Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut

Computing Periods of Projective Hypersurfaces Using Picard-Lefschetz Theory

Diplomarbeit
Leipzig, 5. September 2020

Betreuender Hochschullehrer:
Bernd Sturmfels
MPI für Mathematik in den
Naturwissenschaften

vorgelegt von:
Name: Rafael Mohr
Matrikel-Nr.: 3750533
Studiengang: Mathematik

Acknowledgements

On the academic side of things I want to thank Claus Diem for introducing me to the field of algebraic geometry, for years of educational lectures and enlightening tutorial sessions.

I want to thank Bernd Sturmfels for allowing me to study and to work on this thesis at the Max Planck Institute for Mathematics in the Sciences in Leipzig.

I want to thank Marc Mezzarobba for thoroughly answering all the questions I had about his SageMath package.

Most of all I want to thank Emre Sertöz for his dedicated supervision of this thesis. During our talks I learned countless things about mathematics and its research methodology.

On the personal side of things I want to thank Katrin Kurz for her emotional support during the final stages of this thesis.

First and foremost I am eternally grateful to my parents without whom my studies would not have been possible.

Contents

Introduction

Computing Periods: The Smooth Case

Period Matrices. Let X be a compact Kähler manifold. Famously, the complex geometry of X induces the functorial *Hodge decompostion* on the purely topological cohomology groups of X:

$$H^k(X,\mathbb{Z}) \otimes \mathbb{C} \cong H^k(X,\mathbb{C}) = \bigoplus_{i=0}^{k} H^{i,k-i}(X,\mathbb{C}),$$

with this decomposition satisfying the constraints given in definition 1.1.1.2. One sais that $H^k(X,\mathbb{Z})$ carries a *pure Hodge structure of weight k*. The main goal of this thesis is to compute how this decomposition relates to the lattice

$$H^n(X,\mathbb{Z}) \subset H^n(X,\mathbb{C})$$

in the special case where $X \subset \mathbb{P}^{n+1}$ is a smooth complex projective hypersurface. This relationship is described by a so-called *period matrix* associated to X (see definition 1.1.2.4). We remark that, in this case, the middle cohomology group $H^n(X,\mathbb{Z})$ carries the only "interesting" Hodge structure associated to X in a sense made precise in Remark 1.1.1.5.

In fact, we will treat this problem for the *primitive cohomology* $PH^n(X,\mathbb{C})$ respectively the *primitive homology* $PH_n(X,\mathbb{Z})$ of X. This is the subspace of the (co)homology of X which "does not come from the surrounding projective space". $PH^n(X,\mathbb{Z})$ also carries a Hodge structure as above and can be thought of as the "non-trivial" part of the (co)homology of X.

More precisely we want to solve the following

Problem 0.0.0.1. Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface, cut out by a homogenous polynomial f_X with rational coefficients. Denote by $\langle \cdot, \cdot \rangle$ the canonical dual pairing between $PH_n(X,\mathbb{C})$ and $PH^n(X,\mathbb{C})$. Compute a *period matrix* of X:

$$\mathcal{P}(\mathrm{X}) := \left(\langle \gamma_j, \omega_i \rangle\right)_{1\leq i,j\leq m}$$

where

- $\gamma_1, ..., \gamma_m$ is a basis of $PH_n(X, \mathbb{Z})$.
- $\omega_1, ..., \omega_m$ is a basis of $PH^n(X, \mathbb{C})$ such that for all p there exists an m_p such that $\omega_1, ..., \omega_{m_p}$ span $F^p := \bigoplus_{k \geq p} PH^{k,n-k}(X, \mathbb{C})$.

The values $\langle \gamma_j, \omega_i \rangle$ are called *periods* associated to X.

Before we briefly introduce our method to solve this problem, we discuss two applications of the invariant $\mathcal{P}(X)$.

The Picard Group of a Smooth Projective Hypersurface. Suppose that $X \subset \mathbb{P}^3$, i.e. that X is a surface. Let $\text{Pic}(X)$ be the group of Cartier divisors of X modulo linear equivalence. Any curve in X naturally defines a cycle in $H_2(X, \mathbb{Z})$ and one can prove [Lef50] that the thus constructed map

$$\text{Pic}(X) \hookrightarrow H_2(X, \mathbb{Z})$$

is injective. Moreover, Lefschetz characterized the algebraic cycles in $H_2(X, \mathbb{Z})$ as follows ([Mov20], [Lef50])

Theorem 0.0.0.2 (Lefschetz $(1,1)$-Theorem)**.** *A cycle $\gamma \in H_2(X, \mathbb{Z})$ lies in $\text{Pic}(X)$ if and only if*

$$\langle \gamma, \omega \rangle = 0$$

for every $\omega \in H^{2,0}(X, \mathbb{C})$.

Computation of the Picard group respectively the Picard rank of X is an active area of research (see e.g. [EJ11], [Cha14], [LS19]), especially for smooth projective hypersurfaces of degree 4. We remark that if $\omega \in H^{2,0}(X, \mathbb{C})$ then $\omega \in PH^2(X, \mathbb{C})$. In the case when X is cut out by a degree 4 polynomial, i.e. X is a K3-surface, we have

$$\dim H^{2,0}(X, \mathbb{C}) = 1$$

and hence for a generator ω of $H^{2,0}(X, \mathbb{C})$

$$\begin{aligned} \text{rk}(\text{Pic}(X)) &= \text{rk}_{\mathbb{Z}}\{\gamma \in H_2(X, \mathbb{Z}) \mid \langle \gamma, \omega \rangle = 0\} \\ &= \text{rk}_{\mathbb{Z}}\{\gamma \in PH_2(X, \mathbb{Z}) \mid \langle \gamma, \omega \rangle = 0\} + 1. \end{aligned}$$

Thus if one is able to compute the (possibly transcendental) periods of ω to arbitrary precision, one can compute the Picard rank of X with "high certainty". This approach to the computation of the Picard rank of a smooth projective hypersurface has been investigated in more detail in [LS19].

Torelli Theorems. The *Torelli theorems* (see e.g. [Voi20], [Don83]) assert that for almost all cases, a smooth projective hypersurface is uniquely determined by its periods. The classical Torelli theorem states this for curves and has been extended to, for example, K3-surfaces:

Theorem 0.0.0.3. *Two complex K3-surfaces X and Y are isomorphic if and only if there is an isomorphism $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ preserving both the cup product and the Hodge decomposition.*

We remark that, by definition of the primitive cohomology, this is the case if and only if there exists an isomorphism $PH^2(X, \mathbb{Z}) \cong PH^2(Y, \mathbb{Z})$ preserving both the cup product and the Hodge filtration.

This demonstrates the power of period matrices as a complex-geometric invariant. The central difficulty in solving problem 0.0.0.1 lies in recovering the underlying lattice structure of $PH^n(X, \mathbb{C})$ since it is a purely topological invariant.

Using Pencils of Hypersurfaces to Compute Periods. Due to the power of the period matrix as a complex-geometric invariant, numerous attempts have been made to compute it (see e.g. [TT84], [EJ18] and [Ser19]).

Let us now briefly discuss the main idea of our method to solve problem 0.0.0.1. In [Ser19], the author used a deformation-theoretic approach to compute a period matrix for a smooth projective hypersurface X: The main idea is to deform the given hypersurface X into another smooth hypersurface Y, for which a period matrix is known, and to then use analytic continuation to track the resulting change of periods. We vary this approach based on an idea presented in [ES06]. We deform the given hypersurface X into any hypersurface Y and track the change of periods of X when we "walk around" the singular values of this deformation, i.e. we compute the monodromy action on $PH_n(X, \mathbb{C})$ resulting from this deformation. The theory of how this monodromy action is determined by the geometry of the situation is known as *Picard-Lefschetz theory*. We then want to abuse the fact that this monodromy action must necessarily preserve the integral primitive homology $PH_n(X, \mathbb{Z})$. In [HKS20], the authors combined the approach presented in [Ser19] with neural networks to find deformations into smooth hypersurfaces Y with known period matrix for which computationally tracking the resulting change of periods would be as quick as possible. We demonstrate the potential of our strategy by discussing an example given in [HKS20] in section 3.3.3 for which the authors of [HKS20] were not able to compute a period matrix in any reasonable time. We managed to compute a period matrix for this example in roughly three and a half hours.

Let us now introduce our approach to solve problem 0.0.0.1 in slightly more detail:

1. Suppose that $f_X \in \mathbb{Q}[z_0, ..., z_{n+1}]$. Choose any other $f_Y \in \mathbb{Q}[z_0, ..., z_{n+1}]$ with $\deg(f_Y) = \deg(f_X)$ such that f_Y defines a hypersurface different from X. Put X and $Y := \{f_Y = 0\}$ in a *pencil of hypersurfaces* by declaring

$$f_t := tf_X + (1-t)f_Y,\ t \in \mathbb{C};$$
$$X_t := \{f_t = 0\}.$$

2. Let $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ be the fibration associated to the pencil. Compute

sections $\omega_1(t), ..., \omega_m(t)$ of the cohomology bundle of the smooth part π of $\overline{\pi}$ such that $\omega_i := \omega_i(1)$ form a basis of $PH^n(X, \mathbb{C})$ as above.

3. $\overline{\pi}$ will have finitely many singular values. Let $S \subset \mathbb{C}$ be the set of singular values of $\overline{\pi}$. Compute the monodromy action of $\pi_1(\mathbb{P}^1 - S, 1)$ on $PH_n(X, \mathbb{C})$ in terms of the basis $\omega_1^*, ..., \omega_m^*$ of $PH_n(X, \mathbb{C})$, represented computationally by the standard unit vectors.

   Here we choose paths in $\pi_1(\mathbb{P}^1 - S, 1)$ such that each path encloses exactly one element s in S.

4. Since the monodromy action $\pi_1(1, \mathbb{P}^1 - S) \to \mathrm{GL}(PH_n(X, \mathbb{C}))$ arises purely topologically and preserves intersection products, the monodromy representation must factor over $\mathrm{SL}\left(PH_n(X, \mathbb{Z})\right)$. We want to use this fact to now compute a monodromy invariant lattice consisting of *integral vectors*, i.e. vectors of the form

$$\left(\langle \gamma, \omega_1 \rangle, ..., \langle \gamma, \omega_m \rangle\right)^{\intercal}, \ \gamma \in PH_n(X, \mathbb{Z}).$$

Let us now discuss how each of the chapters in this thesis pertains to these steps.

In step 2 we use the classical technique of *Griffiths residues* and *Griffiths-Dwork reduction* ([Gri69], [Dwo62]) , see section 1.2. This allows one to represent elements in $PH^n(X, \mathbb{C})$ explicitly as differential forms on $\mathbb{P}^{n+1}$ with poles on X modulo certain exact forms. These forms in turn are given as elements of a vector space of polynomials. The pole order of such a differential form turns out to be compatible with the Hodge filtration on $PH^n(X, \mathbb{C})$ (see lemma 1.2.1.3). Griffiths-Dwork reduction allows one to decide when the residues of two such forms in $PH^n(X, \mathbb{C})$ are linearly independent.

This technique is again used in step 3: For each of the sections $\omega_i(t)$ we compute a differential operator $\mathcal{D}_i \in \mathbb{C}(t)\left[\frac{d}{dt}\right]$ in t called the *Picard-Fuchs operator* of $\omega_i(t)$. This operator has the property that it annihilates $\omega_i(t)$ and is minimal in order with respect to this property. This means in particular (see lemma 1.1.5.2) that its solution space at $t = 1$ is spanned by the functions

$$\langle \omega_j^*, \omega_i(t) \rangle, \ j = 1, ..., m. \tag{1}$$

where ω_j denotes the multi-valued section of the primitive cohomology bundle of $\overline{\pi}$ given by parallel transport of $\omega_j(1)$. Then we can numerically compute the analytic continuation of each of these functions along our chosen paths in $\pi_1(\mathbb{P}^1 - S, 1)$ using code presented in [Mez16] and relate this easily to the monodromy action on $PH_n(X, \mathbb{C})$ (see section 1.1.6).

We ideally want to decide when a monodromy operator T at a given singular value s can be used to compute an integral vector before computing T. This is because computation of the Picard-Fuchs equations and computation of analytic continuation for them are by far the most computationally expensive steps in our method. Specifically, we want to know whether there exists a polynomial $q \in \mathbb{Z}[x]$ such that

$$\mathrm{rk}(q(T)) = 1.$$

If $\mathrm{rk}(q(T)) = 1$ then

$$v = (\langle \delta, \omega_i \rangle)_{i=1}^{m} \, ; \; \delta \in PH_n(X, \mathbb{C})$$

does not lie in the kernel of $q(T)$, $q(T)v$ is a scalar multiple of an integral vector, i.e.

$$w := q(T)v = \lambda \left(\langle \gamma, \omega_i \rangle \right)_{i=1}^{m}$$

for some $\gamma \in PH_n(X, \mathbb{Z})$ and unknown $\lambda \in \mathbb{C}$. Once we have obtained such a w we can apply more monodromy matrices to w until we have computed a basis of $PH_n(X, \mathbb{Z})$. These monodromy matrices may have to be computed using other pencils of hypersurfaces through X, in which case we make sure to carry over our basis basis of primitive cohomology $\omega_1, ..., \omega_m$. The result is a period matrix $\mathcal{P}(X)$ for X, with the entire basis $\omega_1, ..., \omega_m$ changed to $\lambda\omega_1, ..., \lambda\omega_m$.

In order to decide whether such a q exists we want to gain information about the Jordan normal form of T before computing T.

If we are in the surface case, i.e. $X \subset \mathbb{P}^3$ and if the singular fiber X_s carries only isolated singular points, we will discuss in chapter 2 how T almost entirely depends only on the types of these singular points in a sense made precise in chapter 2. In some cases, we will then know enough about the Jordan normal form of T by computing certain invariants of X_s coming from singularity theory (see sections 2.1.1 and 2.1.4), even before computing the Picard-Fuchs equations.

If we are not in the surface case or if singularity theory does not suffice to predict the Jordan normal form of T to a sufficient extent, we can use the follwing: Even though T arises purely topologically, the theory of *Hodge structures* and *variations of Hodge structures* (see section 1.1) puts restrictions on the Jordan normal form of T and connects the Jordan normal form of T to the local theory of the Picard-Fuchs equations at the singular points of $\overline{\pi}$. Using this, we will then be able to predict the Jordan normal form of T almost completely after computing a few Picard-Fuchs equations (see theorem 3.1.3.5) but still before doing any analytic continuation.

For a detailed discussion of our method see chapter 3.

Computing Periods: The Singular Case

Period Matrices in the Singular Case. The second contribution of this thesis is as follows: Suppose that X is again a projective hypersurface, this time singular. Deligne [Del75] showed that in this case the cohomology group $H^n(X, \mathbb{Z})$ carries a functorial *mixed Hodge structure* (see definition 1.1.3.1). This can be thought of as a Hodge structure with several parts of different weight in contrast to a pure Hodge structure which only contains elements of one fixed weight. Such a mixed Hodge structure can be parametrized by a period matrix $\mathcal{P}(X)$ analogous to the smooth case together with the data of a few matrices $(W_k)_{k=0}^n$ with entries in $\mathbb{Q}$ each giving the part "of weight k" of this Hodge structure by computing $\mathcal{P}(X)W_k$.

