Multiple binomial sums

MPRI – Efficient algorithms in computer algebra

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A binomial identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

How would you prove it?

Proof. Let $[n] = \{1, \ldots, n\}$. There is a bijection between

$$\{S \subseteq [2n] \mid \#S = n\} \text{ and } \{A, B \subseteq [n] \mid \#A + \#B = n\}$$

$$S \mapsto \big(\underbrace{S \cap [n]}_{\mathrm{lo}(S)}, \underbrace{(S - n) \cap [n]}_{\mathrm{hi}(S)} \big)$$

$$A \cup (B + n) \hookleftarrow (A, B)$$

A slightly trickier one

$$\sum_{k=0} 2^{n-2k} \binom{n}{2k} \binom{2k}{k} = \binom{2n}{n}.$$

How would you prove it?

Proof. There is a bijection between

$$\{S \subseteq [2n] \mid \#s = n\} \text{ and } \{A, B, C \subseteq [n] \text{ disjoint} \mid \#A + \#B + 2\#C = n\}$$

$$s \mapsto (\mathrm{lo}(s) \setminus \mathrm{hi}(S), \mathrm{hi}(S) \setminus \mathrm{lo}(S), \mathrm{lo}(S) \cap \mathrm{hi}(S))$$

$$A \cup (B + n) \cup C \cup (C + n) \leftarrow\!\shortmid (A, B, C)$$

A computer-aided proof (exercise)

Let $u(n,k) = 2^{n-2k}\binom{n}{2k}\binom{2k}{k}$. To find a recurrence relation of order one on $\sum_k u(n,k)$, we look for $p \in \mathbb{Q}(n)$ and $R \in \mathbb{Q}(n,k)$ such that

$$u(n+1,k) + p(n)u(n,k) = R(k+1)u(n,k+1) - R(k)u(n,k).$$

Divide by $u(n,k)$ and we obtain

$$\frac{(n-2k)(n-2k-1)}{4(k+1)^2}R(k+1) - R(k) = \frac{2(n+1)}{n+1-2k} + p(n).$$

If there is such an R, its denominator is constrained! (See Abramov's algorithm.) Indeed, let A be the poles of R (in an algebraic closure of $\mathbb{Q}(n)$). We have

$$A \subseteq (A-1) \cup \left\{-1, \tfrac{n+1}{2}\right\} \text{ and } A-1 \subseteq A \cup \left\{\tfrac{n-1}{2}, \tfrac{n}{2}, \tfrac{n+1}{2}\right\}.$$

It follows that $A \subseteq \left\{\frac{n+1}{2}\right\}$. Finer analysis of the exponents shows that $R(k) = S(k)/(n+1-2k)$, for some $S \in \mathbb{Q}(n)[k]$.
Let us switch to Maple.

More identities (Brent, Ohtsuka, Osborn, Prodinger 2014)

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3 - j^3| = \frac{2n^2(5n-2)}{4n-1}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^5 - j^5| = \frac{2n^2(43n^3-70n^2+36n-6)}{(4n-1)(4n-3)}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^7 - j^7| = \frac{2n^2(531n^5-1960n^4+2800n^3-1952n^2+668n-90)}{(4n-1)(4n-3)(4n-5)}\binom{4n}{2n}$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|ij(i^2 - j^2)| = \frac{2n^3(n-1)}{2n-1}\binom{2n}{n}^2$$

$$\sum_{i,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^3j^3(i^2 - j^2)| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-3)(2n-1)}\binom{2n}{n}^2$$

A complicated one (Le Borgne)

$$
\begin{aligned}
1 + F_n^{-1,-1} + 2F_n^{0,0} - F_n^{0,1} + F_n^{1,0} - 3F_n^{1,1} + F_n^{1,2} - F_n^{3,1} + 3F_n^{3,2} \\
- F_n^{3,3} - 2F_n^{4,2} + F_n^{4,3} - F_n^{5,2} = \sum_{m=0}^{n} \frac{\binom{n+2}{m}\binom{n+2}{m+1}\binom{n+2}{m+2}}{\binom{n+2}{1}\binom{n+2}{2}},
\end{aligned}
$$

$$
\text{where } F_n^{a,b} = \sum_{d=0}^{n-1}\sum_{c=0}^{d-a}\binom{d-a-c}{c}\binom{n}{d-a-c}\left(\binom{n+d+1-2a-2c+2b}{n-a-c+b} - \binom{n+d+1-2a-2c+2b}{n+1-a-c+b}\right).
$$

Automation is nice to have...

