Numerical periods and effective algebraic geometry

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joint work with Marc Mezzarobba, Eric Pichon-Pharabod, Mohab Safey El Din, Emre Sertöz, and Pierre Vanhove

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Section 1

Picard–Fuchs equations

Periods

$$\alpha \quad = \int_\gamma F(x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

* F is a rational function
* γ is a complex n-cycle on which F is continuous

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- contains information about the geometry of the denominator of F
- often not computable exactly, need hundreds or thousands of digits
- in this regime, direct numerical recipes do not work well

Relative periods

$$\alpha(t) = \int_\gamma F_t(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

∗ F_t is a rational function of t and $x_1, \ldots, x_n$
∗ γ is a complex n-cycle on which F_t is continuous ($t \in U$)

- contains information about the geometry of the denominator of F_t, as a familiy depending on t
- computable exactly up to finitely many constants

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Picard–Fuchs equations

There are polynomials $p_0(t), \ldots, p_r(t) \neq 0$ such that

$$p_r(t)\alpha^{(r)}(t) + \cdots + p_1(t)\alpha'(t) + p_0(t)\alpha(t) = 0.$$

Differential equations as a data structure

What can we compute using diff. eq. to represent functions?

* addition, multiplication, composition with algebraic functions

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- **numerical analytic continuation** with certified precision (Chudnovsky & Chudnovsky, 1990; Mezzarobba, 2010; van der Hoeven, 1999)

```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
                      [0.78539816339744831 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1], path=[0,i+1,2*i,i-1,0,1])
              [3.9269908169872415 +/- 4.81e-17] + [+/- 4.63e-21]*I
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- **numerical integration**

Computation of Picard–Fuchs equations

$$E(t) \triangleq \oint \sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x = \frac{1}{2\pi i}\oint \overbrace{\frac{1}{1-\frac{1-t^2x^2}{\left(1-x^2\right)y^2}}}^{F(t,x,y)}\mathrm{d}x\mathrm{d}y$$

Theorem (Euler, 1733)

$(t-t^3)E'' + (1-t^2)E' + tE = 0$

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Proof. Observe that

$$(t-t^3)\frac{\partial^2 F}{\partial t^2} + (1-t^2)\frac{\partial F}{\partial t} + tF =$$

$$\frac{\partial}{\partial x}\left(-\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2}\right) + \frac{\partial}{\partial y}\left(\frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2}\right) \qquad \square$$

(Chyzak, 2000; Koutschan, 2010; Lairez, 2016)

Section 2

Computing volume of semi-algebraic sets

joint work with Marc Mezzarobba and Mohab Safey El Din

The semiring of volumes

$$\mathbb{V} \triangleq \left\{ \mathrm{vol}(A) \,\middle|\, A \subset \mathbb{R}^n \text{ compact semialgebraic defined over } \mathbb{Q} \right\}$$

- $\mathrm{vol}(A) + \mathrm{vol}(B) = \mathrm{vol}\left(A \times [0,1] \cup B \times [1,2]\right)$
- $\mathrm{vol}(A)\,\mathrm{vol}(B) = \mathrm{vol}(A \times B)$

$\Rightarrow$ $\mathbb{V}$ is a semiring.

Kontsevich–Zagier periods $\triangleq (\mathbb{V} - \mathbb{V}) + (\mathbb{V} - \mathbb{V})i$

Theorem (Lairez, Mezzarobba, & Safey El Din, 2019)

On input $A = \{f_1 \geq 0, \ldots, f_r \geq 0\}$ and $p > 0$,
we can compute $\mathrm{vol}(A) \pm 2^{-p}$ in time $f(A) p \log(p)^{3+\epsilon}$.

Case of one equation, smooth boundary

- $f \in \mathbb{R}[x_1, \ldots, x_n]$
- $X \triangleq \{x \in \mathbb{C}^n \mid f(x) = 0\}$
- Assumption: X is smooth.
- $A \triangleq \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$
- $\partial A = \{x \in \mathbb{R}^n \mid f(x) = 0\} = X \cap \mathbb{R}^n$