Using Pencils of Hypersurfaces in the Singular Case. In the case $\dim(\mathrm{Sing}(X)) = 0$ we provide a method to compute both $\mathcal{P}(X)$ and $(W_k)_{k=1}^n$ based on previous unpublished work by the supervisor of this thesis:

1. We again put X in a pencil of hypersurfaces as above by declaring

$$f_t := (1-t)f_X + tf_Y$$

   with f_X and f_Y defined as above where $Y := \{f_Y = 0\}$ is smooth.

2. The primitive cohomology bundle of the smooth part of the associated fibration $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ defines a so-called *variation of Hodge structures* (see definition 1.1.4.1). This induces a mixed Hodge structure over $t = 0$, called the *Schmid limit mixed Hodge structure* (see section 1.1.8). If one is given a period matrix $\mathcal{P}(Y)$ of Y one can compute this mixed Hodge structure as a datum $(\mathcal{P}_\infty, (W_k)_{k=0}^{2n-1})$ as above (see section 3.2).

3. One can then use a simple compatibility between the Schmid limit mixed Hodge structure and the mixed Hodge structure on $H^n(X, \mathbb{Z})$ [Ste76] to compute the mixed Hodge structure on $H^n(X, \mathbb{Z})$ (see section 2.1.5).

As far as we know, we are the first to give a way to compute periods matrices in the singular case.

Chapter 1

Preliminaries

We first recall some of the basic definitions and concepts of Hodge theory. Our main sources of exposition are [Voi02], [Huy05], [PS08] and [Gri68].

1.1 Hodge Structures and Periods

1.1.1. Hodge Structures. Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface. Let $H^n(X, \mathbb{Z})$ be the n-th singular cohomology group of X with coefficients in $\mathbb{Z}$. As X is in particular a Kähler manifold, Hodge theory yields a decomposition

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^n_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{k=0}^{n} H^{n-k,k}(X, \mathbb{C}).$$

satisfying certain constraints (see definition 1.1.1.2).

Remark 1.1.1.1. Here, the space $H^{n-k,k}(X, \mathbb{C})$ can be thought of as differential forms ω with "$n-k$ dz's and k $d\bar{z}$'s" which satisfy $\Delta\omega = 0$ where $\Delta = \partial\partial^* + \partial^*\partial$ is the Laplace operator and ∂ is the holomorphic part of the canonical differential d on the differential forms on X.

This decomposition is a complex-geometric invariant of X whereas the cohomology groups of X themselves just come "from the topology" of X. The geometric occurence of this decomposition motivates the following definition:

Definition 1.1.1.2. A *pure Hodge structure (HS) of weight* n is a tuple $\mathfrak{H} := (H_{\mathbb{Z}}, n, \{H^{p,q}\})$ where

1. $H_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank.
2. n is an integer, called the *weight* of $\mathfrak{H}$.
3. $\{H^{p,q}\}$ is a collection of subspaces of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ satisfying

(a) $H^{p,q} = 0$ unless $p + q = n$.
(b) $\overline{H^{p,q}} = H^{q,p}$.
(c) $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$

In the situation of definition 1.1.1.2 we will sometimes say that $H_{\mathbb{Z}}$ *carries a pure HS of weight n*.

Definition 1.1.1.3. A *Hodge filtration* of weight k on a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank is a collection of subspaces $\{F^p\}$ of $H_{\mathbb{C}}$, defining a finite decreasing filtration of $H_{\mathbb{C}}$ satisfying

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}} \quad \forall p \in \mathbb{Z}.$$

The two definitions turn out two be equivalent, i.e. any pure HS of weight k defines a Hodge filtration via

$$F^p = \bigoplus_{r \geq p} H^{r,n-r}$$

and a Hodge filtration determines a pure HS via

$$H^{p,q} = F^p \cap \overline{F^q}$$

respectively.

Definition 1.1.1.4. A *morphism of pure HS's* is a morphism of the underlying $\mathbb{Z}$-modules such that its $\mathbb{C}$-linear extension preserves the Hodge filtration.

Remark 1.1.1.5. Let X be as above. As the Hodge decomposition on the cohomology groups of X is functorial with respect to holomorphic maps, i.e. the induced maps on cohomology are morphisms of pure Hodge structures, the only "interesting" cohomological complex-geometric invariant of X is the pure Hodge structure on the middle cohomology $H^n(X, \mathbb{C})$ of X. This is since we have $H^k(\mathbb{P}^{n+1}, \mathbb{Z}) \cong H^k(X, \mathbb{Z})$ for all $k \neq n$ via the canonical inclusion $X \subset \mathbb{P}^{n+1}$ by the Lefschetz theorem on hyperplane sections and the hard Lefschetz theorem. Hence we will only be interested in computing the pure HS on the middle cohomology of X.

1.1.2. Primitive Cohomology and Polarization. In the case of a smooth projective hypersurface X, the so-called middle *primitive cohomology* is the part that "does not come from the surrounding projective space". It is a subspace of $H^n(X, \mathbb{Z})$ that may be explicitly represented as a vector space of polynomials called *Griffiths residues* (see section 1.2), thus making it available for computational manipulation. Additionally, it carries a certain bilinear form which restricts the monodromy on the primitive cohomology of X if X is contained in a family as a fiber. In more detail: The restriction of the standard Kähler class κ of $\mathbb{P}^{n+1}$ determines a Kähler class κ_X on X. Let $L\colon H^n(X, \mathbb{Z}) \to H^{n+2}(X, \mathbb{Z})$, $L(\omega) = \kappa_X \wedge \omega$, be the corresponding Lefschetz operator. We define

Definition 1.1.2.1. The middle *primitive cohomology on X* is given by

$$PH^n(X,\mathbb{Z}) = \ker(L).$$

Remark 1.1.2.2. Dually, one may of course define the middle *primitive homology $PH_n(X,\mathbb{Z})$* of X.

Since $H^k(\mathbb{P}^{n+1},\mathbb{Z}) = 0$ for odd k we have $H^n(X,\mathbb{Z}) = PH^n(X,\mathbb{Z})$ if n is odd. Moreover

Lemma 1.1.2.3 ([Ser19]). *If n is even there exists an algebraic class $v \in H^{n/2,n/2}(X,\mathbb{Z})$ such that $H^n(X,\mathbb{Z}) = PH^n(X,\mathbb{Z}) \oplus \mathbb{Z}\langle v\rangle$.*

Thus $PH^n(X,\mathbb{Z})$ also carries a pure HS by restricting the HS on $H^n(X,\mathbb{Z})$. The HS on $PH^n(X,\mathbb{Z})$ is the HS that we will compute in the end.

We will computationally represent this HS as follows: Denote by $\langle\cdot,\cdot\rangle : V \times V^* \to \mathbb{C}$ the canonical pairing between a complex vector space V and its dual V^*. One may represent the Hodge structure on $PH^n(X,\mathbb{Z})$ by a *period matrix* and thus make such a Hodge structure representable by a computer:

Definition 1.1.2.4. Fix a basis $\gamma_1,...,\gamma_m$ of $PH_n(X,\mathbb{Z})$ and a basis $\omega_1,...,\omega_m$ of $PH^n(X,\mathbb{C})$ such that for all p there exists an m_p such that $\omega_1,...,\omega_{m_p}$ span $F^pH^n(X,\mathbb{C})$. Then we call the matrix

$$\mathcal{P}(X) := \left(\langle\gamma_j,\omega_i\rangle\right)_{1\leq i,j\leq m}$$

a *period matrix* associated to X. For any $\omega \in PH^n(X,\mathbb{C})$ and $\gamma \in PH_n(X,\mathbb{Z})$ we call values of the form $\langle\gamma,\omega\rangle$ *periods* of X. Additionally, we call $(\langle\gamma_j,\omega\rangle)_{j=1}^m$ a *period vector* of $\omega \in PH^n(X,\mathbb{C})$ and $(\langle\gamma,\omega_i\rangle)_{i=1}^m$ an *integral vector* for $\omega_1,...,\omega_m$.

Additionally, as mentioned above, one has a certain bilinear form on the middle primitive cohomology of X which restricts the monodromy on families containing X as a fiber: The definition

$$Q(\omega_1,\omega_2) := (-1)^{n(n-1)/2}\langle[X],\omega_1\wedge\omega_2\rangle;$$
$$\omega_1,\omega_2 \in PH^n(X,\mathbb{C})$$

turns the pure HS on $PH^n(X,\mathbb{Z})$ into a *polarized Hodge structure* defined as follows:

Definition 1.1.2.5. A pure HS $(H_\mathbb{Z}, n, \{H^{p,q}\})$ is called *polarized* if a non-degenerate bilinear form Q on $H_\mathbb{Z}$ is fixed and its extension to $H_\mathbb{C}$ satisfies the following properties

1. $Q(F^p, F^{n-p+1}) = 0$ for all p.
2. The form given by $\sum_{p+q=n} i^{p-q}Q|_{H^{p,q}}$ is a hermitian form on $H_\mathbb{C}$ where $i \in \mathbb{C}$ is the imaginary unit.

1.1.3. Mixed Hodge Structures. Deligne [Del75] showed that if X is a projective (but not necessarily smooth) variety then the integral cohomology groups of X carry functorial *mixed Hodge structures*:

Definition 1.1.3.1. A *mixed Hodge structure* is a tuple $\mathfrak{H} := (H_{\mathbb{Z}}, F^{\bullet}, W_{\bullet})$ where

1. $H_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank.
2. $W_{\bullet}$ is a finite increasing filtration of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$, called the *weight filtration* of $\mathfrak{H}$.
3. $F^{\bullet}$ is a finite decreasing filtation of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$, called the *Hodge filtration* of $\mathfrak{H}$, such that the filtration induced by $F^{\bullet}$ on the complexification of
$$\mathrm{Gr}_l^W H_{\mathbb{Q}} := W_l / W_{l-1}$$
satisfies the above properties of a Hodge filtration of a pure HS of weight l.

Remark 1.1.3.2. If X is again projective, then the MHS on $H^k(X, \mathbb{Z})$ satisfies $W_l = 0$ unless $l \in \mathbb{N}_0 \cap [0, k]$ and $F^p = 0$ unless $p \in \mathbb{N}_0 \cap [0, k]$. Additionally, one has that $W_k = H^k(X, \mathbb{Q})$ and $F^0 H^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$.

Analogously to morphisms of pure HS's we have morphisms of MHS's:

Definition 1.1.3.3. A *morphism of MHS's* is given by a morphism of the underlying $\mathbb{Z}$-modules such that its $\mathbb{Q}$-linear respectively $\mathbb{C}$-linear extension preserves the weight respectively the Hodge filtration.

1.1.4. Variations of Hodge structures. Suppose now that we are given a family of projective varieties, for example a family of projective hypersurfaces via the equation
$$\mu f(z) + \nu g(z) = 0, \ [\mu : \nu] \in \mathbb{P}^1$$
where $f, g \in \mathbb{C}[z_0, ..., z_{n+1}]_d$ for some $d > 0$ and where f and g define different hypersurfaces. We call such a family a *pencil of hypersurfaces in* $\mathbb{P}^{n+1}$. Let
$$\overline{\mathfrak{X}} := \{(z, [\mu : \nu]) \in \mathbb{P}^{n+1} \times \mathbb{P}^1 \mid \mu f(z) + \nu g(z) = 0\}$$
and let $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ be the second projection. The locus of singular values of $\overline{\pi}$ is Zariski-closed in $\mathbb{P}^1$ and hence is equal to a finite set $S \subset \mathbb{P}^1$. Let $\mathfrak{X} := \overline{\mathfrak{X}} - \overline{\pi}^{-1}(S)$ and let $\pi : \mathfrak{X} \to \mathbb{P}^1 - S$ be the restriction of $\overline{\pi}$ to $\mathfrak{X}$. By Ehrenmann's fibration theorem, π is a $\mathcal{C}^{\infty}$-fibration. Hence all fibers of π are diffeomorphic via a choice of an appropriate path in $\mathbb{P}^1 - S$. This identification however is not holomorphic. Therefore one can think of this as being given a smooth manifold, diffeomorphic to a fiber of π, with a varying complex structure on it. The primitive cohomology bundle induced by π thus determines a *polarized variation of Hodge structures*:

Definition 1.1.4.1. A *polarized variation of Hodge structures of weight* k $\mathscr{H} := (\mathbf{H}_{\mathbb{Z}}, \mathcal{F}^\bullet, Q)$ (VHS) on some complex manifold B consists of the following data:

1. a local system $\mathbf{H}_{\mathbb{Z}}$ of free, finite rank $\mathbb{Z}$-modules on B.
2. a finite decreasing filtration $\mathcal{F}^\bullet$ of the vector bundle $\mathcal{H} := \mathbf{H}_{\mathbb{Z}} \otimes \mathcal{O}_B$ by holomorphic subvectorbundles such that $\mathcal{F}^\bullet$ induces a pure HS of weight k on each fiber $\mathbf{H}_{\mathbb{Z},t}$, $t \in B$, and such that the canonical connection ∇ on $\mathcal{H}$, given by the canonical differential d on $\mathcal{O}_B$, satisfies *Griffiths transversality*:
$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B.$$
3. A flat, nondegenerate bilinear form Q on $\mathcal{H}$, symmetric if k is even and skew if k is odd, which takes integral values on $\mathbf{H}_{\mathbb{Z}}$ and which polarizes the Hodge structure on each $\mathbf{H}_{\mathbb{Z},t}$.

For convenience, we make the following definition:

Definition 1.1.4.2. Suppose that $B = \mathbb{P}^1 - S$ where S is a finite set. Let $s \in S$ and $t_0 \in B$. Let γ be a closed path starting and ending at t_0 such that s is the only point in S it encloses and going around s. We call the monodromy operator on $\mathbf{H}_{\mathbb{Z},t_0}$ given by γ the monodromy *at* s.

As mentioned above, the polarization of a VHS puts strong restrictions on the Jordan normal form of any monodromy operator T: As T must preserve the polarization of the VHS, and hence also a hermitian form, all eigenvalues of T must have absolute value 1. As T preserves a lattice, its characteristic polynomial has integral coefficients. Thus all eigenvalues of T must be roots of unity, proving the first part of the following

Theorem 1.1.4.3 (Monodromy theorem, [PS08])**.** *All eigenvalues of T are roots of unity. Let m be chosen minimally such that T^m is unipotent, i.e. there exists some k such that $(T^m - \mathrm{Id})^k = 0$. If $l := \max\{p - q \mid \mathcal{H}^{p,q}_{t_0} \neq 0\}$ then $(T^m - \mathrm{Id})^{l+1} = 0$.*

1.1.5. Picard–Fuchs Equations. Let now again $B := \mathbb{P}^1 - S$ where S is a finite set of points in $\mathbb{P}^1$. Fix a complex parameter t centered at some $s \in S$ on $\mathbb{P}^1$ and denote by $\frac{d}{dt}$ the composition of ∇ and contraction with the vector field given by t. The *Picard-Fuchs equations* are differential operators associated to sections $\omega(t)$ of $\mathcal{H}$ connecting the monodromy on $\mathcal{H}$ to analytic continuation for solutions to these equations. This is what will allow us to compute the monodromy on the primitive homology bundle of a pencil of hypersurfaces. They are defined as follows:

Definition 1.1.5.1. The monic differential operator $\mathcal{D}_\omega \in \mathbb{C}(t)\left[\frac{d}{dt}\right]$ annihilating a given section $\omega(t)$ of $\mathcal{H}$ and minimal in order with respect to this property is called the *Picard–Fuchs equation* of $\omega(t)$.