Motivation from computer science

[50] Develop computer programs for simplifying sums that involve binomial coefficients.

Exercise 1.2.6.63
The Art of Computer Programming
Knuth (1968)

Motivation from number theory

Let $\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}$.
Can you prove that $\zeta(3) \notin \mathbb{Q}$? (Apéry 1979)

For $n \geq 0$, let

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \text{ and } l_n = \operatorname{lcm}(1, 2, \ldots, n)^3.$$

There is some integer sequence (b_n) such that $b_n - 2l_n a_n \zeta(3) \longrightarrow 0$.

It implies that $\zeta(3) \notin \mathbb{Q}$.

Desired algorithms for binomial sums

Deciding equality

input $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$

output *true*

Simplification

input $\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i}$

output $(2n+1)\binom{2n}{n}^2$

Computation of a recurrence relation

input $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$

output $n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5) u_{n-1}$

The algebra of binomial sums

The formal grammar of binomial sums

$\bigcirc \longrightarrow$ integer linear combination of variables

$$\square \longrightarrow \binom{\bigcirc}{\bigcirc} \qquad\qquad \square \longrightarrow \mathrm{Cst}^{\bigcirc}$$

$$\square \longrightarrow \square + \square \qquad\qquad \square \longrightarrow \square \cdot \square$$

$$\square \longrightarrow \sum_{n=\bigcirc}^{\bigcirc} \square$$

The algebra of binomial sums

Let $\mathbb{S}$ be the algebra of functions $\mathbb{Z}^{(\mathbb{N})} \to \mathbb{C}$.
The algebra of binomial sums, denoted $\mathcal{B}$, is the smallest subalgebra of $\mathcal{S}$ such that

(a) The Kronecker delta sequence $n, \dots \mapsto \delta_n$,
defined by $\delta_0 = 1$ and $\delta_n = 0$ if $n \neq 0$, is in $\mathcal{B}$.

(b) The geometric sequences $n, \dots \mapsto C^n$, for all $C \in \mathbb{C} \setminus \{0\}$, are in $\mathcal{B}$.

(c) The binomial sequence $n, k, \dots \mapsto \binom{n}{k}$ is in $\mathcal{B}$.

(d) If $\lambda : \mathbb{Z}^d \to \mathbb{Z}^e$ is an affine map and if $u \in \mathcal{B}$,
then $n_1, n_2, \dots \mapsto u_{\lambda(n_1,\dots,n_d),0,\dots}$ is in $\mathcal{B}$.

(e) If $u \in \mathcal{B}$, then the following directed indefinite sum is in $\mathcal{B}$:

$$n_1, \dots, n_d, m, \dots \mapsto \sum_{k=0}^{m} u_{n_1,\dots,n_d,k}.$$

Main result

Theorem

Let u be a binomial sum. Then $(u_n)_{n\in\mathbb{Z}}$ is P-recursive.
In other words, there are polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)u_n + p_1(n)u_{n+1} + \cdots + p_r(n)u_{n+r} = 0.$$

Moreover, this result is *effective*: there is an algorithm to compute a recurrence relation as above.

Corollary

Equality of binomial sums is decidable.

A. Bostan, P. Lairez, B. Salvy (May 1, 2017). "Multiple Binomial Sums". In: *J. Symb. Comput.* 80, pp. 351–386. DOI: 10/ggck6p

Laurent series

For a field K, let $K((x)) \doteq \cup_{N\geq 0} x^{-N} K[[x]]$, the field of *Laurent series over* K.
It is a field. Why?