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$$\begin{aligned}
\mathrm{vol}(A) = \int_A 1 \mathrm{d}x_1 \cdots \mathrm{d}x_n &\overset{\text{Stokes}}{=} \int_{\partial A} x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&\overset{\text{Cauchy}}{=} \int_{\partial A} \left(\frac{1}{2\pi i} \oint_{\text{circle around } p} \frac{x_1}{f} \frac{\partial f}{\partial x_1} \mathrm{d}v \right) \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \frac{1}{2\pi i} \oint_{\mathrm{Tube}(\partial A)} \frac{x_1}{f} \frac{\partial f}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_n.
\end{aligned}$$

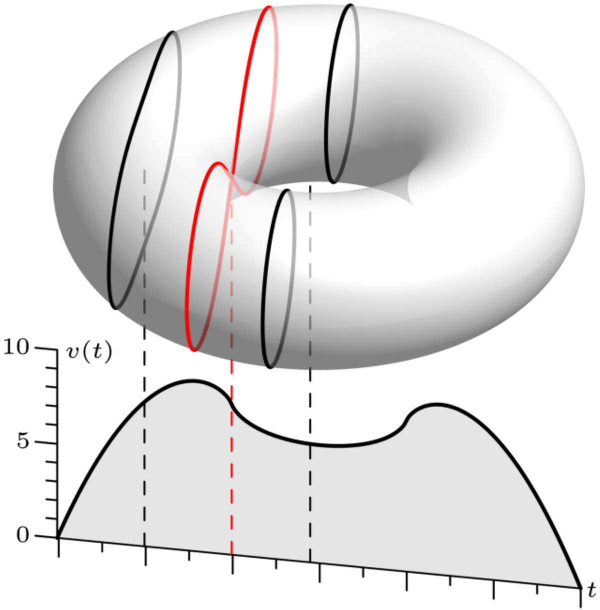
This is a period!

Volume of a slice

- $f \in \mathbb{R}[x_1, \dots, x_n]$
- $A_t \triangleq A \cap \{x_n = t\} \subset \mathbb{R}^{n-1}$

- $t \mapsto \mathrm{vol}(A_t)$ is continuous and piecewise analytic
- $\mathrm{vol}(A) = \int_{-\infty}^{\infty} \mathrm{vol}(A_t)\mathrm{d}t$
- $\mathrm{vol}(A_t) = \underbrace{\frac{1}{2\pi i} \oint_{\mathrm{Tube}(\partial A_t)} \frac{x_1}{f|_{x_n=t}} \frac{\partial f|_{x_n=t}}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}}$

↯ satisfies a Picard-Fuchs equation!


10
5
0
$v(t)$
t

Algorithm (single equation, compact case)

def *volume*($\{f \geq 0\}$):

    [symbolic integration]

  compute a differential equation (E) for $\oint \frac{x_1}{f|_{x_n=t}} \frac{\partial f|_{x_n=t}}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}$

    [real algebraic geometry]

  compute $\Sigma \subset \mathbb{R}$ such that $\mathrm{vol}(A_\bullet)$ is analytic on $\mathbb{R} \setminus \Sigma$

  $v \leftarrow 0$

  for each I bounded component of $\mathbb{R} \setminus \Sigma$**:**

      [induction on dimension]

    evaluate $\mathrm{vol}(A_\bullet)$ at sufficiently many points in I

    deduce initial conditions for $\mathrm{vol}(A_\bullet)$

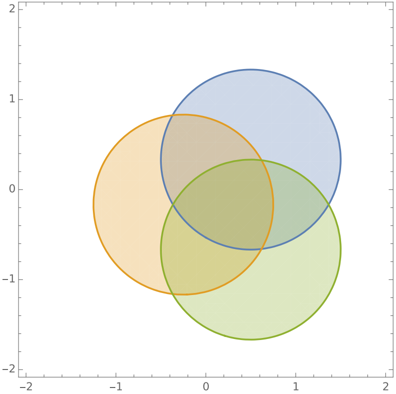
    $v \leftarrow v + \int_I \mathrm{vol}(A_t)\mathrm{d}t$

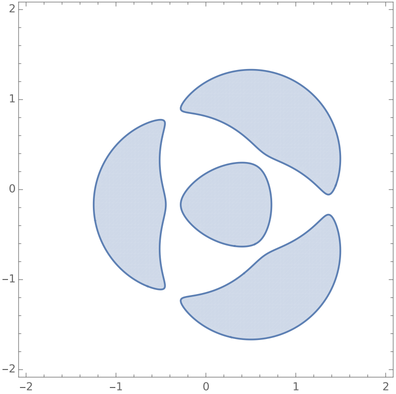
  return v

Several inequalities

* $f_1, \dots, f_r \in \mathbb{Q}[x_1, \dots, x_n]$

$$\mathrm{vol}\,\{f_1 \geq 0, \dots, f_r \geq 0\} = \lim_{\epsilon \to 0^+} \mathrm{vol}\,(\text{some c.c. of } \{f_1 \cdots f_r \geq \epsilon\})$$





Section 3

Periods of quartic surfaces

joint work with Emre Sertöz

Periods of a quartic surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

Let $\gamma_1, \dots, \gamma_{22}$ be a basis of $H_2(X, \mathbb{Z})$,
and let $\omega_X \in \Omega^2(X)$ be the unique holomorphic 2-form on X.

The *periods* of X are the complex numbers $\alpha_1, \dots, \alpha_{22}$ defined – up to scaling and choice of basis – by

$$\alpha_i \stackrel{\text{def}}{=} \oint_{\gamma_i} \omega_X = \frac{1}{2\pi i} \oint_{\mathrm{Tube}(\gamma_i)} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{f|_{w=1}}$$

Periods determine the Néron-Severi group

The Néron-Severi group of X (a smooth quartic surface) is the sublattice of $H_2(X, \mathbb{Z})$ generated by the classes of algebraic curves on X.

Theorem (Lefschetz, 1924)

$$\mathrm{NS}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \;\middle|\; \int_\gamma \omega_X = 0 \right\}$$

In coordinates, $\mathrm{NS}(X) \simeq \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}$.
This is the lattice of *integer relations between the periods*.

The NS group determine the possible degree and genus of all the algebraic curves lying on X.

The very generic case

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w,x,y,z]_4 \setminus$ (countable union of algebraic hypersurfaces).
Then $\mathrm{NS}(X_f) = \mathbb{Z} \cdot$ (hyperplane section).

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Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w,x,y,z]_4$ such that $\mathrm{NS}(X_f) = \mathbb{Z} \cdot h$.