Fix some $t_0 \in B$. Let $\mathcal{H}^*$ be the dual bundle of $\mathcal{H}$. Fix a basis $\alpha_1, ..., \alpha_m$ of $\mathcal{H}^*_{t_0}$. These extend to flat sections of $\mathcal{H}^*$ over any simply connected neighborhood U of t_0 in B. Over U we may hence write

$$\omega(t) = \sum_{k=1}^{m} \langle \alpha_k, \omega(t) \rangle \alpha_k.$$

We have the following

Lemma 1.1.5.2 ([Ser19])**.** *The periods of $\omega(t)$, i.e. the functions $\langle \alpha_k, \omega(t) \rangle$, span the solution space of $\mathcal{D}_\omega$ near t_0.*

Proof. By the flatness of the sections determined by the α_k, we have

$$\frac{d}{dt}\omega(t) = \sum_{k=1}^{m} \left(\frac{d}{dt} \langle \alpha_k, \omega(t) \rangle \right) \alpha_k.$$

so that the functions $\langle \alpha_k, \omega(t) \rangle$, $k = 1, ..., m$, are annihilated by $\mathcal{D}_\omega$. Hence the periods of $\omega(t)$ are contained in the solution space of $\mathcal{D}_\omega$. Define

$$\sigma_k(t) := \langle \alpha_k, \omega(t) \rangle, \; k = 1, ..., m;$$
$$\sigma(t) = (\sigma_1(t), ..., \sigma_m(t)).$$

Let $l := \mathrm{ord}(\mathcal{D}_\omega)$. Then the space of solutions to the equation $\mathcal{D}_\omega f = 0$ has dimension l. Now, by the minimality of $\mathcal{D}_\omega$, the vectors $\sigma(t), \sigma'(t), ..., \sigma^{(l-1)}(t)$ are linearly independent. Therefore, the space spanned by the $\sigma_i(t)$ has at least dimension l since linear relations are preserved by differentiating. □

1.1.6. Monodromy of VHS's and Analytic Continuation. Denote by $\mathrm{sol}(\mathcal{D}_\omega)$ the local system of solutions of $\mathcal{D}_\omega$. Let T_ω be the operator on $\mathrm{sol}(\mathcal{D}_\omega)_{t_0}$ given by analytic continuation along a closed path from t_0 to t_0 around $s \in S$. Let T be the monodromy operator on $\mathbf{H}^*_{\mathbb{Z},t_0}$ given by the same path from t_0 to t_0. Then, as we have the morphism of local systems $p_\omega : \mathbf{H}^*_{\mathbb{Z}} \to \mathrm{sol}(\mathcal{D}_\omega)$ given by $\alpha \mapsto \langle \alpha, \omega(t) \rangle$, we have the following compatibility between the monodromy operator T and analytic continuation T_ω:

$$p_\omega(T\alpha) = T_\omega p_\omega(\alpha); \; \alpha \in \mathbf{H}_{\mathbb{Z},t_0}.$$

As mentioned above, computation of the Picard-Fuchs equations and their analytic continuation operators will then allow us to compute the monodromy operator on a fiber of the homology bundle of a given pencil of hypersurfaces, using this compatibility (see section 3.1.4).

1.1.7. The Regularity of the Picard–Fuchs equations. Furthermore, the *regularity* of the Picard-Fuchs equations is key to predicting the Jordan normal form of T after computing a few Picard-Fuchs equations and before analytic continuation (see theorem 3.1.3.5):

Definition 1.1.7.1. Let $\mathcal{D} \in \mathbb{C}(t)\left[\frac{d}{dt}\right]$ be monic. A point $s \in \mathbb{C}$ is said to be a *singular point* of $\mathcal{D}$ if any of the coefficients of $\mathcal{D}$ has a pole at s. A singular point is said to be a *regular singular point* of $\mathcal{D}$ if $\mathcal{D}$ admits a full basis of solutions of the form

$$g_i(t) := (t-s)^\nu \left(\sigma_0(t) + \sigma_1(t)\log(t-s) + ... + \sigma_p(t)\frac{(\log(t-s))^p}{p!}\right)$$

around s where $\nu \in \mathbb{Q}$ and the σ_i are analytic functions around s. We say that $\mathcal{D}$ is *regular* if all singular points are regular.

Luckily, all Picard-Fuchs equations occuring in our computations will be regular by the following theorem:

Theorem 1.1.7.2 (Regularity theorem, [Gri70])**.** *Any Picard-Fuchs operator coming from a VHS is regular.*

1.1.8. The Schmid Limit MHS. Fix a disc Δ around some $s \in S$ and let $\Delta^* := \Delta - \{s\}$. Abusing notation, we denote the restriction of the VHS $\mathscr{H}$ still by $\mathscr{H}$. This VHS induces a certain MHS over s, called the *Schmid limit MHS*. As we will briefly discuss later, if $\mathscr{H}$ arises from a smooth map $\pi : \mathfrak{X} \to \Delta$, as in the case of a pencil of hypersurfaces in $\mathbb{P}^{n+1}$, where $\mathfrak{X}$ is a compact Kähler manifold, and all fibers X_t of π are smooth for $t \neq 0$, one may relate the canonical mixed Hodge structure on X_0 due to Deligne [Del75] and the Schmid limit MHS of $\mathscr{H}$ if the singular locus of X_0 is zero-dimensional [Ste76]. The Schmid limit MHS is constructed as follows: Fix any $t_0 \in \Delta^*$ and let T be the monodromy operator on $\mathbf{H}_{\mathbb{Z},t_0}$ at s. Suppose without loss of generality that Δ has unit radius and let

$$\begin{aligned} e : \mathfrak{h} &\to \Delta; \\ u &\mapsto e^{2\pi i u} \end{aligned}$$

be the universal cover of Δ by the upper halfplane $\mathfrak{h}$. Since $\mathfrak{h}$ is simply connected the vector bundle $e^*\mathcal{H}$ trivializes to $\mathcal{H}_{t_0} \times \mathfrak{h}$ by a choice of a branch of the logarithm $\log(t)$. By this construction any $\alpha \in \mathcal{H}_{t_0}$ hence defines a flat multivalued section of $\mathcal{H}$ over Δ^*. We may thus define:

Definition 1.1.8.1 ([Sch73])**.** Let m be as in the previous definition and l as in theorem 1.1.4.3. Let

$$N := \log(T^m) = \sum_{k=1}^{l} (-1)^{k+1}\frac{(T^m - \mathrm{Id})^k}{k}.$$

The *untwisting operator* φ is given by

$$\varphi(\alpha)(t) := \exp\left(-\frac{1}{2\pi i}\log(t)N\right)\alpha; \quad \alpha \in \mathcal{H}_{t_0}.$$

Let $w : \Delta^* \to \Delta^*$ be given by $u \mapsto t = u^m$. The monodromy operator of $w^*\mathcal{H}$ is T^m. An easy computation then shows

Lemma 1.1.8.2. *$\varphi(\alpha)(t) \in H^0(\Delta^*, w^*\mathcal{H})$ for any $\alpha \in \mathcal{H}_{t_0}$, i.e. $\varphi(\alpha(t))$ defines a single-valued global section of $w^*\mathcal{H}$.*

We will later use this simple fact to relate the regularity of the Picard-Fuchs equations to the Jordan normal form of T (see theorem 3.1.3.5).

To define the Schmid limit MHS, we first introduce its weight filtration:

Definition 1.1.8.3. Let N be a nilpotent operator on a vector space V. Let l be chosen minimally such that $N^{l+1} = 0$. We define the weight filtration $W(N, \bullet)$ associated to N recursively as follows:

1. $W_{2l} = V$, $W_{-1} = 0$.
2. $W_{l+i} = \{v \in V \mid N^{i+1}v \in W_{l-i-2}\}$.
3. $W_{l-i} = N^i W_{l+1}$.

Now, since the canonical connection on $w^*\mathcal{H}$ is flat one may extend $w^*\mathcal{H}$ to a vector bundle $w^*\mathcal{H}_\Delta$ over Δ via the untwisting operator and push-forward via $\Delta^* \hookrightarrow \Delta$ [PS08, Proposition 11.3, p. 255]. Additionally the subbundles $\mathcal{F}^p$ extend to bundles $\mathcal{F}^p_\Delta$ on Δ in such a way that:

Theorem 1.1.8.4 ([Sch73]). *The tuple*

$$w^*\mathscr{H}_\infty := (w^*\mathcal{H}_{\Delta,0,\mathbb{Z}}, w^*\mathcal{F}^\bullet_\Delta(0), W(N, \bullet))$$

is a MHS, called the Schmid limit MHS of $\mathscr{H}$.

Remark 1.1.8.5. One may parametrize the pure Hodge structures of weight k and with $\dim F^p$ fixed for every p on a given $\mathbb{Z}$-module by a subvariety of a product of Grassmanians by viewing each F^p as a point in a suitable Grassmanian. This subvariety is called the *period domain* associated to $H_\mathbb{Z}$, k and the collection $\dim F^\bullet$. A VHS then induces a map into this variety called the *period map*. The Hodge filtration of the Schmid limit MHS is then given by the limit of this map (for details, see [Sch73] or [Gri70]).

1.2 Griffiths Residues

Our main sources for this section are [Gri69], [Ser19] and [Lai16].

1.2.1. The Griffiths residue map. Let $f_X \in \mathbb{C}[z_0, ..., z_{n+1}]_d$ be a homogenous polynomial of degree d and suppose that the projective hypersurface $X := \{f_X = 0\} \subset \mathbb{P}^{n+1}$ is smooth.

As our goal is to compute a period matrix for the Hodge structure on the middle cohomology of X, we need a way to do linear algebra on the space $H^n(X, \mathbb{C})$, mainly to compute Picard-Fuchs equations. Griffiths residues are a way of representing the space $H^n(X,\mathbb{C})$ and its Hodge filtration by a vector space of polynomials. Let

$$V := H^0(\mathbb{P}^{n+1} - X, \Omega_{\mathbb{P}^{n+1}});$$
$$V_q := H^0(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}(q[X])).$$

Then one has

$$V = \bigoplus_{q \geq 0} V_q.$$

Let $\mathrm{vol}_{\mathbb{P}^{n+1}}$ be the standard volume form on $\mathbb{P}^{n+1}$. Since

$$\Omega_{\mathbb{P}^{n+1}} \cong \mathcal{O}_{\mathbb{P}^{n+1}}(-n-2)$$

we have

$$\Omega_{P^{n+1}}(q[X]) \cong \Omega_{P^{n+1}}(qd) \cong \mathcal{O}_{P^{n+1}}(qd - n - 2)$$

and so

Lemma 1.2.1.1. *Any element in V_q may be written as a quotient*

$$\frac{p}{f_X^q} \mathrm{vol}_{\mathbb{P}^{n+1}};\ p \in \mathbb{C}[z_0, ..., z_{n+1}]_{qd-n-2}.$$

For a cycle $\gamma \in H_n(X,\mathbb{Z})$, let $\tau(\gamma)$ be a thin tube around a representative of γ. We have a well defined map

$$H^{n+1}(\mathbb{P}^{n+1} - X, \mathbb{C}) \to PH^n(X,\mathbb{C})$$
$$\omega \mapsto \left(\gamma \mapsto \frac{1}{2\pi i}\int_{\tau(\gamma)} \omega\right)$$

which turns out to be an isomorphism. By Serre duality and de Rham's theorem we also have a natural map

$$V \to H^{n+1}(\mathbb{P}^{n+1} - X, \mathbb{C}).$$

Definition 1.2.1.2. The *Griffiths residue map* res is given by the composition of the above two maps

$$\mathrm{res} : V \to H^{n+1}(\mathbb{P}^{n+1} - X, \mathbb{C}) \xrightarrow{\sim} PH^n(X,\mathbb{C})$$

By Stokes' theorem any form of degree n on $\mathbb{P}^{n+1} - X$ must map to an element in the kernel of the residue map under the canonical derivation map d on the corresponding de Rham-complex. This can however be strengthened [Gri69, theorems 4.2, 4.3 and 8.1]:

Lemma 1.2.1.3. *The Griffiths residue map* res *satisfies the following properties:*

1. res *is surjective. Additionally*

$$\ker(\mathrm{res}\,|_{V_q}) = d\left(H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}((q-1)[X]))\right)$$

2. *We have*

$$\mathrm{res}(V_q) = F^{n-q+1}PH^n(X, \mathbb{C}).$$

1.2.2. Griffiths-Dwork reduction. Define $W_q := H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(q[X]))$. Since we will represent classes in the middle cohomology of X by a polynomial as in lemma 1.2.1.1 we want to be able to decide when the differential form given by such a polynomial lies in the kernel of the residue map, i.e. when it lies in the space $d\left(\bigoplus_{k\geq 0} W_k\right)$. To this end we observe the following: Let $J(f_X)$ be the Jacobian ideal of f_X in $\mathbb{C}[z_0, ..., z_{n+1}]$, i.e. the ideal generated by the first partial derivatives of f_X. Denote by ∂_{z_j} differentiation by z_j. By lemma 1.2.1.3 the differential form given by an element p/f_X^q has residue 0 if and only if

$$\frac{p}{f_X^q} \in \sum_{j=0}^{n+1} \partial_{z_j}\left(\mathbb{C}[z_0, ..., z_{n+1}, 1/f_X]\right).$$

Now observe that for polynomials b_j

$$\sum_{j=0}^{n+1} \partial_{z_j}\left(\frac{b_j}{f_X^{q-1}}\right) = \frac{\sum_{j=0}^{n+1} \partial_{z_j} b_j}{f_X^{q-1}} - (q-1)\frac{\sum_{j=0}^{n+1} b_j \partial_{z_j} f_X}{f_X^q}.$$

Hence if $p \in J(f_X)$ then the differential form given by p/f_X^q is equivalent modulo exact forms to a form in V_{q-1}. Griffiths [Gri69] showed that the converse also holds if the hypersurface given by f_X is smooth. This gives us a reduction procedure to maximally reduce a differential form given by a polynomial and decide whether it has residue 0. This procedure is called *Griffiths-Dwork reduction*:

1. Compute a Gröbner basis G of $J(f_X)$.
2. Given any $\omega \in V$, write $\omega = \sum_{q=1}^{r} \frac{p_q}{f_X^q} \mathrm{vol}_{\mathbb{P}^{n+1}}$ where p_q is homogenous of degree $qd - n - 2$.
3. Compute the remainder q_r of p_r modulo $J(f_X)$ using G and write $p_r - q_r = \sum_{j=0}^{n+1} b_j \partial_{z_j} f_X$. By the previous observation, ω is equivalent modulo exact forms to

$$\omega' = \left(\frac{q_r}{f_X^r} + \frac{p_{r-1} + q_{r-1}}{f_X^{r-1}} + \sum_{q=1}^{r-2} \frac{p_q}{f_X^q}\right) \mathrm{vol}_{\mathbb{P}^{n+1}}$$

   where $q_{r-1} = \frac{1}{r-1} \sum_{j=0}^{n+1} \partial_{z_j} b_j$.

