclay, Inria, France

joint work with E. Pichon-Pharabod, E. Sertöz, and P. Vanhove

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today's menu: elliptic curves and quartic surfaces

1. The 2 periods of an elliptic curve

2. Picard–Fuchs equations

3. The 22 periods of a quartic surface

4. How to compute periods faster?

The endomorphism ring of an elliptic curve

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Problem

Is $\mathrm{End}(X)$ nontrivial ($\neq \mathbb{Z}$)?

Most elliptic curves does not have a nontrivial endomorphism.

Algebraic approach

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* Main approach by reduction modulo p (e.g. Cremona & Sutherland, 2023).

Analytic approach

An elliptic curve is a torus.

* $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$
* $\mathrm{End}(X) \simeq \{z \in \mathbb{C} \mid z\Lambda = \Lambda\}$

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Proposition

$\mathrm{End}(X)$ is nontrivial if and only if the equations

$$\begin{cases} z\,\alpha_1 = a\,\alpha_1 + b\,\alpha_2 \\ z\,\alpha_2 = c\,\alpha_1 + d\,\alpha_2 \end{cases}$$

has a nontrivial solution, $z \in \mathbb{C}$ and $a, b, c, d \in \mathbb{Z}$.

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Proposition

$\mathrm{End}(X)$ is nontrivial if and only if the equation

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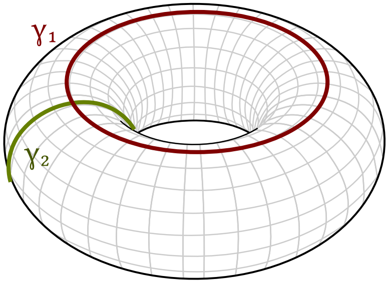
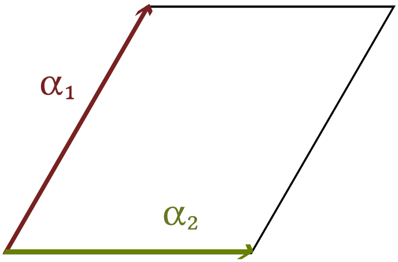
We can find integer relations between complex numbers given only high precision approximations using the Lenstra–Lenstra–Lovász algorithm.

Computation of the periods





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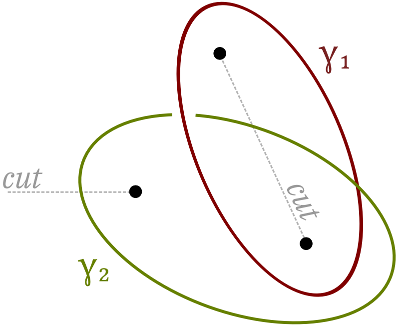

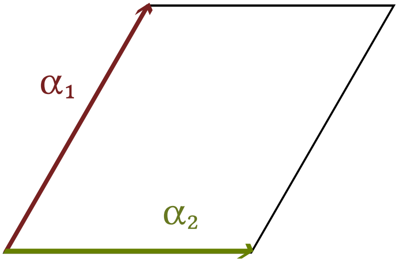

$$\alpha_i = \int_{\gamma_i} \omega_X = \int_{\gamma_i} \frac{\mathrm{d}x}{\sqrt{x^3 + ax + b}}$$

Demo!

High precision quadrature uncovers the endomorphism ring of elliptic curves.

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heuristic algorithm, only provides a safe bet.
No known way to trick the heuristic.

∗ Possibility to certify *a posteriori* (Costa et al., 2019), at the cost of simplicity of course

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Periods

$$\alpha = \int_\gamma F(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

- F is a rational function
- γ is a complex n-cycle on which F is continuous

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⚠ in this regime, direct numerical recipes do not work well

Relative periods

$$\alpha(t) = \int_\gamma F_t(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

- F_t is a rational function of t and $x_1, \ldots, x_n$
- γ is a complex n-cycle on which F_t is continuous ($t \in U$)

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Picard–Fuchs equations

There are polynomials $p_0(t), \dots, p_r(t) \neq 0$ such that

$$p_r(t)\alpha^{(r)}(t) + \cdots + p_1(t)\alpha'(t) + p_0(t)\alpha(t) = 0.$$

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```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
                          [0.78539816339744831 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1], path=[0,i+1,2*i,i-1,0,1])
                 [3.9269908169872415 +/- 4.81e-17] + [+/- 4.63e-21]*I
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- **numerical integration**