We will work in the field of *iterated Laurent series*

$$\mathbb{C}((x_1, \dots, x_n)) \doteq \mathbb{C}((x_n))((x_{n-1})) \cdots ((x_1)).$$

It means: expand first with respect to x_1, then x_2, etc.

For a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $R \in \mathbb{C}((x_1, \dots, x_n))$ we denote $[\mathbf{x}^\alpha] R$ the coefficient of $x_n^{\alpha_n}$ in the coefficient of $x_{n-1}^{\alpha_{n-1}}$ of [...] the coefficient of $x_1^{\alpha_1}$ in R.

$\mathbb{C}(x_1, \dots, x_n) \subset \mathbb{C}((x_1, \dots, x_n))$, so we now know what is the coefficient of a monomial in a rational function!

Exercise

What is the coefficient of 1 in $\frac{x_1}{x_1+x_2}$?

What is the coefficient of 1 in $\frac{x_2}{x_1+x_2}$?

An intermediary representation

Lemma

Every binomial sum is a linear combination of sequences of the form

$$n_1, n_2, \ldots \mapsto [1] R_0 R_1^{n_1} \cdots R_d^{n_d},$$

for some $R_0, \ldots, R_d \in \mathbb{C}(x_1, \ldots, x_r)$.

Proof?

Residues and diagonals

For $R \in \mathbb{C}((t, x_1, \ldots, x_n))$, let

$$\operatorname{res}_{x_1,\ldots,x_n} R \doteq \sum_{k \in \mathbb{Z}} \left([x_1^{-1} \cdots x_n^{-1} t^k] R \right) t^k.$$

For $R \in \mathbb{C}[\![x_0, \ldots, x_n]\!]$, let

$$\operatorname{diag} R \doteq \sum_{k \in \mathbb{Z}} \left([x_0^k \cdots x_n^k] R \right) t^k.$$

Lemma

$$\operatorname{diag} R = \operatorname{res}_{x_1,\ldots,x_n} \frac{1}{x_1 \cdots x_n} R\left(\frac{t}{x_1 \cdots x_n}, x_1, \ldots, x_n \right).$$

Coefficient representation of binomial sums

Theorem (Bostan, Lairez, Salvy 2017)

Let $(u_n)_{n\geq 0}$ be a sequence and let $f(t) = \sum_n u_n t^n$ its generating function. The following are equivalent:

1. *(u_n) is a binomial sum;*
2. *$f(t) = \operatorname{res}_{x_1,\ldots,x_n} R$, for some $R \in \mathbb{C}(t, x_1, \ldots, x_n)$;*
3. *$f(t) = \operatorname{diag} R$, for some $R \in \mathbb{C}(x_0, \ldots, x_n) \cap \mathbb{C}[[x_0, \ldots, x_n]]$.*

A class of formal power series with interesting closure properties

Let $\mathcal{D} \subseteq \mathbb{C}[[t]]$ be the set of all $\operatorname{diag} R$ for some *rational* power series $R(x_0, \ldots, x_n)$.
Then

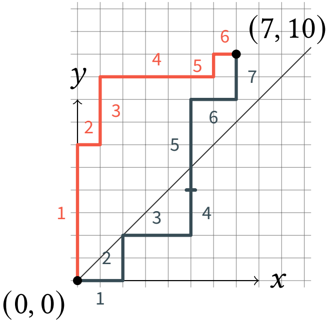
- $\mathcal{D}$ is a subalgebra
  Proof. $\operatorname{diag}(R(x_0, \ldots, x_n)) \operatorname{diag}(S(y_0, \ldots, y_m)) =$
  $\operatorname{diag}(R(x_0 y_1 \cdots y_m, x_1, \ldots, x_n) S(x_0 \cdots x_n, y_1, \ldots, y_m))$
- $\mathcal{D}$ is closed under Hadamard product
  Proof.
  $\operatorname{diag}(R(x_0, \ldots, x_n)) \odot \operatorname{diag}(S(y_0, \ldots, y_m)) = \operatorname{diag}(R(x_0, \ldots, x_n) S(y_0, \ldots, y_m))$
- $\mathcal{D}$ is closed under differentiation
  Proof. $(\operatorname{diag} R(x_0, \ldots, x_n))' = \operatorname{diag}\left(\frac{1}{x_1 \cdots x_n} \frac{\partial R}{\partial x_0}\right)$
- $\mathcal{D}$ contains only D-finite element (Christol 1985; Lipshitz 1988)
- $\mathcal{D}$ all algebraic power series (Furstenberg 1967)