4. Proceed inductively by applying the same reduction method to the term of pole order $r-1$ of ω'. Repeat until ω is fully reduced.

Denote the resulting differential form by $[\omega]_{\mathrm{GD}}$. By the above characterization of exact forms in V by Griffiths we then have

Lemma 1.2.2.1 ([Ser19]). *Let $\omega \in V$. Then $\operatorname{res}(\omega) = 0$ if and only if $[\omega]_{GD} = 0$. Hence a family $[\omega_1]_{GD}, ..., [\omega_k]_{GD} \in V$ is linearly independent if and only if $\operatorname{res}(\omega_1), ..., \operatorname{res}(\omega_k)$ are linearly independent.*

Chapter 2

Singularity Theory

Our main sources of exposition in this chapter are [Dim92], [Mil68] and [Kul98].

For this chapter, we fix the following conventions and definitions: Let $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ be a pencil of projective hypersurfaces in $\mathbb{P}^{n+1}$ as defined in section 1.1.4. Let $S \subset \mathbb{P}^1$ be its set of singular values. Pick some singular value $s \in S$, a complex parameter t centered at s and a disc Δ around s such that s is the only singular value of $\overline{\pi}$ in Δ. Let $\overline{\pi}_s : \overline{\mathfrak{X}}|_\Delta \to \Delta$ be the restriction of $\overline{\pi}$ to the family $\mathfrak{X}$ over Δ. We assume that the fiber $X_0 := \overline{\pi}_s^{-1}(\{s\})$ has finitely many singular points, i.e. that the singular locus of X_0 is zero-dimensional. Fix any $t_0 \in \Delta^* := \Delta - \{s\}$ and let T be the monodromy operator on the fiber $H^n(X_{t_0}, \mathbb{Z})$ at s.

The goal of this chapter is twofold:

We first want to discuss how, under the additional assumption that $\overline{\mathfrak{X}}|_\Delta$ is smooth, the Jordan normal form of T depends, up to the discrepancy between the geometric and algebraic multiplicity of the eigenvalue 1, only on the so-called *right equivalence types* of the singular points of X_0. This, together with a computation of the number of singular points, the *Milnor number* and the *corank* of X_0 then allows us to make a prediction, before computing T, whether there exists a polynomial $q \in \mathbb{Z}[x]$ such that $\mathrm{rk}(q(T)) = 1$. Then $q(T)v$ is a scalar multiple of an integral cycle for any v not in the kernel of T.

We then give the theoretical foundation to compute a period matrix of X_0 using a result by Steenbrink [Ste76]: If X_0 is as above and $\overline{\mathfrak{X}}|_\Delta$ is smooth then the MHS on $H^n(X_0, \mathbb{Z})$ coincides with the restriction of the Schmid limit MHS on $H^n(X_{t_0}, \mathbb{Z})$, where $t_0 \in \Delta^*$, to $\ker(T - \mathrm{Id})$.

2.1 The Milnor Number and the Milnor Fibration of a Singular Point

2.1.1. The Milnor Number. Fix a polynomial $f \in \mathbb{C}[x_1, ..., x_{n+1}]$. Let $V(f) = \{p \in \mathbb{C}^{n+1} \mid f(p) = 0\}$ be the zero locus of f.

Definition 2.1.1.1. Let $p \in V(f)$ and let $J(f)$ be the jacobian ideal of f. The *Milnor number* of f at p is defined by

$$\mu_p(f) := \dim_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{C}^{n+1},p} / J(f) \right).$$

Remark 2.1.1.2. Clearly, $\mu_p(f) = 0$ if f is smooth at p. Also, by the Nullstellensatz, $\mu_p(f) < \infty$ if p is an isolated singular point of f.

For our purposes, we are interested in computing the number

$$\mu(f) = \sum_{p \in V(f)} \mu_p(f)$$

and the number of singular points $n_{\mathrm{sing}}(f)$ in $V(f)$. Both these numbers are finite if the number of singular points in $V(f)$ is finite. Denote $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, ..., x_{n+1}]$.

Lemma 2.1.1.3. *Suppose that $V(f)$ has finitely many singular points. Then the number of singular points $n_{sing}(f)$ in $V(f)$ is given by*

$$n_{sing}(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[\mathbf{x}]}{\sqrt{(f, J(f))}}.$$

Proof. The algebra $A := \mathbb{C}[\mathbf{x}] / \sqrt{(f, J(f))}$ is reduced, artinian and has finitely many maximal ideals. An easy application of the chinese remainder theorem then shows that the number above coincides with the number of maximal ideals in A. □

Lemma 2.1.1.4 ([Bod04])**.** *Suppose that $V(f)$ has finitely many singular points. Then there exists $m \in \mathbb{N}$ such that*

$$\mu(f) = \dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}] / (f^m, J(f)).$$

Furthermore, if some $k \in \mathbb{N}$ satisfies

$$\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}] / (f^k, J(f)) = \dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}] / (f^{k+1}, J(f)),$$

then we can choose $m = k$ in the above formula for $\mu(f)$.

Proof. Assume for now that $V(f)$ contains exactly one singular point p with corresponding maximal ideal $\mathfrak{m}$. Now, by the definition of $\mu_p(f)$, it is enough to show that there exists $m \in \mathbb{N}$ with $f^m \in J(f)\mathbb{C}[\mathbf{x}]_{\mathfrak{m}}$ for the first part of the lemma. By the Nullstellensatz there exists some $k \in \mathbb{N}$ such that

$$\mathfrak{m}^k \subset J(f)\mathbb{C}[\mathbf{x}]_{\mathfrak{m}} \subset \mathfrak{m}$$

But since $f \in \mathfrak{m}^2$ there exists $m \in \mathbb{N}$ such that $f^m \in J(f)\mathbb{C}[\mathbf{x}]_{\mathfrak{m}}$. This shows the first claim of the lemma. If $V(f)$ contains more than one singular point, we simply apply the above argument to every singular point and take the maximum over all k's.

To prove the second part we move to the following more abstract situation: Let $R = \mathbb{C}[\mathbf{x}]_{\mathfrak{m}}$, let $I \subset R$ be an ideal in R with $\mathfrak{m}^k \subset I \subset \mathfrak{m}$ for some $k \in \mathbb{N}$. Let $f \in \mathfrak{m}$. Suppose there exists $l \in \mathbb{N}$ with

$$\dim_C R/(f^l, I) = \dim_C R/(f^{l+1}, I)$$

Then, since we are dealing with finite dimensional $\mathbb{C}$-algebras, we have

$$f^l \in (f^{l+1}, I)$$

This implies that for some $a \in R$

$$f^l - af^{l+1} = f^l(1 - af) \in I$$

But since $1 - af$ is a unit in R it follows that $f^l \in I$, finishing the proof. □

2.1.2. Isolated Hypersurface Singularities. Let us briefly define what we mean by an isolated hypersurface singularity.

Definition 2.1.2.1. Let f be as above and assume that $p \in V(f)$ is an isolated singular point of f. We call the tuple (f, p) an *isolated hypersurface singularity* (IHS) in $\mathbb{C}^{n+1}$.

We now introduce a notion of isomorphisms of isolated hypersurface singularities:

Definition 2.1.2.2. Two isolated hypersurface singularities (f, p) and (g, q) in $\mathbb{C}^{n+1}$ are said to be *right equivalent* if there exists an open neighborhood U of p in $\mathbb{C}^{n+1}$, an open neighborhood V of q in $\mathbb{C}^{n+1}$ and a biholomorphic map $\psi : U \to V$ such that $\psi(p) = q$ and

$$g = f \circ \psi.$$

2.1.3. The Milnor Fibration. The Milnor fibration is a powerful invariant associated to an IHS and serves us to describe the monodromy operator on (co-)homology at a singular value of a pencil, if the corresponding fiber has only isolated singular points.

Heuristically, if f is a local equation of X_0 around p and p is the only singular point of X_0 then we should only observe nontrivial monodromy in a small neighborhood of p in $\overline{\mathfrak{X}}|_{\Delta^*}$. This idea is made precise by the theory of the *Milnor fibration* of f:

Theorem 2.1.3.1 (Milnor's fibration theorem, [Mil68]). *Denote by B_ε an open ball around p in $\mathbb{C}^{n+1}$, let Δ denote a disc around 0 in $\mathbb{C}$ and let $\Delta^* := \Delta - \{0\}$. Then the map*

$$\varphi : \mathfrak{M} := B_\varepsilon \cap f^{-1}(\Delta^*) \to \Delta^*;$$
$$varphi(x) := f(x),$$

is a $\mathcal{C}^\infty$*-fibration for* ε *chosen so small that all fibers* $f^{-1}(t)$*,* $t \in \Delta^*$*, are transversal to* B_ε *after possibly shrinking* Δ*. Its general fiber* M_t*,* $t \in \Delta^*$*, is homotopy equivalent to a bouquet of* $\mu_p(f)$ n*-Spheres.*

Definition 2.1.3.2. We call the map $\varphi : \mathfrak{M} \to \Delta^*$ the *Milnor fibration* of f at p.

Remark 2.1.3.3. The Milnor fibration is only defined up to diffeomorphism and potentially rescaling Δ. This is, however, enough for all applications.

Remark 2.1.3.4. Clearly, two right-equivalent IHS's have diffeomorphic Milnor fibrations.

2.1.4. The Corank of an Isolated Hypersurface Singularity. We introduce another invariant of an isolated hypersurface singularity:

Definition 2.1.4.1. Let (f, p) be an IHS in $\mathbb{C}^{n+1}$. The *corank* of (f, p) is the corank of the Hessian matrix of f at p.

The corank of an isolated hypersurface singularity tells us how many variables are "interesting":

Theorem 2.1.4.2 (Generalized Morse lemma, [Arn98])**.** *Let* (f, p) *be an isolated hypersurface singularity in* $\mathbb{C}^{n+1}$ *of corank* r*. Then* (f, p) *is right equivalent to* $(g, 0)$ *where* g *is of the form*

$$g = h + x_{(n+1)-r}^2 + \dots + x_{n+1}^2$$

where h *is a polynomial in* $\mathbb{C}[x_1, \dots, x_r]$*, vanishing to order at least 3 in the point* $0 \in \mathbb{C}^{n+1}$*.*

Clearly, we can compute the corank of an isolated hypersurface singularity as follows:

Lemma 2.1.4.3. *Let* (f, p) *be as above. Assume that* p *is the only singular point of* f *in* $V(f)$*. Let* $M_{k,f}$ *be the ideal generated by the minors of order* k *of the Hessian matrix of* f*. Then we have*

$$\operatorname{corank}(f, p) = n + 1 - \max\{k \mid 1 \notin (f, J(f), M_{k,f})\}.$$

2.1.5. The Milnor Fibration in a Global Setting. Let us now describe how the Milnor fibration of an isolated hypersurface singularity plays a role in a pencil of hypersurface in $\mathbb{P}^{n+1}$ and how one may relate the MHS on X_0 and the Schmid limit MHS of the VHS given by $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$. For now, we additionally assume that X_0 has exactly one singular point p.

Lemma 2.1.5.1. *Suppose that* $\overline{\mathfrak{X}}|_\Delta$ *is smooth. Let* $p \in X_0$ *be an isolated singular point of* X_0 *and let* f *be a local equation for* X_0 *around* p *in local coordinates* $x_1, \dots, x_{n+1}$*. Then, after possibly shrinking* Δ *and for a sufficiently small ball* B *around* p *in* $\overline{\mathfrak{X}}$*, the fibration* $\overline{\pi}_s|_{B - \overline{\pi}_s^{-1}(0)} : B - \overline{\pi}_s^{-1}(0) \to \Delta^*$ *is diffeomorphic to the Milnor fibration of* f *at* p*.*

Proof. In the local coordinate t we may write

$$\overline{\mathfrak{X}}|_\Delta = \{(z,t) \in \mathbb{P}^{n+1} \times \Delta \mid f_0(z) - t f_1(z) = 0\},$$

where $f_0, f_1 \in \mathbb{C}[z_0, ..., z_{n+1}]_d$ for some $d > 0$ and $X_0 = \{f_0 = 0\} \subset \mathbb{P}^{n+1}$. Now, clearly, $\overline{\mathfrak{X}}|_\Delta$ is smooth if and only if $f_1(p) \neq 0$. Thus we can choose a local trivialization of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$ in an open ball B around p such that in the local coordinates $x_1, ..., x_{n+1}$ given by this trivialization we have $g_1(x_1, ..., x_{n+1}) = 1$ where g_1 is a local equation for $X_1 := \{f_1 = 0\}$. Thus

$$\overline{\mathfrak{X}}|_B \cong \{(x,t) \in \mathbb{C}^{n+1} \times \mathbb{C} \mid f(x) = t\}.$$

The result follows after possibly shrinking B and Δ. □

We now describe how, in the situation of lemma 2.1.5.1, the monodromy operator T on $H^n(X_{t_0}, \mathbb{Z})$, $t_0 \in \Delta^*$, is almost entirely determined by the monodromy operator T_{loc} on the cohomology of the Milnor fiber $M_{t_0} \subset X_{t_0}$ where $M_{t_0} := X_{t_0} \cap B$ with B defined as in lemma 2.1.5.1. First of all, one may give an explicit geometric description of the space $\ker(T - \text{Id})$, namely the *local invariant cycles theorem* [de 15]:

Theorem 2.1.5.2. *One has $H^n(X_0, \mathbb{Z}) \cong H^n(\overline{\mathfrak{X}}|_\Delta, \mathbb{Z})$ via a retraction by deformation. The map*

$$H^n(X_0, \mathbb{Z}) \to H^n(X_{t_0}, \mathbb{Z})$$

given by the composition of this isomorphism and the restriction map, i.e. the map $H^n(\mathfrak{X}|_\Delta, \mathbb{Z}) \to H^n(X_{t_0} t, \mathbb{Z})$ given by the inclustion $X_{t_0} \hookrightarrow \overline{\mathfrak{X}}$, surjects to the kernel of $T - \text{Id}$.