Computation of Picard–Fuchs equations

$$E(t) \triangleq \oint \sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x = \frac{1}{2\pi i}\oint \overbrace{\frac{1}{1-\frac{1-t^2x^2}{(1-x^2)y^2}}}^{F(t,x,y)}\,\mathrm{d}x\mathrm{d}y$$

Theorem (Euler, 1733)

$(t-t^3)E'' + (1-t^2)E' + tE = 0$

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Proof. Observe that

$$(t-t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF =$$
$$\frac{\partial}{\partial x}\left(-\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2}\right) + \frac{\partial}{\partial y}\left(\frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2}\right)$$ □

(Chyzak, 2000; Koutschan, 2010; Lairez, 2016)

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Curves on a surface

Let $f \in \mathbb{C}[w, x, y, z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

* X contains algebraic curves.
* *Trivial* curves are those obtained by intersecting X with another surface.

Problem

Does X contain a nontrivial curve?

The very generic case

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w, x, y, z]_4 \setminus$ (countable union of algebraic hypersurfaces).
Then X_f contains only trivial curves.

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Theorem (van Luijk, 2007)

Let $f = 2w^4 + w^3z + w^2x^2 + 2w^2xy + 2w^2xz - w^2y^2 + w^2z^2 + \cdots$
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Theorem (Lairez & Sertöz, 2019)

Let $f = wx^3 + w^3y + xz^3 + y^4 + z^4$. Then X_f contains only trivial curves.

Algebraic approach

Reduction to countably many polynomial systems.

$$
\begin{aligned}
\{\text{lines in } X\} &= \left\{(u,v) \in (\mathbb{C}^4)^2 \,\middle|\, u \wedge v \neq 0 \text{ and } \forall t, f(u+tv) = 0\right\} /\!\sim \\
\{\text{conic curves in } X\} &= \Big\{(u,v,w) \in (\mathbb{C}^4)^3 \,\Big| \\
&\qquad u \wedge v \wedge w \neq 0 \text{ and } \forall t, f(u+tv+t^2 w) = 0\Big\} /\!\sim \\
\{\text{twisted cubics in } X\} &= \Big\{(u_0,\dots,u_3) \in (\mathbb{C}^4)^4 \,\Big| \\
&\qquad u_0 \wedge \cdots \wedge u_3 \neq 0 \text{ and } \forall t, f\left(\sum_{i=0}^{3} u_i t^i\right) = 0\Big\} /\!\sim \\
\{\text{deg. 4 gen. 1 c. in } X\} &= \Big\{(g_1,g_2,h_1,h_2) \in (\mathbb{C}[\mathbf{x}]_2)^4 \,\Big| \\
&\qquad g_1 \text{ and } g_2 \text{ generic and } f = h_1 g_1 + h_2 g_2\Big\} /\!\sim
\end{aligned}
$$

Periods of a quartic surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

Let $\gamma_1, \ldots, \gamma_{22}$ be a basis of $H_2(X,\mathbb{Z})$,
and let $\omega_X \in \Omega^2(X)$ be the unique holomorphic 2-form on X.

The *periods* of X are the complex numbers $\alpha_1, \ldots, \alpha_{22}$ defined – up to scaling and choice of basis – by

$$\alpha_i \stackrel{\text{def}}{=} \oint_{\gamma_i} \omega_X = \frac{1}{2\pi i} \oint_{\text{Tube}(\gamma_i)} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{f|_{w=1}}$$

Periods determine the Néron-Severi group

The Néron-Severi group of X (a smooth quartic surface) is the sublattice of $H_2(X, \mathbb{Z})$ generated by the classes of algebraic curves on X.

Theorem (Lefschetz, 1924)

$$\mathrm{NS}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \;\middle|\; \int_\gamma \omega_X = 0 \right\}$$

In coordinates, $\mathrm{NS}(X) \simeq \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1 \alpha_1 + \cdots + u_{22} \alpha_{22} = 0 \right\}$.
This is the lattice of *integer relations between the periods*.