Univariate residues are algebraic

Theorem

Let $R \in \mathbb{C}(t, x)$. The Laurent series $\operatorname{res}_x R$ is algebraic: there is a non zero polynomial $P(t, y)$ such that $P(t, \operatorname{res}_x R) = 0$.

Example from combinatorics: rook paths



$a_{i,j} \doteq$ number of paths from $(0,0)$ to (i,j)

Easy recurrence relation:

$$a_{i,j} = \sum_{k<i} a_{k,j} + \sum_{k<j} a_{i,k}$$

What about $a_{n,n}$?

Recurrence relations (also in higher dimension)

- dimension 2

$$9nu_n - (14 + 10n)u_{n+1} + (2 + n)u_{n+2} = 0$$

- dimension 3

$$\begin{aligned} &-192n^2(1 + n)(88 + 35n)u_n \\ &+(1 + n)(54864 + 100586n + 59889n^2 + 11305n^3)u_{n+1} \\ &-(2 + n)(43362 + 63493n + 30114n^2 + 4655n^3)u_{n+2} \\ &+2(2 + n)(3 + n)^2(53 + 35n)u_{n+3} = 0 \end{aligned}$$

- dimension 4

$$\begin{aligned} &5000n^3(1 + n)^2(2705080 + 3705334n + 1884813n^2 + 421590n^3 + 34983n^4)u_n \\ &-(1 + n)^2(80002536960 + 282970075928n + \cdots + 6386508141n^6 + 393838614n^7)u_{n+1} \\ &+2(2 + n)(143370725280 + 500351938492n + \cdots + 2636030943n^7 + 131501097n^8)u_{n+2} \\ &-(3 + n)^2(26836974336 + 80191745800n + 100381179794n^2 + \cdots + 44148546n^7)u_{n+3} \\ &+2(3 + n)^2(4 + n)^3(497952 + 1060546n + 829941n^2 + 281658n^3 + 34983n^4)u_{n+4} = 0 \end{aligned}$$

Dimension 2

$$a_{i,j} = \sum_{k<i} a_{k,j} + \sum_{k<j} a_{i,k} \Rightarrow \sum_{i,j\geq 0} a_{i,j} x^i y^j = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}$$

$$\begin{aligned}
\sum_{n\geq 0} a_{n,n} t^n &= \operatorname{diag}\left(\frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}} \right) \\
&= \operatorname{res}_x \frac{(1-x)(t-x)}{x(3tx - 2x^2 - 2t + x)} \\
&= \sum_{n\geq 0} \left(\sum_{p=0}^{n} \sum_{q=0}^{n} \binom{p+q}{p} \binom{n-1}{p-1} \binom{n-1}{q-1} \right) t^n \\
&\stackrel{?}{=} \frac{1}{2} - \frac{1-t}{2\sqrt{1 - 10t + 9t^2}}
\end{aligned}$$

Hermite's reduction

Lemma

For any $R \in \mathbb{C}(t,x)$, $\operatorname{res}_x \frac{\partial R}{\partial x} = 0$.

Hermite's reduction

For any $R = a/b^N \in \mathbb{C}(t,x)$, there is a unique polynomial $c \in \mathbb{C}(t)[x]$ such that $\deg_x c < \deg_x b$ and

$$R = \frac{c}{b} + \frac{\partial A}{\partial x}$$

for some $A \in \mathbb{C}(t,x)$.