Moreover, X_0 is homeomorphic to a cone over $X_0 \cap \partial B$ with vertex p [Kul98]. Hence

$$\check{H}^k(X_0, \mathbb{Z}) \cong H^k(X_{t_0}, M_{t_0}; \mathbb{Z}) \quad \forall k,$$

where $\check{H}^k(X_0, \mathbb{Z})$ is the k-th reduced cohomology group and $H^k(X_{t_0}, M_{t_0}; \mathbb{Z})$ denotes the k-th relative cohomology group of the pair (X_{t_0}, M_{t_0}). This is since $\mathfrak{X}_\Delta - B \to \Delta$ is a smooth fibration. Now, since M_{t_0} is homotopy equivalent to a bouquet of n-spheres, we have $H^k(M_{t_0}, \mathbb{Z}) = 0$ unless $k = 0$ or $k = n$. The long exact sequence of the pair (X_{t_0}, M_{t_0}) hence yields the commutative diagram (see e.g. [Rah16, remark 4.2.3])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(X_0,\mathbb{Z}) & \longrightarrow & H^n(X_{t_0},\mathbb{Z}) & \longrightarrow & H^n(M_{t_0},\mathbb{Z}) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \text{Id}} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T_{\text{loc}}} & & \\ 0 & \longrightarrow & H^n(X_0,\mathbb{Z}) & \longrightarrow & H^n(X_{t_0},\mathbb{Z}) & \longrightarrow & H^n(M_{t_0},\mathbb{Z}) & \longrightarrow & \dots \end{array} \qquad (1)$$

with exact rows where the diagram commutes again because of the smoothness of the fibration $\mathfrak{X}|_\Delta - B \to \Delta$ [Kul98]. Now we have

Lemma 2.1.5.3. *The Jordan normal form of T is determined up to the geometric multiplicity of the eigenvalue 1 by the right equivalence type of the hypersurface singularity given by X_0 and p.*

Proof. This immediately follows from the above diagram. Let $\iota : M_{t_0} \hookrightarrow X_{t_0}$ be the inclusion map. If $v \in H^n(X_{t_0}, \mathbb{Z})$ is a generalized eigenvector for some eigenvalue $\lambda \neq 1$ of T, then $\iota^* v \neq 0$ by Theorem 2.1.5.2. We have $(T - \lambda \operatorname{Id})^k v = 0$ for some k so the same holds for T_{loc} instead of T by the commutativity of the above diagram. But T_{loc} only depends on the right equivalence class of the isolated hypersurface singularity given by X_0 and p. □

Remark 2.1.5.4. All of these observations apply analogously if X_0 carries finitely many, but more than one, singular point. In this case the Jordan Normal form of T carries several distinct blocks, each associated to an operator T_{loc} on the cohomology of the Milnor fibration of a distinct singular point of X_0.

According to [Ste76], there exists a MHS on $H^n(M_{t_0}, \mathbb{Z})$, such that the first (or second) row of the diagram (1) becomes an exact sequence of MHS's where $H^n(X_0, \mathbb{Z})$ carries the canonical MHS and $H^n(X_{t_0}, \mathbb{Z})$ carries the Schmid limit MHS. This will enable us to compute the MHS on $H^n(X_0, \mathbb{Z})$ if X_0 carries finitely many singular points. More precisely, using this together with theorem 2.1.5.2 and remark 2.1.5.4 yields

Theorem 2.1.5.5. *Suppose that $\overline{\mathfrak{X}}|_\Delta$ is smooth and suppose that X_0 carries finitely many isolated singular points. Then the MHS on $H^n(X_0, \mathbb{Z})$ is isomorphic to the Schmid limit MHS restricted to* $\ker(T - \operatorname{Id})$.

2.2 A Few Computationally Useful Cases

We discuss a few special isolated hypersurface singularities that can occur in a pencil of projective hypersurfaces that will be computationally useful to us.

2.2.1. The Simple Node. The simplest isolated hypersurface singularity is the so-called *simple node*:

Definition 2.2.1.1. The simple node is the (up to right equivalence) unique isolated hypersurface singularity with Milnor number 1.

Remark 2.2.1.2. This turns out to be well-defined: Any hypersurface singularity in $\mathbb{C}^{n+1}$ with Milnor number 1 turns out to have corank $n+1$ and is thus is given by the equation

$$x_1^2 + ... + x_{n+1}^2 = 0$$

after a suitable change of coordinates by theorem 2.1.4.2.

Observe that if a pencil carries an X as in remark 2.2.1.3 at a singular value s then the monodromy operator T at s on homology satisfies

$$\operatorname{rk}(T - \operatorname{Id}) = 1$$

by the commutative diagram in section 2.1.5. This is since, in this case, the map $H^n(X_{t_0}, \mathbb{Z}) \to H^n(M_{t_0}, \mathbb{Z})$ turns out to be surjective [Voi02]. In particular, for any complex cycle γ not in the kernel of $T - \mathrm{Id}$,

$$(T - \mathrm{Id})\gamma$$

is a scalar multiple of an integral cycle. This enables us to compute periods in theory.

Remark 2.2.1.3. A general singular hypersurface $X \subset \mathbb{P}^{n+1}$ of fixed degree d carries exactly one singular point which is a simple node: The set of hypersurfaces of degree d in $\mathbb{P}^{n+1}$ are parametrized by a projective space $\mathbb{P}_{n,d}$ of dimension

$$\binom{n+d+1}{d}.$$

This projective space carries the *discriminant variety* $D_{n,d}$ of singular hypersurfaces in $\mathbb{P}^{n+1}$. $D_{n,d}$ carries the set $D^0_{n,d}$ of hypersufaces X as above which turns out to be Zariski open in $D_{n,d}$ [Voi02, vol. 2, p. 45].

Pencils of projective hypersurfaces which only carry the types of singular hypersurfaces as in the previous remark are called *Lefschetz pencils*:

Definition 2.2.1.4. A *Lefschetz pencil* is a pencil of projective hypersurfaces $\overline{\pi}$ such that the corresponding line in $\mathbb{P}_{n,d}$ intersects $D_{n,d}$ transversally and intersects $D_{n,d}$ in $D^0_{n,d}$.

It follows from remark 2.2.1.3 that a general pencil through a fixed smooth hypersurface is a Lefschetz pencil.

It is important to note, however, that Lefschetz pencils are computationally impractical. This is due to the following

Theorem 2.2.1.5 ([Voi02])**.** *The monodromy representation on a primitive homology group arising from a Lefschetz pencil is irreducible.*

It immediately follows that the Picard-Fuchs equations of a Lefschetz pencil must have maximal order: The map p_ω defined as in section 1.1.6 must be bijective as the kernel of it is a monodromy equivariant subspace. This makes the Picard-Fuchs equations in this case "too big" to handle computationally in reasonable time.

2.2.2. Simple Surface Singularities. Suppose that the hypersurface we are interested in computing periods for lies in $\mathbb{P}^3$. There is a special class of isolated hypersurface singularities in $\mathbb{C}^3$ which will come in handy in this case:

Definition 2.2.2.1. A *simple surface singularity* is an isolated hypersurface singularity in $\mathbb{C}^3$ such that the intersection form on the middle homology of its Milnor fiber is negative definite.

Remark 2.2.2.2. There are many equivalent definitions for simple surface singularities. Among them is an explicit list of right equivalence types. See [Dur79] for details.

Suppose now that we are in the situation of section 2.1.5 with $n = 2$ and that the central fiber X_0 together with its unique singular point p defines a simple surface singularity of odd Milnor number μ. It follows that T_{loc} must have eigenvalue 1 or -1 since T_{loc} acts on $H^n(M_{t_0}, \mathbb{C})$ which is of odd dimension μ. We can, however, exclude the occurence of the eigenvalue 1 for T_{loc} by the following theorem:

Theorem 2.2.2.3 ([Arn98])**.** *The monodromy on the homology of the Milnor fiber of an isolated hypersurface singularity has eigenvalue* 1 *if and only if the intersection product on the homology of the Milnor fiber degenerates.*

Moreover, one can use *spectral theory* of singularities to show that in this case the eigenvalue -1 occurs with multiplicity 1 [Arn98]. It follows that if $p \in \mathbb{Z}[x]$ is the characteristic polynomial of T and $q(x) := p(x)/(x-1) \in \mathbb{Z}[x]$ then

$$\operatorname{rk} q(T) = 1.$$

Furthermore spectral theory additionally yields

Lemma 2.2.2.4. *Suppose that X_0 has exactly one singular points and that X_0 together with its singular point defines a simple surface singularity. Then T has eigenvalue -1 if and only if the Milnor number μ of X_0 is odd. Furthermore, T is diagonizable.*

Proof. We use the notation of 2.1.5. Spectral theory shows that T_{loc} is always diagonizable and has eigenvalue -1 if and only if μ is odd (see [Arn98] for details). But the same must then hold for T since the morphism $H^n(X_{t_0}, \mathbb{Z}) \to H^n(M_{t_0}, \mathbb{Z})$ in the commutative diagram (1) is surjective by theorem 2.2.2.3. □

We can compute the corank of the IHS defined by X_0 and p to detect a certain class of simple surface singularities:

Definition 2.2.2.5. We define the singularity $A_n := (f, 0)$ for $n \in \mathbb{N}$ where

$$f = x_1^{n+1} + x_2^2 + x_3^2.$$

By the generalized Morse lemma 2.1.4.2, these are precisely the surface singularities in $\mathbb{C}^3$ of corank 1 and can thus be detected computationally, using lemma 2.1.4.3. Additionally

Lemma 2.2.2.6 ([Dur79])**.** *The singularities A_n are simple for every $n \in \mathbb{N}$.*

Furthermore, one can use an explicit classification list to show

Lemma 2.2.2.7 ([Dur79])**.** *Any IHS in $\mathbb{C}^3$ of Milnor number ≤ 8 is simple.*

Chapter 3

Using Pencils of Hypersurfaces to Compute Period Matrices

3.1 The Smooth Case

We now give a rough description of our method to compute periods of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ cut out by some homogeneous polynomial $f_X \in \mathbb{Q}[z_0, ..., z_{n+1}]$. We start by fixing an arbitrary homogeneous $f_Y \in \mathbb{Q}[z_0, ..., z_{n+1}]$ of the same degree as f_X and cutting out a hypersurface $Y \subset \mathbb{P}^{n+1}$ with $Y \neq X$. Let $\overline{\pi}\colon \overline{\mathfrak{X}} \to \mathbb{P}^1$ be the pencil determined by the equation

$$tf_X + (1-t)f_Y = 0;\ t \in \mathbb{C}.$$

The overall computational strategy is now as follows: Let $\mathcal{H}$ be the primitive cohomology bundle associated to $\overline{\pi}$. Suppose that the monodromy operator on $\mathcal{H}^*_{t_0} = PH_n(X, \mathbb{C})$ at some singular value s of $\overline{\pi}$ is given by a matrix T in terms of some basis $\alpha_1, ..., \alpha_m$ of $PH_n(X, \mathbb{C})$ which are represented by "period coordinates"

$$(\langle \alpha_i, \omega_1 \rangle, ..., \langle \alpha_i, \omega_m \rangle)$$

where the ω_k are as in definition 1.1.2.4. Suppose that there is some polynomial q with integral coefficients such that

$$\mathrm{rk}(q(T)) = 1.$$

Suppose that v is not in the kernel of $q(T)$. Then

$$q(T)v$$

is a scalar multiple of an integral vector w since T preserves $PH_n(X, \mathbb{Z})$. Let $q(T)v = \lambda w$. Then $q(T)v$ is a period for the basis $\lambda\omega_1, ..., \lambda\omega_m$ which obviously still satisfies the requirements for a basis of $PH^n(X, \mathbb{C})$ given in definition 1.1.2.4. Thus we will from now on refer to scalar multiples of integral vectors as integral vectors as well. We have three likely situations where this happens:

1. The singular fiber associated to T carries exactly one singular point which is a simple node. In this case we have $\mathrm{rk}(T - \mathrm{Id}) = 1$ (see section 2.2.1).
2. T has eigenvalue -1 with multiplicity 1. In this case q is the characteristic polynomial of T divided by $x + 1$.
3. The Jordan normal form of T has a single Jordan block of maximal size. In this case

$$q(x) = (x^l - 1)^k$$

for suitable l and k.

Ideally, we want to predict any of these situations before doing any potentially expensive computation of Picard-Fuchs equations or analytic continuation. The first situation can be detected by a Milnor number computation by the definition of the simple node. If $n = 2$ the second situation can sometimes be predicted using singularity theory (see section 3.1.1) by a computation of numbers of singular points and milnor numbers of singular fibers occuring in $\overline{\pi}$.

If a computation of the Milnor number and number of singular points of a singular fiber X_s of $\overline{\pi}$ does not yield enough information to decide if the first or second situation occurs or if we are not in the surface case, we can still predict the Jordan normal of T to some extent by computing a few Picard-Fuchs equations (see section 3.1.3) thus detecting the third situation.

3.1.1. Computing the Singular Values, their Associated Number of Singular Points and their Milnor Numbers. To compute the singular values of $\overline{\pi}$, we need to compute those values of t where the Jacobian of

$$f_t = tf_X + (1 - t)f_Y$$

w.r.t. the variables $z_0, ..., z_{n+1}$ has a zero. This is a simple matter of computing the saturation of the corresponding jacobian ideal w.r.t. the ideal $(z_0, ..., z_{n+1})$ and elimination using Gröbner bases. The singular values are then represented as roots of a polynomial $p \in \mathbb{Q}[t]$. We compute its factorization into irreducible factors

$$p = p_1 ... p_l$$

over $\mathbb{Q}$ and compute the number of singular points n_s of a singular fiber X_s by computing the Hilbert polynomial of the ideal

$$\sqrt{(f_i, J(f_i))}$$

(see lemma 2.1.1.3) where $p_i(s) = 0$ and f_i is the image of f_t in the field $\mathbb{Q}[t]/(p_i)$. Similarly, to compute the Milnor number μ_s of X_s (i.e. the sum of all Milnor numbers of all singular points of X_s) we compute the Hilbert polynomial of the ideal

$$(f_i^k, J(f_i))$$

where k is chosen as in Lemma 2.1.1.4.

Remark 3.1.1.1. We found during our experiments that, to keep the degree and order of the Picard-Fuchs equations as low as possible, using pencils with few singular values was usually the best choice. In this case the time it took to compute the above Hilbert polynomials was negligible in comparison to the time it took to compute Picard-Fuchs equations and monodromy operators.

Analogously, we can compute the corank c_s of the singular fiber X_s if $n_s = 1$ using lemma 2.1.4.3.

Suppose that the singular locus of X_s is zero-dimensional and suppose that we are in the surface case, i.e. $X \subset \mathbb{P}^3$. Suppose that we have computed the number of singular points n_s of X_s, the Milnor number μ_s of X_s and the corank c_s of X_s if $n_s = 1$ and μ_s is odd. Now we want to check whether the monodromy operator T can be used to compute a an integral vector.

1. If $n_s = 1$ and $\mu_s = 1$ then X_s carries a simple node and we can use T to compute an integral vector since $\mathrm{rk}(T - \mathrm{Id}) = 1$ (see section 2.2.1). Note that this works even if $n \neq 2$.

2. If $n_s = 1$ and $\mu_s \leq 8$ is odd, then X_s carries a unique singular point which is simple by lemma 2.2.2.7. Therefore T has eigenvalue -1 with multiplicity 1 and we can use T to compute an integral vector. If $n_s = 1$ and $\mu_s \leq 8$ is even then neither of the three situations mentioned at the beginning of this chapter occurs (see lemma 2.2.2.4) and we can discard X_s.

3. If $n_s = 1$, $\mu_s \geq 9$ is odd and $c_s = 1$ then X_s carries a unique singular point which is simple by lemma 2.2.2.6 and theorem 2.1.4.2. T can be used to compute an integral vector as in item 1.