The NS group determine the possible degree and genus of all the algebraic curves lying on X.

The Fermat hypersurface

Let $f = w^4 + x^4 + y^4 + z^4$.
The vector of periods is

$$\begin{pmatrix} 1 & i & i & i & i & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -i & -i & -i & -i & -i & -i & 0 \end{pmatrix}$$

$$\operatorname{rank} \mathrm{NS}(X_f) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \{\text{periods}\} = 20.$$

Indeed there are 48 lines on X_f spanning a sublattice of $H_2(X, \mathbb{Z})$ of rank 20.

Numerical computation of periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

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⚠ Afflicted by the size of the PF equation (generically order 21 and degree ≥ 1000), the algorithm does not always terminate in reasonnable time.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \dots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

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2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \,\middle|\, \sum_i u_i [10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$, this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.

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3. Compute a LLL-reduced basis of L
4. Output the *short* vectors

What is a short vector?

Let $f = 3x^3z - 2x^2y^2 + xz^3 - 8y^4 - 8w^4$.

With 100 digits of precision on the periods, here is a LLL-reduced basis of the lattice L (last 5 columns omitted).

$$
\left[\begin{array}{cccccccccccccc rrr c}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1669083212117905913652734&0&1937019641160560221317687&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1669083212117905913652734&1937019641160560221317687&\dots\\
\hline
1&0&0&-1&0&0&0&1&1&0&0&0&0&0&-1465118299011954436717 89&844784290445878224678 23&-3659802286906301049192 96&\dots\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&-3371677202526783102581 77&2241101519734039462214 21&-7431169559364872799105 52&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&3570314792535223114836 50&7680663376663510994327 48&9405259947193910799984 35&\dots\\
0&0&0&0&0&1&0&0&1&0&1&0&0&0&-5527566718288541531149 05&-1260182482795835854860 71&5350958119531659172108 63&\dots\\
0&-1&1&0&0&0&0&0&1&0&0&-1&0&0&1043354311299086458251 33&-2316162845853183635708 49&5027304085859624110253 06&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-6491595864302031736926 32&7707848679670711009456 65&-2152014469737999315531 272&\dots\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&2777479839347976908352 05&-2862573987306137296638 4&-6387321794083584799900 97&\dots\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&1465118299011954436717 90&-844784290445878224678 23&3659802286906301049192 96&\dots\\
0&0&0&0&0&0&0&0&0&0&0&-1&1&1&2508991467754066459367 61&5756150300112560313950 07&-1148300124261040782472 91&\dots\\
0&1&0&0&0&0&0&1&0&0&-1&0&0&0&1043354311299086458251 33&-2316162845853183635708 49&5027304085859624110253 07&\dots\\
0&0&0&0&0&0&-1&0&0&0&0&0&1&-1&-1406449504434545869194 39&-3930582062123501406142 35&4299330808339302082915 57&\dots\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&5949330706001409509615 61&2731561038203141265890 96&-6718459918484982233168 74&\dots\\
0&0&0&0&1&0&0&-1&0&0&0&0&0&0&3371677202526783102581 77&-2241101519734039462214 21&7431169559364872799105 52&\dots\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-8243171548389966819846 21&1771197631974658877549 38&-2367923009246437407024 32&\dots\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0&3793441190239651081048 33&-7697229643267340511839 5&6063667760411549738045 41&\dots\\
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0&0&0&0&0&0&0&0&0&1&0&-1&0&0&-2777479839347976908352 06&2862573987306137296638 4&6387321794083584799900 97&\dots\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&-6902523593067784274510 0&4571029143435868632583 66&6606523468775867078488 17&\dots
\end{array}\right]
$$

A triple alternative

- ϟ Certified error bounds!
- ∗ assume that the periods are known $\pm\beta^{-1}$

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Lemma

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I do not know how to deal with 2, there are quartic surfaces with NS group minimaly generated by arbitrary large elements (Mori, 1984).

But we can do something about 3.

Separation of periods

Let $f \in \mathbb{Q}[w, x, y, z]_4$
and let $\alpha_1, \dots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

1. The 2 periods of an elliptic curve

2. Picard–Fuchs equations

3. The 22 periods of a quartic surface

4. How to compute periods faster?

Direct integration?