Computing Hermite's reduction

input $a/b^N \in \mathbb{C}(t,x)$ with $a, b \in \mathbb{C}(t)[x]$ and b squarefree

output $c \in \mathbb{C}(t)[x]$ such that $\dfrac{a}{b^N} = \dfrac{c}{b} + \dfrac{\partial A}{\partial x}$ for some $A \in \mathbb{C}(t,x)$

procedure HermiteRed(a/b^N)
    if $N = 0$ **then**
        return 0
    else if $N = 1$ **then**
        compute the Euclidean division $a = qb + r$
        return r/b
    else
        compute $u, v \in \mathbb{C}(t)[x]$ such that $a = ub + v\frac{\partial b}{\partial x}$
        return HermiteRed $\left(\dfrac{u}{b^{N-1}} + \dfrac{1}{N-1}\dfrac{1}{b^{N-1}}\dfrac{\partial v}{\partial x}\right)$

C. Hermite (1872). "Sur l'intégration des fractions rationnelles". In: *Ann. Sci. École Norm. Sup.* 2nd ser. 1, pp. 215–218

Computing univariate residues

input $a/b^N \in \mathbb{C}(t,x)$ with $a, b \in \mathbb{C}(t)[x]$ and b squarefree

output a differential equation for $\operatorname{res}_x \frac{a}{b^N}$

procedure ResidueToDiffEq(a/b^N)
  $r \leftarrow 0$
  $c_0 \leftarrow \mathrm{HermiteRed}(a/b^N)$
  while $c_0, \dots, c_r$ are linearly independent over $\mathbb{C}(t)$ **do**
    $r \leftarrow r + 1$
    $c_r \leftarrow \mathrm{HermiteRed}\left(\frac{\partial}{\partial t} \frac{c_{r-1}}{b}\right)$
  compute $p_0, \dots, p_r \in \mathbb{C}(t)$, not all zero, with $\sum_{i=0}^{r} p_i(t) c_i(t,x) = 0$
  return $p_r(t) y^{(r)} + \cdots + p_1(t) y' + p_0 y = 0$

Inverse of a Laurent power series

We say that a monomial $x_0^{i_0} \cdots x_n^{i_n}$ is *small* if the first nonzero exponent i_k is positive. Given two monomials m and n, we say that $m < n$ if the monomial m/n is small.
$\cdots < x_0^3 < x_0^2 < x_0 < \cdots < x_1^3 < x_1^2 < x_1 < \cdots < 1 < x_1^{-1} < \cdots < x_0^{-1} < x_0^{-2} < \cdots$

For a Laurent series $R \in \mathbb{C}((x_0, \ldots, x_n))$, let $\operatorname{lm}(p)$ be the largest of its monomial.

Lemma

Let $R \in \mathbb{C}((x_0, \ldots, x_n))$.

If $\operatorname{lm}(R) < 1$, *then* $\dfrac{1}{1-R} = \sum_{k=0}^{\infty} R^k$.

If $\operatorname{lm}(R) > 1$, *then* $\dfrac{1}{1-R} = -\dfrac{1}{R} \sum_{k=0}^{\infty} R^{-k}$.

Simplification of residues

Let $R = \frac{a}{(x_i-\rho)^k}$, with $a, \rho \in \mathbb{C}(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Then

$$\operatorname{res}_{x_i} R = \begin{cases} 0 & \text{if } k > 1 \text{ or } \operatorname{Im}(\rho) > x_i \\ a & \text{otherwise.} \end{cases}$$

Lemma

Let $R = a/b \in \mathbb{C}(t, x_1, \ldots, x_n)$ such that the denominator b factors into factors of degree at most one with respect to x_i. Then there is a rational function $S \in \mathbb{C}(t, x_1, \ldots, x_{i-1}, x_i, \ldots, x_n)$ such that

$$\operatorname{res}_{x_1,\ldots,x_n} R = \operatorname{res}_{x_1,\ldots,x_i,x_{i-1},\ldots,x_n} S.$$

Very effective in practice!

A recent example

Problem

Prove that

$$\sum_{k=1}^{m+s-1}\left(\sum_{r=1}^{s}\binom{s}{r}\binom{k-1}{r-1}\frac{b-1}{(-b)^r}\sum_{i=0}^{r-1-\max(k-m,0)}(-b)^i\binom{r-1}{i}\right)(bx)^k \leq 0$$

for any integers $m, s \geq 0$, any $b > 1$ and any $x \in [0, 1)$.

C. Koutschan, E. Wong (Sept. 1, 2021). “Creative Telescoping on Multiple Sums”. In: *Math.Comput.Sci.* 15.3, pp. 483–498. DOI: 10/gkc49d

Residues of rational functions are D-finite

Theorem

Let $R \in \mathbb{C}(t, x_1, \ldots, x_n)$ and $f(t) = \operatorname{res}_{x_1,\ldots,x_n} R$. Then f is D-finite: there exist polynomials $p_0(t), \ldots, p_r(t)$, not all zero, such that

$$p_d(t) f^{(d)}(t) + \cdots + p_1(t) f'(t) + p_0(t) f(t) = 0.$$

And there is an algorithm to compute them.

Corollary

If $(u_n)_{n\geq 0}$ is a binomial sum, then it is P-recursive: there exists polynomials $q_0(n), \ldots, q_r(n)$, not all zero, such that for any $n \geq 0$

$$q_0(n)u(n) + q_1(n)u(n+1) + \ldots + q_r(n)u(n+r) = 0.$$

A computational handle on residues

Residues of derivatives

For any $A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$,

$$\operatorname{res}_{x_1,\dots,x_n}\left(\frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n}\right) = 0.$$

Corollary

Let $R \in \mathbb{C}(t, x_1, \dots, x_n)$ and $f(t) = \operatorname{res}_{x_1,\dots,x_n} R$. Let $p_0, \dots, p_r \in \mathbb{C}(t)$ and $A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$.

$$\sum_{k=0}^{r} p_k(t)\frac{\partial^k R}{\partial t^k} = \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i} \Rightarrow \sum_{k=0}^{r} p_k(t) f^{(k)}(t) = 0.$$

Proof (Lipshitz 1988)

Let $R = \frac{a}{p} \in \mathbb{C}(t, x_1, \ldots, x_n)$. Let $\delta = \max(\deg_x a, \deg_x p)$.

- Let $V_N = \left\{ \frac{b}{p^N} \,\middle|\, b \in \mathbb{C}(t)[x_1, \ldots, x_n]_{\leq N\delta} \right\}$
  Let $D_N = \left\{ \frac{\partial^{\alpha+\beta_1+\cdots+\beta_n} R}{\partial t^\alpha \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \,\middle|\, \alpha + \beta_1 + \cdots + \beta_n \leq N - 1 \right\} \subseteq V_N$
- $\dim_{\mathbb{C}(t)} V_N = \binom{N\delta+n}{n} \sim_{N\to\infty} \frac{1}{n!}(N\delta)^n$ and $\#D_N = \binom{N+n}{n+1} \sim_{N\to\infty} \frac{1}{(n+1)!} N^{n+1}$
  So $\dim V_N = o(\#D_N)$ as $N \to \infty$.
- For some N large enough, the elements of D_N are not linearly independent. So there are some $c_{\alpha,\underline{\beta}} \in \mathbb{C}(t)$ such that $(*)$ $\sum_{\alpha,\underline{\beta}} c_{\alpha,\underline{\beta}} \frac{\partial^{\alpha+\underline{\beta}} R}{\partial t^\alpha \partial x^{\underline{\beta}}} = 0$.
- Then

$$\sum_\alpha c_{\alpha,0}(t) \frac{\partial^\alpha R}{\partial t^\alpha} = \sum_{i=1}^n \frac{\partial[\ldots]}{\partial x_i}.$$

- Therefore, $\operatorname{res}_{x_1,\ldots,x_n} R$ is D-finite. Wait... really?

Proof, a small fix

It may be possible that all the $c_{\alpha,0}$ are zero, in which case we cannot conclude that $\operatorname{res}_{x_1,\dots,x_n} R$ is D-finite.