4. Note that the Milnor number $\mu_p(X_s)$ of any singular point p of X_s by definition satisfies

   $$\mu_p(X_s) \leq \mu_s - n_s + 1.$$

   This is because $\mu_s = \sum_{p \in \mathrm{Sing}(X_0)} \mu_p(X_s)$. If $n_s = 2$, μ_s is even and $\mu_p(X_s) \leq 8$ using the above inequality, then we cannot use T for period computation as the eigenvalue -1 occurs with multiplicity 0 or more than 1 by Remark 2.1.5.4. This is since in this case X_s carries two singular points, both of which define simple singularities and both of which have either odd or even Milnor number. We discard X_s in this case.

5. if $n_s = \mu_s > 1$ then X_s carries n_s simple nodes (i.e. multiple copies of A_1). Therefore we can discard X_s because the Jordan normal form of T is of the form

$$\begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

by lemma 2.2.2.4 where the $-\mathrm{Id}$-block is of size n_s.

Remark 3.1.1.2. Note that the above criteria may not be enough to decide whether any of the three situations mentioned at the beginning of this chapter occurs. For example the IHS D_9 of Milnor number 9 and corank 2 is simple and can thus be used to compute an integral vector in the sense of item 3 above. On the other hand the IHS X_9, also of Milnor number 9 and corank 2, turns out to not be useful in the sense of the three situations mentioned at the beginning of this chapter (see [Arn98] for a definition of D_9 and X_9).

If the above criteria fail or if we are not in the surface case then we have theorem 3.1.3.5 to fall back on. We do, however, need to compute a few Picard-Fuchs equations to apply this theorem.

3.1.2. Picard–Fuchs Equations: Implementational Details. We give some of the details on how differential operators in $\mathbb{C}(t)\left[\frac{d}{dt}\right]$ and analytic computation for them are implemented. Our main computational tool is the package *Numerical Evalutation of D-finite Functions for ore_algebra* written by Marc Mezzarobba [Mez16] [1]. This package enables us to both compute analytic continuation matrices to arbitrary precision with exact error bounds and to compute a *collection of canonical local monomials* (see definition 3.1.2.1) at a singular point of a Picard-Fuchs equation, all of which are regular by theorem 1.1.7.2. Let $\mathcal{D} \in \mathbb{C}(t)\left[\frac{d}{dt}\right]$. Let $k := \mathrm{ord}(\mathcal{D})$. Internally, a solution f to the equation $\mathcal{D}f = 0$ at a regular point $p \in \mathbb{C}$ is represented by its Taylor coefficients about p:

$$\left(f(p), f'(p), ..., \frac{1}{(k-1)!} f^{(k-1)}(p)\right)$$

This uniquely determines f as a solution to $\mathcal{D}f = 0$.

Conversely, at a regular singular point $q \in \mathbb{C}$ a solution to the equation $\mathcal{D}f = 0$ may be uniquely written as an expansion

$$f(t) = \sum_{i=1}^{k} a_i g_i(t),\ a_i \in \mathbb{C}$$

where the g_i are as in definition 1.1.7.1. They are power series in terms of monomials of the form

$$m_i(t) = (t-q)^{\nu} \log(t-q)^k$$

[1] `marc.mezzarobba.net/code/ore_algebra_analytic/`

where ν ranges over the roots of the so-called indicial polynomial of $\mathcal{D}$ at q and k is less than the multiplicity of the root ν (see [Mez16] for details).

Definition 3.1.2.1. We call the collection of $m_i(t)$ as above the *collection of canonical local monomials* of $\mathcal{D}$ at q.

f is then represented by the vector

$$(a_1, ..., a_k).$$

3.1.3. Differentially Generated Bases. We introduce certain bases of a fiber of the primitive cohomology bundle associated to pencils of projective hypersurface. These bases consist of differentials of sections of the primitive cohomology bundle associated to a pencil, evaluated at the fiber in question. Computing such a basis will allow us to both

1. Predict the Jordan normal form of a monodromy operator on primitive homology at a singular value of the pencil before doing any analytic continuation using the regularity of the Picard-Fuchs equations.
2. Compute the monodromy operator on primitive homology at any smooth value of the pencil by analytic continuation using section 1.1.6.

We use the same notation as at the beginning of this chapter. Let $\mathscr{H}$ be the VHS associated to the primitive cohomology bundle of $\overline{\pi}$. Fix any smooth value $t_0 \in \mathbb{P}^1$ of $\overline{\pi}$. Let s be a singular value of $\overline{\pi}$. Suppose that we have computed sections $\omega_1(t), ..., \omega_m(t)$ of the VHS $\mathscr{H}$ generating the associated vector bundle $\mathcal{H}$ at $t_0 = 1$, representing Griffiths residues of monomials of suitable degree in $\mathbb{C}[\mathbf{z}]/J(f_X)$. We use the notation

$$\omega := \omega(t_0)$$

for a section ω of the cohomology bundle.

Definition 3.1.3.1. We introduce the following notation for a section $\omega(t)$ of the primitive cohomology bundle of $\overline{\pi}$:

$$\omega^{(k)}(t) := \frac{1}{k!}\left(\frac{d}{dt}\right)^k \omega(t).$$

Now we can perform the following two steps:

1. We compute the Picard-Fuchs equations $\mathcal{D}_{\omega_1}, ..., \mathcal{D}_{\omega_k}$ of a subset $\omega_1(t), ..., \omega_k(t)$ of the above set of sections, such that

   $$\omega_1(t), \omega_1^{(1)}(t), ..., \omega_1^{(\mathrm{ord}(\mathcal{D}_{\omega_1})-1)}(t), \omega_2(t), \omega_2^{(1)}(t), ..., \omega_k^{(\mathrm{ord}(\mathcal{D}_{\omega_k})-1)}(t) \ (*)$$

   generate $\mathcal{H}$ at t_0. This can be done by repeatedly differentiating and applying GD-reduction, using lemma 1.2.2.1. Observe that, for any $\alpha \in PH_n(X_{t_0}, \mathbb{C})$,

   $$\langle \alpha, \omega_i^{(j)} \rangle$$

is equal to the j-th Taylor coefficient of the solution $\langle \alpha, \omega_i(t) \rangle$ to $\mathcal{D}_{\omega_i} f = 0$, i.e. each $\omega_i^{(j)}$ corresponds to a "Taylor coordinate" of $\mathcal{D}_{\omega_i}$.

2. Pick a subset $\mathcal{B}$ consisting of sections $\omega_{i_l}^{(j_l)}(t)$ as in $(*)$ such that $\mathcal{B}$ gives a basis of $PH^n(X_{t_0}, \mathbb{C})$ when evaluated at t_0. The point is that now each form in $\mathcal{B}$ uniquely corresponds to a "Taylor coordinate" corresponding to some differential operator.

Definition 3.1.3.2. We call a set of sections $\mathcal{B}$ as above a *differentially generated basis* of $\overline{\mathfrak{X}}$ at t_0.

Recall that all eigenvalues of a monodromy operator at a singular value are roots of unity by theorem 1.1.4.3.

First of all, we need the following two lemmas.

Lemma 3.1.3.3. *Applying the untwisting operator to a cycle $\gamma \in PH_n(X, \mathbb{C})$ corresponds to substituting $\log(t) = 0$ in the representation of γ given by the functions $\langle \gamma, \omega_{i_l}^{(j_l)}(t) \rangle$ associated to γ.*

Proof. The functions $\langle \gamma, \omega_{i_l}^{(j_l)}(t) \rangle$ have to become singlevalued after applying the untwisting operator by lemma 1.1.8.2. This means that after applying the untwisting operator, these funtions can no longer contain any logarithmic terms. On the other hand, by definition 1.1.8.1, the untwisting operator only adds multiples of powers of $\log(t)$. □

Lemma 3.1.3.4. *Let A and B be two square matrices of the same size such that $AB = BA$. If $\ker(A) \subseteq \ker(B)$ then $\ker(A^k) \subseteq \ker(B^k)$ for every $k \in \mathbb{N}$.*

Proof. This is an easy proof by induction. □

We can then put restrictions on the Jordan normal form of the monodromy operator at any singular value s as follows: Let $\mathcal{D}_1, ..., \mathcal{D}_m$ be the differential operators corresponding to the indices i_l in the notation as above. Let $\mathcal{M}$ be the collection of canonical local monomials occuring in the solution spaces of the $\mathcal{D}_i$ at s. Let $n_{\beta,k}$ be the number of canonical local monomials in $\mathcal{M}$ of the form

$$(t-s)^{\alpha/\beta} \log^{j-1}(t-s), \; j = 1, ..., k.$$

where $\alpha, \beta \in \mathbb{Z}$ are coprime. Furthermore, let $I_{\beta,k}$ be the set of indices i_l where such monomials occur in $\mathrm{sol}(\mathcal{D}_{i_l})$ at s for fixed β and k. Let

$$m_{\beta,k} := \sum_{i \in I_{\beta,k}} |\{ j \mid 0 \le j \le \mathrm{ord}(\mathcal{D}_i) - 1, \; \omega_i^{(j)}(t) \notin \mathcal{B} \}|$$

where $|\cdot|$ denotes the cardinality of a set.

We are now ready to prove the following

Theorem 3.1.3.5. *Let T be the monodromy operator on $PH_n(X_{t_0},\mathbb{C})$ at s. Denote by*

$$\mathrm{Eig}(\lambda,k) = \{\delta \in PH_n(X_{t_0},\mathbb{C}) \mid (T-\lambda\,\mathrm{Id})^k\delta = 0\}$$

the generalized eigenspace of index k for an eigenvalue λ of T. Denote by E the set of all eigenvalues of T of multiplicative order β. Then

$$n_{\beta,k} - m_{\beta,k} \leq \sum_{\lambda\in E} \dim \mathrm{Eig}(\lambda,k) \leq n_{\beta,k}.$$

Proof. Step (i): We assume first that T is unipotent, i.e. that the only eigenvalue of T is $\lambda = 1$. Hence $\beta = 1$. We claim that $\gamma \in \mathrm{Eig}(1,k)$ if and only if the only canonical local monomials in $\mathcal{M}$ appearing in the functions $\langle\gamma,\omega_{i_l}^{(j_l)}(t)\rangle$ are of the form $(t-s)^\alpha \log^{j-1}(t-s)$ for $\alpha \in \mathbb{Z}$ and $j = 1,...,k$. Let us prove the only if-direction by induction over k. Assume therefore that $\gamma \in \mathrm{Eig}(1,k)$. The statement is clear if $k = 1$ since in this case γ has to be singlevalued. Assume the statement is true for every $l \in \{1,...,k-1\}$. Let

$$N = \log(T) = \sum_{k=1}^{r-1} (-1)^{k+1} \frac{(T-\mathrm{Id})^k}{k}$$

where r is chosen minimally such that $(T-\mathrm{Id})^r = 0$. By definition 1.1.8.1 of the untwisting operator φ we then have

$$\varphi(\gamma)(t) = \gamma + \alpha_1 \log(t) N\gamma + ... + \alpha_{k-1} \log^{k-1}(t) N^{k-1}\gamma \tag{3.1}$$

where $\alpha_j = (-1)^j \frac{1}{j!(2\pi i)^j}$. Note that $N^k\gamma = 0$ since N may be written as a sum over powers of $(T-\mathrm{Id})^k$. Also note that

$$N^i\gamma \in \mathrm{Eig}(1,k-i) \quad \forall i = 0,...,k-1$$

by a similar argument. By the induction hypothesis the functions $\langle N^i\gamma,\omega_{i_l}^{(j_l)}(t)\rangle$ associated to $N^i\gamma$ can therefore only contain canonical local monomials of the form

$$(t-s)^\alpha \log^{j-1}(t-s);\ \alpha \in \mathbb{Z},\ j = 1,...,k-i-1.$$

But by lemma 3.1.3.3 φ has to cancel the logarithmic terms in the functions $\langle\gamma,\omega_{i_l}^{(j_l)}(t)\rangle$ associated to γ. A comparison of coefficients in 3.1 for the functions $\langle\varphi(\gamma)(t),\omega_{i_l}^{(j_l)}(t)\rangle$ then shows that the functions $\langle\gamma,\omega_{i_l}^{(j_l)}(t)\rangle$ can only contain canonical local monomials of the form

$$(t-s)^\alpha \log^{j-1}(t-s);\ \alpha \in \mathbb{Z},\ j = 1,...,k-1,$$

proving the only-if-direction of the claim. Now for the if-direction: Suppose that the only canonical local monomials occuring in the functions $\langle \gamma, \omega_{i_l}^{(j_l)}(t) \rangle$ are of the form $(t-s)^\alpha \log^{j-1}(t-s)$ for $\alpha \in \mathbb{Z}$ and $j = 1, ..., k$. In this case we have

$$\varphi(\gamma)(t) = \gamma + \alpha_1 \log(t) N\gamma + ... + \alpha_k \log^{r-1}(t) N^{r-1}\gamma.$$

But again by lemma 3.1.3.3 this expression has to cancel the logarithmic terms in the functions $\langle \gamma, \omega_{i_l}^{(j_l)}(t) \rangle$. A comparison of coefficients as above then forces $N^k \gamma = 0$ which shows that $\gamma \in \mathrm{Eig}(1, k)$ by lemma 3.1.3.4.

Step (ii): Now let

$$\begin{aligned} V &= \{\delta \in PH_n(X, \mathbb{C}) \mid \langle \delta, \omega_{i_l}^{(j_l)} \rangle = 0 \ \forall l \text{ such that } i_l \notin I_{1,k}\} \\ &= \mathbb{C}\langle \omega_{i_l}^{(j_l)} \mid \omega_{i_l}^{(j_l)}(t) \in \mathcal{B},\ i_l \in I_{1,k} \rangle^*. \end{aligned}$$

By construction we now have an injection

$$p_{1,k} := \bigoplus_{i \in I_{1,k}} p_{\omega_i} : V \hookrightarrow V' := \bigoplus_{i \in I_{1,k}} \mathrm{sol}(\mathcal{D}_{\omega_i})$$

where the p_{ω_i} are defined as in section 1.1.6. By construction we also have

$$\dim V = \dim V' - m_{1,k}.$$

Now let $W \subset V'$ be the supspace of functions which only contain canonical local monomials of the form $(t-s)^\alpha \log^{j-1}(t-s)$ for $\alpha \in \mathbb{Z}$ and $j = 1, ..., k$. Clearly $\dim W = n_{1,k}$. By the first step of the proof we have

$$\begin{aligned} \mathrm{Eig}(1, k) &\subset V \text{ and} \\ p_{1,k}(\mathrm{Eig}(1, k)) &\subset W. \end{aligned}$$

On the one hand it then follows that

$$\dim(\mathrm{Eig}(1, k)) = \dim(p_{1,k}(\mathrm{Eig}(1, k))) \le \dim W = n_{1,k}$$

and on the other hand we have

$$\begin{aligned} \dim(\mathrm{Eig}(1, k)) &= \dim(p_{1,k}(\mathrm{Eig}(1, k))) \\ &= \dim(p_{1,k}(V) \cap W) \\ &\ge \dim V + \dim W - \dim V' = n_{1,k} - m_{1,k} \end{aligned}$$

where the second equality holds again by step (i). This proves the result for $\beta = 1$. If $\beta \neq 1$ then we restrict to the local subsystem on a disc around s given by $V_\beta := \bigoplus_{\lambda \in E, k \in \mathbb{N}} \mathrm{Eig}(\lambda, k)$ and the action of T on V_β. We then pull back this local subsystem by $w \mapsto t = w^\beta$, reducing to the unipotent case. □

Remark 3.1.3.6. The utility of theorem 3.1.3.5 is twofold: If we are in the surface case (i.e. $X \subset \mathbb{P}^3$) then we can still predict the Jordan normal form of T if computation of the singularity-theoretic invariants as in section 3.1.1 does not let us decide whether T will be useful for computing an integral vector. Secondly, theorem 3.1.3.5 is true in any dimension.