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Theorem (van Luijk, 2007)

Let $f = 2w^4 + w^3z + w^2x^2 + 2w^2xy + 2w^2xz - w^2y^2 + w^2z^2 + \cdots$
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Theorem (Lairez & Sertöz, 2019)

Let $f = wx^3 + w^3y + xz^3 + y^4 + z^4$. Then $\mathrm{NS}(X_f) = \mathbb{Z} \cdot h$.

The Fermat hypersurface

Let $f = w^4 + x^4 + y^4 + z^4$.
The vector of periods is

$$\begin{pmatrix} 1 & i & i & i & i & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -i & -i & -i & -i & -i & -i & 0 \end{pmatrix}$$

$$\operatorname{rank} \mathrm{NS}(X_f) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \{\text{periods}\} = 20.$$

Indeed there are 48 lines on X_f spanning a sublattice of $H_2(X, \mathbb{Z})$ of rank 20.

Numerical computation of periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w,x,y,z]_4$
and let $f_t = (1-t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w,x,y,z]_4$.

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⚠ Afflicted by the size of the PF equation (generically order 21 and degree ≥ 1000), the algorithm does not always terminate in reasonnable time.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \dots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

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2. Let $L = \left\{ (\mathbf{u}, x, y) \in \mathbb{Z}^{22+2} \;\middle|\; \sum_i u_i [10^{1000}\alpha_i] = x + y\sqrt{-1} \right\}$, this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.

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3. Compute a LLL-reduced basis of L
4. Output the *short* vectors

What is a short vector?

Let $f = 3x^3z - 2x^2y^2 + xz^3 - 8y^4 - 8w^4$.

With 100 digits of precision on the periods, here is a LLL-reduced basis of the lattice L (last 5 columns omitted).

$$
\left[\begin{array}{cccccccccccccc rrr c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1669083212117905913652734 & 0 & 1937019641160560221317687 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1669083212117905913652734 & 1937019641160560221317687 & \dots \\
\hline
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -146511829901195443671789 & 84478429044587822467823 & -365980228690630104919296 & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -337167720252678310258177 & 224110151973403946221421 & -743116955936487279910552 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 357031479253522311483650 & 768066337666351099432748 & 940525994719391079998435 & \dots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -552756671828854153114905 & -126018248279583585486071 & 535095811953165917210863 & \dots \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 104335431129908645825133 & -231616284585318363570849 & 502730408585962411025306 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -649159586430203173692632 & 770784867967071100945665 & -2152014469737999315531272 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 277747983934797690835205 & -28625739873061372966384 & -638732179408358479990097 & \dots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 146511829901195443671790 & -84478429044587822467823 & 365980228690630104919296 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 250899146775406645936761 & 575615030011256031395007 & -114830012426104078247291 & \dots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 104335431129908645825133 & -231616284585318363570849 & 502730408585962411025307 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -140644950443454586919439 & -393058206212350140614235 & 429933080833930208291557 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 594933070600140950961561 & 273156103820314126589096 & -671845991848498223316874 & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 337167720252678310258177 & -224110151973403946221421 & 743116955936487279910552 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -824317154838996681984621 & 177119763197465887754938 & -236792300924643740702432 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 379344119023965108104833 & -76972296432673405118395 & 606366776041154973804541 & \dots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 552756671828854153114905 & 126018248279583585486070 & -535095811953165917210864 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -140644950443454586919440 & -393058206212350140614234 & 429933080833930208291557 & \dots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -104335431129908645825133 & 231616284585318363570849 & -502730408585962411025307 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -467285675585474370500971 & -950623161465256990213520 & -1255629063127217210042702 & \dots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -146511829901195443671790 & 84478429044587822467823 & -365980228690630104919296 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -277747983934797690835206 & 28625739873061372966384 & 638732179408358479990097 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -69025235930677842745100 & 457102914343586863258366 & 660652346877586707848817 & \dots
\end{array}\right]
$$

A triple alternative

- Certified error bounds!
- $*$ assume that the periods are known $\pm\beta^{-1}$

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I do not know how to deal with 2, there are quartic surfaces with NS group minimaly generated by arbitrary large elements (Mori, 1984).

But we can do something about 3.

Separation of periods

Let $f \in \mathbb{Q}[w,x,y,z]_4$
and let $\alpha_1, \ldots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

Section 4

How to compute periods faster?
Effective homology computation

joint work with Eric Pichon-Pharabod and Pierre Vanhove

Double integrals *via* Fubini

- $f \in \mathbb{C}[w, x, y, z]_4$ (generic coordinates)
- $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
- $X_t \triangleq X \cap \left\{\frac{w}{x} = t\right\}$ (hyperplane section)
- Consider the surface as a family of curves

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Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \underbrace{\oint_{\text{cycle in } X_t} \frac{\omega_X}{\mathrm{d}t}}_{} .$$

satisfies a Picard–Fuchs equation!

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satisfies a Picard–Fuchs equation!

To be implemented, requires a concrete description of γ.
We need to *compute* $H_2(X, \mathbb{Z})$

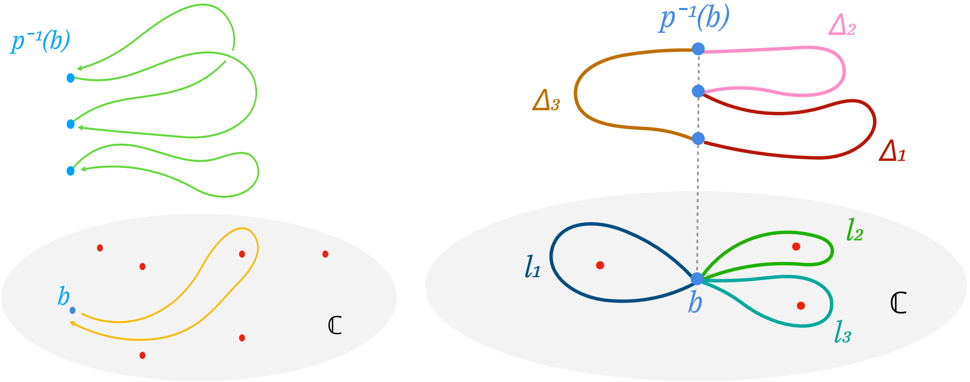
The homology of curves (Tretkoff & Tretkoff, 1984)

- X a complex algebraic curve
- $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map
- $\Sigma \triangleq \{\text{critical values}\}$

- Given a loop in $\mathbb{P}^1(\mathbb{C}) \setminus \Sigma$, starting from a base point b, and a point in the fiber $p^{-1}(b)$, the loop lifts in X uniquely.
- Compute loops in $\mathbb{P}^1(\mathbb{C})$ that lift in a basis of $H_1(X, \mathbb{Z})$

(Costa et al., 2019; Deconinck & van Hoeij, 2001)

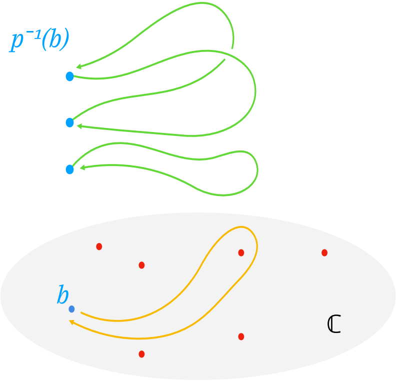
Principle of the method

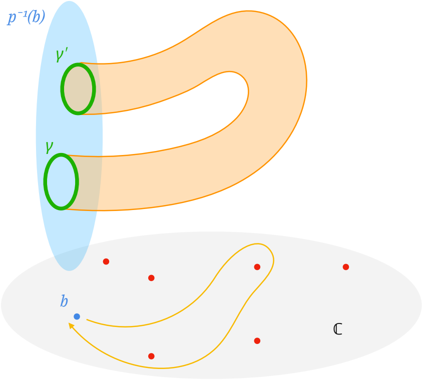


1. compute pieces of paths in X by lifting loops
2. connect them to form loops

Homology of surfaces

	dimension 1	dimension 2
monodromy action	permute the fiber	linear action on $H_1(X)$
lift in X	path	*hosepipe*
computable with	path tracking	numerical ODE solving





Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solution of the Picard–Fuchs equation of the periods of X_t.

- ϟ We can connect hosepipes by integrating a Picard–Fuchs differential equation.

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Thank you!

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