There is some $(\gamma_1, \dots, \gamma_n)$ such that multiplying $(*)$ by $x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ (and permuting with the derivations in an appropriate way), produce a relation with some $c_{\alpha,0}$ nonzero.

But I skip the details here. The important argument is the dimension counting.

Exercise

Find what is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{n}.$$

Tous les moyens sont bons !

1D walk, warm up

Let $a \in [0, 1]$.
$(X_n)_{n \geq 0}$ random walk in $\mathbb{Z}$:

- $X_{n+1} = \begin{cases} X_n - 1 & \text{with probability } \frac{1+a}{2} \\ X_n + 1 & \text{with probability } \frac{1-a}{2} \end{cases}$
- The steps $X_{n+1} - X_n$ are indepedent.

Problem

Find a such that

$$E(a) \stackrel{\text{def}}{=} \mathbb{E}\left[\#\left\{n \mid X_n = X_0\right\}\right] = 2$$

A counting formula

$$E(a) = \sum_{n=0}^{\infty} \mathbb{P}\left[X_n = X_0\right]$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1+a}{2}\right)^n \left(\frac{1-a}{2}\right)^n$$
$$= \frac{1}{a}$$

The 2D problem

Let $a \in [0,1]$.
$(X_n)_{n\geq 0}$ a random walk in $\mathbb{Z}^2$:

- $$X_{n+1} = \begin{cases} X_n - (0,1) & \text{with probability } \frac{1}{4} \\ X_n + (0,1) & \text{with probability } \frac{1}{4} \\ X_n - (1,0) & \text{with probability } \frac{1+a}{4} \\ X_n + (1,0) & \text{with probability } \frac{1-a}{4} \end{cases}$$
- The steps $X_{n+1} - X_n$ are indepedent.

Problem

Find a such

$$E(a) \stackrel{\text{def}}{=} \mathbb{E}\left[\#\left\{n \,|\, X_n = X_0\right\}\right] = 2$$

A counting formula

$$\begin{aligned} E(a) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathbb{P}\left[X_n = X_0\right] \\ &= \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{1+a}{4}\right)^k \left(\frac{1-a}{4}\right)^k \left(\frac{1}{4}\right)^{n-k} \left(\frac{1}{4}\right)^{n-k} \\ &= ??? \end{aligned}$$

Goal

Compute a thousand digits of the solution of the equation $E(a) = 2$ in second!

Method 1 (naïve)

1. Chose some big enough N.

2. Compute the polynomial

$$E_N(a) \stackrel{\text{def}}{=} \sum_{n=0}^{N} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{1+a}{4}\right)^k \left(\frac{1-a}{4}\right)^k \left(\frac{1}{4}\right)^{n-k} \left(\frac{1}{4}\right)^{n-k}.$$

3. Compute the root (?) of the equation $E_N(a) = 0$ in the interval $[0, 1]$.

Method 2 (fast evaluation + dichotomy)

1. Find a recurrence relation on the $p_n(a)$, where

$$p_n(a) \overset{\text{def}}{=} \mathbb{P}[X_n = X_0] = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{1+a}{4}\right)^k \left(\frac{1-a}{4}\right)^k \left(\frac{1}{4}\right)^{n-k} \left(\frac{1}{4}\right)^{n-k}.$$

   This relation makes it possible to compute $E(a)$ numerically efficiently.

2. Solve $E(a) = 2$ by dichotomy.

Method 3 (analytic formula + Newton iteration)

1. We find (!!) that

$$E(a) = \frac{4K\left(\frac{2(1-a^2)^{\frac{1}{4}}}{\sqrt{a^2+2\sqrt{1-a^2}+2}}\right)}{\pi\sqrt{a^2+2\sqrt{1-a^2}+2}},$$

   where K is the elliptic integral of the first kind.

2. Apply the Newton iteration,

$$a_{n+1} = a_n - \frac{E(a_n) - 2}{E'(a_n)}$$

   which will converge very fast to the solution.