3.1.4. Computing Monodromy on Primitive Homology. As in the previous section we fix $\omega_1(t), ..., \omega_m(t)$ and a differentially generated basis $\mathcal{B}$ obtained from the $\omega_i(t)$. We represent $\mathcal{B}$ by a matrix B whose rows are the coordinates of the forms $\omega_1, ..., \omega_m$, where again $\omega_i := \omega_i(t_0)$, in terms of the basis $\mathcal{B}$ evaluated at t_0. Let $p_{i_l} : PH_n(X, \mathbb{C}) \to \mathrm{sol}_1(\mathcal{D}_{\omega_{i_l}})$ be given by $p_{i_l}(\alpha) := \langle \alpha, \omega_{i_l}(t) \rangle$ for each index i_l appearing in $\mathcal{B}$. We represent each $p_{i_l}(\omega_k^*)$ by a column vector of Taylor coefficients as in section 3.1.2 and define the matrix

$$A_{i_l} := \left(p_{i_l}(\omega_1^*), ..., p_{i_l}(\omega_m^*)\right)$$

Let T_{i_l} be the monodromy (analytic continuation) operator at s associated to $\mathcal{D}_{\omega_{i_l}}$ given as a matrix in terms of the standard unit basis. Let

$$T'_{i_l} := T_{i_l} A_{i_l}.$$

It follows that

Lemma 3.1.4.1. *Let T' be the matrix obtained from the T'_{i_l} by taking the j_l'th rows of T_{i_l} for each index pair (i_l, j_l) appearing in $\mathcal{B}$. Then the matrix*

$$T := B^{-1} T'$$

is the matrix of the monodromy operator on $PH_n(X, \mathbb{C})$ at s in terms of the basis $\omega_1^*, ..., \omega_m^*$.

Proof. This follows from the construction of $\mathcal{B}$, the compatability between analytic continuation and monodromy (see section 1.1.6) and the fact that solutions to Picard-Fuchs equations are internally represented by their Taylor coordinates (see section 3.1.2). □

This enables us to compute the monodromy T at s on $PH_n(X, \mathbb{C})$ while computing as little analytic continutation matrices as possible.

3.2 Computing the MHS of a Singular Hypersurface

We sketch how to compute the MHS on the middle cohomology of a singular projective hypersurface if it only has finitely many singular points.

3.2.1. Computing the Hodge Filtration of the Schmid Limit MHS. Suppose that we are given a singular hypersurface $X_0 \subset \mathbb{P}^{n+1}$ cut out by a homogenous polynomial f_0 of degree d. Suppose additionally that X_0 has only finitely many singular points. We want to use theorem 2.1.5.5 to compute the MHS on $H^n(X_0, \mathbb{Z})$. To do that, we fix a pencil of hypersurfaces given by

$$tf_1 + (1-t)f_0 = 0.$$

where f_1 is homogenous of the same degree as f_0, $X_1 := \{f_1 = 0\}$ is smooth and the space $\overline{\mathfrak{X}}|_\Delta$ is smooth for a small disc Δ around $0 \in \mathbb{C}$. Suppose additionally that we are given a period matrix

$$\mathcal{P}(1) = \left(\langle \gamma_j, \omega_i(1)\rangle\right)_{1\le i,j\le m}$$

where $\omega_1(t), ..., \omega_m(t)$ generate the primitive cohomology bundle of $\overline{\mathfrak{X}}|_\Delta$. Suppose also that we are given the monodromy operator T on $H_n(X_1, \mathbb{Z})$ at 0 as a matrix T in terms of the basis $\gamma_1, ..., \gamma_m$. Let $H_\mathbb{Z} := PH_n(X_1, \mathbb{Z})$ and denote by

$$\mathfrak{H} := (H_\mathbb{Z}, F^\bullet_\infty, W_\bullet)$$

the Schmid limit MHS of the primitive cohomology bundle induced by $\overline{\mathfrak{X}}|_\Delta$. We want to compute a matrix parametrizing the Hodge filtration of the Schmid limit MHS, i.e.

$$\mathcal{P}_\infty = \left(\langle \gamma_j, \omega_{i,\infty}\rangle\right)_{1\le i,j\le m}.$$

where for all p there exists an m_p such that $\omega_{1,\infty}, ..., \omega_{m_p,\infty}$ are a basis of F^p_∞. To do that we use remark 1.1.8.5.

Let $\mathcal{D}_{\omega_i}$ be the Picard-Fuchs operators associated to the ω_i and define

$$\delta_i := \operatorname{ord}(\mathcal{D}_{\omega_i}).$$

Let $h_{i1}(t), ..., h_{i\delta_i}(t)$ be the basis of solutions for $\mathcal{D}_{\omega_i} f = 0$ given by initial values at $t_0 = 1$ equal to the standard unit vectors. Let $\mathcal{P}_i(t)$ be the i-th row of the period matrix $\mathcal{P}(t)$. Observe that

$$\mathcal{P}_i(t) = (h_{i1}(t), ..., h_{i\delta_i}(t)) \begin{pmatrix} \mathcal{P}_i(1) \\ \mathcal{P}_i'(1) \\ \vdots \\ \mathcal{P}_i^{(\delta_i-1)}(1) \end{pmatrix}.$$

Note that the derivatives $\mathcal{P}_i^{(k)}(1)$ are readily computed using Griffiths residues: Griffiths-Dwork reduction (see section 1.2.2) allows us to write each derivative $\omega_i^{(j)}(1)$ in coordinates w.r.t. the basis $\omega_1(1), ..., \omega_m(1)$. Let $g_{i1}(t), ..., g_{i\delta_i}(t)$ be the local basis of solutions for $\mathcal{D}_{\omega_i}$ at $s = 0$ whose coordinates in terms of the local canonical monomials correspond to the standard unit vectors. Marc Mezzarobba's code [Mez16] allows us to approximate a matrix A_i to arbitrary precision with rigorous error bounds such that

$$(h_{i1}(t),...,h_{i\delta_i}(t)) = (g_{i1}(t),...,g_{i\delta_i}(t))A_i$$

hence

$$\mathcal{P}_i(t) = (g_{i1}(t),...,g_{i\delta_i}(t))A_i \begin{pmatrix} \mathcal{P}_i(1) \\ \mathcal{P}_i'(1) \\ \vdots \\ \mathcal{P}_i^{(\delta_i-1)}(1) \end{pmatrix}. \qquad (*)$$

Using this identity, we can then write down the p-th coordinate of the period map (see remark 1.1.8.5) in Plücker coordinates by taking the $p \times p$-minors of the submatrix of $\mathcal{P}(t) = (\mathcal{P}_i(t))_{i=1}^m$ consisting of the first m_p rows where $\dim F^p PH^n(X, \mathbb{C}) = m_p$. Applying the untwisting operator corresponds to substituting $\log(t) = 0$ in this representation of the period map by lemma 3.1.3.3.

Let, in this representation,

$$\varphi_p(t) := [\phi_1(t) : ... : \phi_{n_p}(t)] \in \mathrm{Gr}(m_p, PH^n(X, \mathbb{C}))$$

be the p-th coordinate of the period map after applying the untwisting operator where n_p is the number of $p \times p$-minors as above. We then take

$$\lim_{t \to 0} \varphi_p(t) = \lim_{t \to 0} [\phi_1(t) : ... : \phi_{n_p}(t)]$$

as follows: Denote by $\mathrm{val}_t(\phi_i(t))$ the valuation of $\phi_i(t)$ w.r.t. t. Let $r := \min_i \mathrm{val}_t(\phi_i(t))$. Then at least one of the entries of

$$[t^{-r}\phi_1(t) : ... : t^{-r}\phi_{n_p}(t)]$$

contains a constant part so that we may take its limit by substituting $t = 0$. We have then obtained the Hodge filtration of the Schmid limit MHS as a point

$$\varphi(0) := (\varphi_p(0))_{p=0}^n \in \prod_{p=0}^n \mathrm{Gr}(m_p, PH^n(X, \mathbb{C}))$$

in a product of Grassmanians.

We now discuss how to compute the matrix $\mathcal{P}_\infty$. By the definition of Plücker coordinates, each $\varphi_p(0)$ corresponds to the projective line spanned by a multivector

$$v^{(p)} := [v_1^{(p)} : ... : v_{m_p}^{(p)}]$$

which we have expressed in coordinates w.r.t. the basis

$$e_{i_1} \wedge ... \wedge e_{i_{m_p}},\ 1 \le i_1 < ... < i_{m_p} \le m$$

where e_i denotes the i-th unit vector. This multivector corresponds to the subspace

$$F^p_\infty = \{w \in PH^n(X, \mathbb{C}) \mid w \wedge v^{(p)} = 0\}$$

where the condition

$$w \wedge v^{(p)} = 0$$

may be expressed as a linear system of equations by a comparison of coefficients. Note that because

$$F^{p+1}_\infty \subset F^p_\infty \quad \forall p$$

we have

$$w \wedge v^{(p+1)} = 0 \;\Rightarrow\; w \wedge v^{(p)} = 0.$$

This means that we can build the matrix $\mathcal{P}_\infty$ by computing a basis of solutions $w_1, ..., w_{m_n}$ for

$$w \wedge v^{(n)} = 0$$

then adding $\dim F^{n+1}_\infty - \dim F^n_\infty$ linearly independent solutions to

$$w \wedge v^{(n-1)} = 0$$

such that $w_1, ..., w_{m_{n-1}}$ are linearly independent and so on until we have reached F^0_∞. Then we may set

$$\mathcal{P}_\infty = (w_i)^m_{i=1},$$

i.e. the w_i form the rows of $\mathcal{P}_\infty$.

Remark 3.2.1.1. Of course the entries of the linear systems of equations arising by going from Plücker coordinates back to a period matrix are inexact. We do however know their exact rank. The linear independence of solutions may for example be checked by a singular value decomposition with exact error bounds, i.e. by using a SVD to compute ranks of matrices with error terms.

3.2.2. Computing the Weight Filtration. Note that T has integral entries by definition. We thus easily compute the minimal $l \in \mathbb{N}$ such that T^l is unipotent and the minimal index $r \in \mathbb{N}$ such that

$$(T^l - \mathrm{Id})^r = 0.$$

Then we compute

$$N := \log(T) = \sum_{k=1}^{r-1} (-1)^{k+1} \frac{T^l - \mathrm{Id}}{k}.$$

Note that N has rational entries. Using the definition 1.1.8.3 of the spaces $W(N,k)$ and Gaussian elimination we then compute a series of matrices

$$W_k \in \mathbb{Q}^{m \times l_k};\ k = 0, ..., 2n-1$$

where $l_k = \dim W(N,k)$, whose columns are a basis of the space $W(N,k)^*$ in terms of the basis $\gamma_1, ..., \gamma_m$. Then

$$\mathcal{P}_\infty W_k$$

parametrizes the weight k part of the Schmid limit MHS.

3.2.3. Computing the MHS on the Central Fiber. By theorem 2.1.5.5 we now need to only pull down the Schmid limit MHS to $\ker(T - \mathrm{Id})$. Since T has integral entries we compute a matrix M whose columns are a basis of $\ker(T - \mathrm{Id})$ in terms of the $\gamma_1, ..., \gamma_m$. Next we compute matrices W'_k whose columns are a basis of the spaces $W(N,k)^* \cap \ker(T - \mathrm{Id})$ in terms of the basis of $\ker(T - \mathrm{Id})$ given by the columns of M. Note that this only needs to be done for $k = 0, ..., n-1$ since X_0 is projective (see remark 1.1.3.2). Let $l := \dim(\ker(T - \mathrm{Id}))$. We then obtain the MHS on $H^n(X_0, \mathbb{Z})$ as follows:

1. Compute $\mathcal{P}_\infty M$. We have $\mathrm{rk}(\mathcal{P}_\infty M) = l$. We then remove rows from this matrix until the resulting matrix is a square matrix, still of rank l. Even though $\mathcal{P}_\infty M$ has entries with error terms the rank can be checked in each step by using a singular value decomposition as in remark 3.2.1.1. By construction, the resulting matrix $\mathcal{P}(X_0)$ is a period matrix parametrizing the Hodge filtration of the MHS on $H^n(X_0, \mathbb{Z})$.

2. The weight k part of this MHS is given by

$$\mathcal{P}(X_0) W'_k.$$

3.3 Examples

To demonstrate our method we now give, for a few smooth quartic hypersurfaces in $\mathbb{P}^3$, a pencil that can be used to compute an integral vector in the sense of chapter 3. Afterwards, we compute more integral vectors by computing the remaining monodromy matrices of our pencil and applying them to our integral vector. We then send randomly generated pencils through the hypersurface of interest, carry over our basis of primitive cohomology, compute all monodromy matrices and apply them to the integral vectors we have so far obtained until we have computed 21 linearly independent integral vectors. We note that computation of the Picard-Fuchs equations and monodromy matrices is usually faster if the polynomials defining a pencil only differ by one or two monomials. We hence made sure that this occurs while randomly generating the pencils after having exhausted all the monodromy matrices coming from our first pencil.

For each example we record the time it took to compute a differentially generated basis and its associated Picard-Fuchs equations as well the time to compute the respective monodromy matrix that can be used to compute an integral vector in the first pencil. We also record the time it took to get from one integral vector to a basis of integral vectors, using other monodromy matrices.

After we have obtained a full period matrix, we check our result by computing the Picard rank of the hypersurface in question using the obtained periods of a holomorphic form as in [LS19]. Recall that the period matrix we obtain is a period matrix for our originial basis of primitive cohomology scaled by some unknown $\lambda \in \mathbb{C}$ (i.e. we get periods of the form $\lambda\omega$ where ω is our original holomorphic form and λ is unknown).

In each case we compute sections of the primitive cohomology bundle of the respective pencil by choosing monomials $m_1, ..., m_m$ in $\mathbb{Q}[x, y, z, w]$ of suitable degree such that their residues give a basis of the primitive cohomology at $t = 1$. By the compatibility between pole order and Hodge filtration (see lemma 1.2.1.3) they then automatically give a basis of the primitive cohomology at $t = 1$ as in definition 1.1.2.4. In the first and third example, we give the structure of a resulting differentially generated basis to illustrate the concept.

All computations have been performed using the computer algebra system SageMath [Dev20] on a Lenovo ThinkPad T450s with code written by the author. To find analytic continuation paths for monodromy computations we used code adapted from

`github.com/emresertoz/PeriodSuite/blob/master/voronoi_path.sage`.