* Sertöz’ algorithm is very indirect.
* Can we directly compute

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$$\alpha_i = \oint_{\gamma_i} \omega_X?$$

* That's a *double* integral.
* How do we get γ_i?
  How do we compute a basis of the singular homology group $H_2(X)$?

Double integrals *via* Fubini

- $f \in \mathbb{C}[w,x,y,z]_4$ (generic coordinates)
- $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
- $X_t \triangleq X \cap \left\{\frac{w}{x} = t\right\}$ (hyperplane section)
- Consider the surface as a family of curves

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Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \underbrace{\oint_{\text{cycle in } X_t} \frac{\omega_X}{\mathrm{d}t}}_{\text{satisfies a Picard–Fuchs equation!}}.$$

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satisfies a Picard–Fuchs equation!

To be implemented, requires a concrete description of γ.
We need to *compute* $H_2(X, \mathbb{Z})$

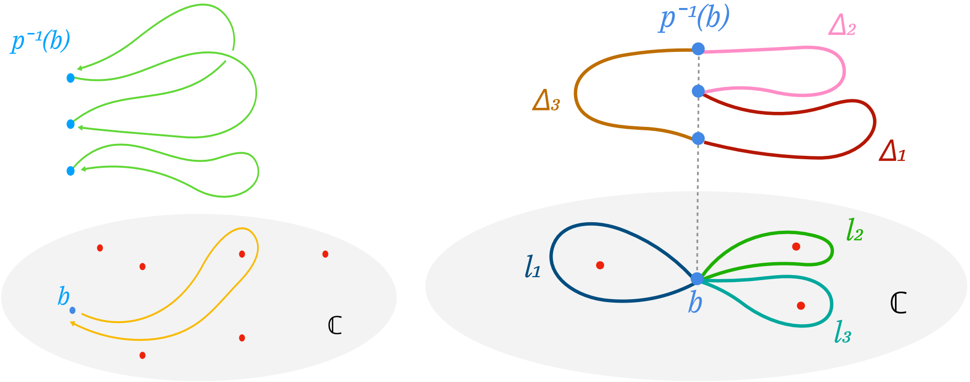
The homology of curves (Tretkoff & Tretkoff, 1984)

- X a complex algebraic curve
- $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map
- $\Sigma \triangleq \{\text{critical values}\}$

- Given a loop in $\mathbb{P}^1(\mathbb{C}) \setminus \Sigma$, starting from a base point b, and a point in the fiber $p^{-1}(b)$, the loop lifts in X uniquely.
- Compute loops in $\mathbb{P}^1(\mathbb{C})$ that lift in a basis of $H_1(X, \mathbb{Z})$

(Costa et al., 2019; Deconinck & van Hoeij, 2001)

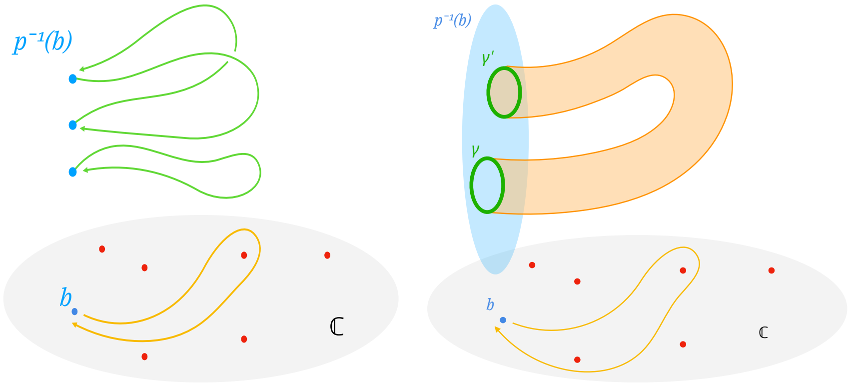
Principle of the method



1. compute pieces of paths in X by lifting loops
2. connect them to form loops

Homology of surfaces

	dimension 1	dimension 2
monodromy action	permute the fiber	linear action on $H_1(X)$
lift in X	path	*hosepipe*
computable with	path tracking	numerical ODE solving



Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solution of the Picard–Fuchs equation of the periods of X_t.

- We can connect hosepipes by integrating a Picard–Fuchs differential equation.

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Thank you!

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