3.3.1. The Fermat Hypersurface. Let $f_1 = x^4 + y^4 + z^4 + w^4$. To demonstrate the prediction of the Jordan normal form of a monodromy matrix via theorem 3.1.3.5 we put $f_0 := x^4 + x^2y^2 + y^4 + xyzw$. f_0 defines a projective hypersurface with a non-isolated singular locus so that we cannot apply any of the results in chapter 2. We let $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ be the pencil of projective hypersurfaces given by the equation

$$(1-t)f_0 + tf_1 = 0;\ t \in \mathbb{C}.$$

Let $\omega_1(t), ..., \omega_{21}(t)$ be the collection of sections of the primitive cohomology bundle computed as described above. We then compute a differentially generated basis: In this case one is of the format

$$\begin{aligned}
&\omega_1(1), ..., \omega_1^{(4)}(1), \omega_2(1), \omega_2^{(1)}(1), \omega_3(1),\\
&\omega_3^{(1)}(1), \omega_4(1), \omega_4^{(1)}(1), \omega_5(1), ..., \omega_5^{(3)}(1),\\
&\omega_6(1), \omega_6^{(1)}(1), \omega_8(1), \omega_8^{(1)}(1), \omega_9(1), \omega_9^{(1)}(1).
\end{aligned}$$

The computation of this basis and the computation of the respective Picard-Fuchs equations took 10.58 seconds. In the language of theorem 3.1.3.5 we find the following corresponding collection of canonical local monomials at $t_0 = 0$ where in this case we computed $m_{\beta,k} = 0$ for all β:

$$\frac{1}{2}\log^2(t), \log(t), 1, t, t^2, 1, t, \frac{1}{t^{\frac{1}{4}}}, t^{\frac{9}{4}}, \frac{1}{t^{\frac{1}{4}}}, t^{\frac{1}{4}}, \frac{1}{\sqrt{t}}, \sqrt{t}, t^{\frac{3}{2}}, t^{\frac{5}{2}}, \frac{1}{t^{\frac{3}{4}}}, t^{\frac{3}{4}}, \frac{1}{t^{\frac{1}{4}}}, t^{\frac{1}{4}}, 1, t.$$

Hence, by theorem 3.1.3.5, the monodromy operator T on primitive homology at 0 has exactly one Jordan block of size 3. The only eigenvalues of T are 2nd and 4th roots of unity also by theorem 3.1.3.5. We hence have

$$\mathrm{rk}(T^4 - \mathrm{Id})^2 = 1$$

i.e. T can be used to compute an integral vector of the Fermat hypersurface $X := \{f_1 = 0\}$. Computing T as in section 3.1.4 took 28.84 seconds.

To compute a full period matrix of the Fermat hypersurface we needed all the monodromy matrices associated to $\overline{\pi}$ as well as 7 additional randomly generated pencils and their respective monodromy matrices. This took the total computation time for a full period matrix to roughly 37 minutes.

Each monodromy matrix in this example was computed with a target error bound of 10^{-100}.

We find the following periods for a holomorphic form on the Fermat hypersurface:

$$\begin{aligned}
&(-128.0, -64.0 + 64.0i, -64.0 + 64.0i, 704.0 + 320.0i,\\
&-64.0 - 64.0i, -192.0 + 64.0i, 128.0i, -160.0 + 32.0i,\\
&-96.0 + 96.0i, -32.0 + 96.0i, -224.0 + 96.0i, -160.0 + 32.0i,\\
&-64.0 + 128.0i, -256.0 + 128.0i, -256.0 - 128.0i,\\
&-192.0 + 192.0i, -352.0 - 32.0i, -64.0 + 192.0i,\\
&128.0 + 128.0i, -160.0, -160.0).
\end{aligned}$$

In this case, ignoring the error bounds of the period vector that was put out, all of these periods are of the form $a + bi$ with $a, b \in \mathbb{Z}$. With these periods, we find that the Picard rank of the Fermat hypersurface is 20, confirming the known result.

3.3.2. A 5-nomial. To demonstrate the singularity-theoretic approach developed in Chapter 2 we put $f_1 = 2w^4 + x^4 + y^4 + y^3z + 4z^4$ and $f_0 = f_1 - 2x^3z$. Let again $\overline{\pi} : \overline{\mathfrak{X}} \to \mathbb{P}^1$ be the pencil given by the equation

$$(1-t)f_0 + tf_1 = 0;\ t \in \mathbb{C}.$$

Using the method described in section 3.1.1 we find that this pencil has 8 singular fibers, all of which have exactly one isolated singular point. 4 of these have Milnor number 6 and the 4 others have Milnor number 3. By lemma 2.2.2.7 each of the fibers with Milnor number 3 carries the singularity A_3. Hence any of the Monodromy operators at these 4 fibers can be used to compute a period vector by lemma 2.2.2.6: Each monodromy operator at the values where the A_3's occur has eigenalue -1 with multiplicity 1.

In this case the computation of a differentially generated basis at $t_0 = 1$ took 20.48 seconds and the computation of one of the monodromy operators with eigenvalue -1 with multiplicity 1 took 37.86 seconds. We find that sometimes a large number of additional randomly generated pencils is required to get to a full basis of integral vectors after having obtained one integral vector: In this case we needed 12 additional randomly generated pencils, taking the total computation time for a full period matrix of $X := \{f_1 = 0\}$ to roughly 70 minutes.

We find the following (rounded) periods for a holomorphic form on X:

$$\begin{aligned}
&(54606.754, 603.32348 + 54003.431i, 603.32348 + 603.32348i,\\
&54606.754, 54003.431 + 603.32348i, 54606.754 - 54606.754i,\\
&54606.754 + 54606.754i, 27605.039 - 301.66174i, 27605.039 + 27001.715i,\\
&603.32348 + 27303.377i, 27605.039 - 301.66174i, 301.66174 + 301.66174i,\\
&27303.377 - 27906.701i, 603.32348, -26700.054 - 26700.054i, -26700.054,\\
&-26700.054 - 27303.377i, -26700.054 - 26700.054i, -26700.054,\\
&-26398.392 - 301.66174i, 603.32348).
\end{aligned}$$

Computing the Picard rank of X with these periods gave a Picard rank of 18. This confirms the Picard rank of X given in the quartic database `pierre.lairez.fr/quarticdb/`.

3.3.3. A Difficult 5-nomial. Let $f_1 := xy^3 + z^4 + x^3w + y^2zw + xw^3$. This is one of the examples given in [HKS20] for which the authors were unable to compute a period matrix in any reasonable time. Here, computations are significantly more expensive than in the last two examples: We could not find a single pencil through f_1 where the Picard-Fuchs equations have order less than 20. One of the more manageable pencils we could find was given by

$$tf_1 + (1-t)f_0 = 0;\ t \in \mathbb{C},$$

where $f_0 := f_1 - xw^3$. This pencil has 12 singular fibers, 11 of which have exactly one singular point which is a simple node. The remaining singular fiber has exacly one singular point of Milnor number 10. Any of the monodromy operators at the values where the simple nodes occur can be used to compute an integral vector (see section 2.2.1). In this case one differentially generated basis is of the format

$$\omega_1(1), ..., \omega_1^{(19)}(1), \omega_2(1)$$

with notation as above. We hence had to compute two Picard-Fuchs equations, namely the PF-equation of $\omega_1(t)$ and the one of $\omega_2(t)$. This took roughly 14 minutes. The computation of one of the monodromy matrices T using this basis as in section 3.1.4 took roughly 11 minutes. An integral vector is hence given by

$$w := (T - \mathrm{Id})v$$

for any v not in the kernel of $(T - \mathrm{Id})$. We chose such a w, computed the remaining monodromy matrices and applied them to w. This took roughly 134 minutes. We note that in this case we had to increase the target error bound during analytic continuation to 10^{-200}: Putting in 10^{-100} caused the computed monodromy matrices to have error bounds of about 10^{300}, making them unusable.

In this case we only needed one additional randomly generated pencil to obtain a full period matrix. Computation of the needed Picard-Fuchs equations for this pencil took roughly 10 minutes and computation of all the monodromy matrices took roughly 78 minutes.

The thus obtained periods of a holomorphic form on $X := \{f_1 = 0\}$ are

$$\begin{aligned}
&(0.0000000, -0.03022660 + 0.003431382i, -0.02743533 - 0.01161599i,\\
&-0.06763545 - 0.01486758i, -0.06729886 + 0.0004327823i,\\
&-0.08611067 - 0.01785912i, -0.08971974 - 0.03133698i,\\
&-0.09879659 - 0.01734328i, -0.07728164 - 0.02878898i,\\
&-0.07156117 - 0.04509659i, -0.05856580 - 0.02877231i,\\
&-0.04660392 - 0.02475775i, -0.01338820 - 0.03067981i,\\
&-0.01663474 + 0.009457657i, -0.06486841 - 0.03921600i,\\
&-0.06881764 + 0.009609022i, -0.07589850 + 0.006222306i,\\
&-0.1036695 - 0.009209903i, -0.005560690 + 0.006579079i,\\
&-0.002920762 - 0.02605885i, -0.03835328 - 0.006622818i).
\end{aligned}$$

Using these periods, we find that X has Picard rank 2. This confirms the likely Picard rank of X given in [HKS20].

Bibliography

[Arn98] V.I. Arnol'd. *Singularity Theory 1*. 1. ed., 2. printing. Berlin: Springer, 1998.

[Bod04] Arnaud Bodin. "Computation of Milnor Numbers and Critical Values at Infinity". In: *Journal of Symbolic Computation* 38.5 (2004), pp. 1417–1427.

[Cha14] F. Charles. "On the Picard number of K3 surfaces over number fields". In: *Algebra Number Theory* 8 (2014), pp. 1–17.

[de 15] Mark Andrea A. de Cataldo. "Perverse sheaves and the topology of algebraic varieties (2015 PCMI)". In: *arXiv e-prints*, arXiv:1506.03642 (June 2015).

[Del75] Pierre Deligne. "Théorie de Hodge". In: *Publ. de l'Inst. des Hautes Etudes Sci* 44 (1975), pp. 79–85.

[Dev20] The Sage Developers. *SageMath, the Sage Mathematics Software System, version 9.1*. `http://www.sagemath.org`. 2020.

[Dim92] Alexandru Dimca. *Singularities and Topology of Hypersurfaces*. Universitext. New York: Springer, 1992.

[Don83] Ron Donagi. "Generic torelli for projective hypersurfaces". In: *Compositio Mathematica* 50.2-3 (1983), pp. 325–353.

[Dur79] Alan Durfee. "Fifteen Characterizations of Rational Double Points and Simple Critical Points". In: *L'Enseignement Matheématique* 25 (1979).

[Dwo62] Bernhard Dwork. "On the zeta function of a hypersurface". In: *Inst. Hautes Études Sci. Publ. Math.* 12 (1962), pp. 5–62.

[EJ11] A.-S. Eisenhans and J. Jahnel. "On the computation of the Picard group for K3 surfaces". In: *Math. Proc. Camb. Philos. Soc.* 151 (2011), pp. 263–270.

[EJ18] Andreas-Stephan Elsenhans and Jörg Jahnel. "Real and complex multiplication on K3 surfaces via period integration". In: *arXiv e-prints*, arXiv:1802.10210 (Feb. 2018).

[ES06] Christian van Enckevort and Duco van Straten. "Monodromy calculations of fourth order equations of Calabi-Yau type". In: *AMS/IP Studies in Advanced Mathematics* (2006), pp. 539–559.

[Gri68] Phillip A. Griffiths. "Periods of Integrals on Algebraic Manifolds, I. (Construction and Properties of the Modular Varieties)". In: *American Journal of Mathematics* 90.2 (1968), p. 568.

[Gri69] Phillip A. Griffiths. "On the Periods of Certain Rational Integrals. I, II." In: *Ann. of Math* 2.90 (1969), 460-495 and 496–541.

[Gri70] Phillip A. Griffiths. "Periods of Integrals on Algebraic Manifolds: Summary of Main Results and Discussion of Open Problems". In: *Bull. Amer. Math. Soc.* 76.2 (1970), pp. 228–296.

[HKS20] Kathryn Heal, Avinash Kulkarni and Emre Can Sertöz. "Deep Learning Gauss-Manin Connections". In: *arXiv e-prints*, arXiv:2007.13786 (July 2020).

[Huy05] Daniel Huybrechts. *Complex Geometry: An Introduction*. Universitext. Berlin: Springer, 2005.

[Kul98] Valentine Kulikov. *Mixed Hodge Structures and Singularities*. Cambridge University Press, 1998.

[Lai16] Pierre Lairez. "Computing Periods of Rational Integrals". In: *Math. Comput.* 85 (2016), pp. 1719–1752.

[LS19] Pierre Lairez and Emre Can Sertöz. "A Numerical Transcendental Method in Algebraic Geometry: Computation of Picard Groups and Related Invariants". In: *SIAM Journal on Applied Algebra and Geometry* 3.4 (2019), pp. 559–584.

[Lef50] Solomon Lefschetz. *L'analysis situs et la géométrie algébrique*. Paris: Gauthier-Villars, 1950.

[Mez16] Marc Mezzarobba. "Rigorous Multiple-Precision Evaluation of D-Finite Functions in SageMath". In: *arXiv e-prints*, arXiv:1607.01967 (July 2016).

[Mil68] John Milnor. "Singular Points of Hypersurfaces". In: *Ann. of Math. Studies* 61 (1968).

[Mov20] Hossein Movasati. "A Course in Hodge Theory. With Emphasis on Multiple Integrals". `w3.impa.br/~hossein/myarticles/hodgetheory.pdf`. July 2020.

[PS08] Chris A.M. Peters and Joseph H.M. Steenbrink. *Mixed Hodge Structures*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 2008.

[Rah16] Mohammad Reza Rahmati. *Hodge Theory of Isolated Hypersurface Singularities*. LAP LAMPERT Academic Publishing, 2016.

[Sch73] Wilfried Schmid. "Variation of Hodge Structure: The Singularities of the Period Mapping". In: *Inventiones math.* 22 (1973), pp. 211–319.

[Ser19] Emre Can Sertöz. "Computing Periods of Hypersurfaces". In: *Math. Comput.* 88 (2019), pp. 2987–3022.

[Ste76] Joseph H. M. Steenbrink. "Mixed Hodge Structure on the Vanishing Cohomology". In: *Symposium in Mathematics*. Nordic Summer School/NAVF. Oslo, Aug. 1976.

[TT84] C. L. Tretkoff and M. D. Tretkoff. "Combinatorial group theory, Riemann surfaces and differential equations". In: *Contributions to group theory* 33 (1984).

[Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2002.

[Voi20] Claire Voisin. "Schiffer variations and the generic Torelli theorem for hypersurfaces". In: *arXiv e-prints*, arXiv:2004.09310 (Apr. 2020).

Ich versichere, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe, insbesondere sind wörtliche oder sinngemäße Zitate als solche gekennzeichnet. Mir ist bekannt, dass Zuwiderhandlung auch nachträglich zur Aberkennung des Abschlusses führen kann. Ich versichere, dass das elektronische Exemplar mit den gedruckten Exemplaren übereinstimmt.

Ort: Datum: Unterschrift: