Lattice reduction Experimental mathematics MPRI – Efficient algorithms in computer algebra

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Outline

1. Lattice reduction

2. BBP formulas

3. Disproof of Mertens' conjecture

Euclidean lattices

Definition

An *Euclidean lattice* is a discrete subgroup of \mathbb{R}^n (with its usual norm). An *Euclidean lattice* is a group \mathbb{Z}^n with a positive definite quadratic form.

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A polynomial-time solvable problem

Input A lattice $\Lambda \subseteq \mathbb{Z}^n$ Output $f \in \Lambda$ nonzero such that $||f|| \le 2^{\frac{n-1}{2}} \min \{||g|| | g \in \Lambda \text{ nonzero}\}$

Volume and *i*th minimum

Let *L* be a lattice with basis f_1, \ldots, f_r . For B > 0, let $L_B = \langle x \in L \mid ||x|| \leq B \rangle$.

Volume

 $\operatorname{vol}(L)^2 = \det \left(f_i \cdot f_j \right)_{1 \le i,j \le r}$

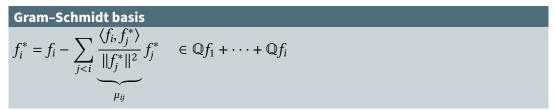
k th minimum

$$\lambda_k(L) = \min \{ B \mid \operatorname{rk} L_B \ge k \}$$

= min $\left\{ \max_{1 \le i \le k} \| v_i \| \mid v_1, \dots, v_k \in L \text{ linearly independent} \right\}$

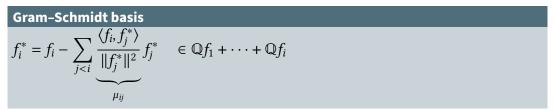
Gram-Schmidt orthogonalization

Let $f_1, \ldots, f_r \in \mathbb{Z}^n$ and let $L = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_r$ be the generated lattice.



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Lemma

 $\operatorname{vol}(L) = \prod_{i=1}^r \|f_i^*\|$

GSO and short vectors

Lemma

For any nonzero $g \in L$, $||g|| \ge \min \{ ||f_1^*||, \dots, ||f_r^*|| \}.$

Proof. Write $g = \sum_{i=1}^{s} a_i f_i$, with $a_i \in \mathbb{Z}$, $1 \le s \le r$ and $a_s \ne 0$. In the GS basis, we have

$$g = a_s f_s^* + [\cdots] f_{s-1}^* + \cdots + [\ldots] f_1^*,$$

and in particular, $||g||^2 \ge a_s^2 ||f_s^*||^2$.

Reduced bases

Definition

A basis $f_1, ..., f_r$ is reduced if (i) $|\mu_{ij}| \le \frac{1}{2}$, for any $1 \le j < i$ (size-reduced); (ii) $||f_{i-1}^*||^2 \le 2||f_i^*||^2$ for any *i*.

I follow Gathen, Gerhard (2013) for the definition. More commonly, condition (ii) is replaced by the stronger, with $\delta \in (\frac{1}{4}, 1)$

 $(\delta-\mu_{i,i-1}^2)\|f_{i-1}^*\|^2\leq \|f_i^*\|^2.$

It took 200 years to develop this definition. It's not at all clear that it's strong enough to be interesting, or weak enough for such bases to exist and be computable in polynomial time.

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Let $\lambda_k(L) = \min \{B \mid \operatorname{rk} L_B \ge k\} = \min \{\max_{1 \le i \le k} ||v_i|| \mid v_1, \ldots, v_k \in L \text{ lin. indep.}\}.$ Let f_1, \ldots, f_r be a reduced basis.

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Lemma

For any $1 \le k \le r$, $\min_{k \le j \le r} ||f_j^*|| \le \lambda_k(L)$.

Proof. Let $v_1, \ldots, v_k \in L$ be linearly independent. Because of the linear independence, there is at least one v_i which is not in $\operatorname{Vect}(f_1, \ldots, f_{k-1})$. By a previous argument, $||v_i|| \ge \min_{k \le j \le r} ||f_j^*||$.

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For any $1 \le k \le r$, $||f_k|| \le 2^{\frac{r-1}{2}} \lambda_k(L)$.

Proof. For $k \leq j \leq r$, we have $||f_j^*|| \geq 2^{\frac{k-j}{2}} ||f_k^*||$. Moreover,

$$||f_k||^2 \le ||f_k^*||^2 + \sum_{j < k} \mu_{kj}^2 ||f_j^*||^2 \le 2^{k-1} ||f_k^*||^2,$$

and the claim follows.

Theorem

Let f_1, \ldots, f_r be a reduced basis of a lattice L. For B > 0, let L_B = Lattice $\{g \in L \mid ||g|| \le B\}$. Let $\kappa = 2^{\frac{r-1}{2}}$ and $\mu \ge \kappa$. **4** Let B > 0 such that $L_B = L_{\mu B}$. Then there is a k such that (i) $||f_i|| \le \kappa B$ for all $i \le k$;

- (ii) $||f_i|| > \mu B$ for all i > k;
- (iii) f_1, \ldots, f_k is a basis of L_B .

Proof. Let $k = \operatorname{rank}(L_B)$. By definition $\lambda_k(L) \leq B$. The hypothesis $L_B = L_{\mu B}$ means that $\lambda_{k+1}(L) > \mu B$. For any $i \leq k$, we have

 $||f_i|| \le \kappa \lambda_i(L) \le \kappa \lambda_k(L) \le \kappa B.$

This proves (i). In particular, $f_i \in L_{\kappa B} \subseteq L_{\mu B} = L_B$, so f_1, \ldots, f_k is a basis of L_B . This proves (iii). For any j > k, the family f_1, \ldots, f_k, f_j is free, so

 $\max\left\{\|f_1\|,\ldots,\|f_k\|,\|f_j\|\right\} \ge \lambda_{k+1}(L) > \mu B.$

Combining with the previous inequality, this implies that $||f_j|| \ge \mu B$. This proves (ii).

The LLL algorithm

```
def LLL(f_1, ..., f_r):
   compute the GS information (the ||f_i^*||^2 and \mu_{ij} coefficients)
   i \leftarrow 2; \quad N_{\text{iter}} \leftarrow 0; \quad N_{\text{swap}} \leftarrow 0
   while i < r:
          N_{\text{itor}} \leftarrow N_{\text{itor}} + 1
          for j from i − 1 to 1:
                 f_i \leftarrow f_i - |\mu_{ii}| f_i
                  update the GS information
          if ||f_{i-1}^*||^2 > 2||f_i^*||^2:
                 swap(f_{i-1}, f_i)
                  update the GS information
                  i \leftarrow \max(i-1,2); \quad N_{\text{swap}} \leftarrow N_{\text{swap}} + 1
          else:
                  i \leftarrow i + 1
   return f_1, \ldots, f_r
```

Correction of the LLL algorithm

Proposition

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After the loop on line 6, it is clear that f_1, \ldots, f_i is size-reduced.

At the begining of each iteration of the "while"-loop, it is also clear that f_1, \ldots, f_{i-1} is reduced.

So if the algorithm terminates, it outputs a reduced basis.

Theorem (A. K. Lenstra, H. W. Lenstra, Lovàsz 1982)

The LLL algorithm terminates in polynomial time.

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If a swap happens, let g_1, \ldots, g_i denote the basis after the swap. Then

$$g_{i-1}^* = f_i^* + \mu_{i,i-1} f_{i-1}^*$$

and so

$$\|g_{i-1}^*\|^2 \le \|f_i^*\|^2 + \frac{1}{4}\|f_{i-1}^*\|^2 \le \frac{3}{4}\|f_{i-1}^*\|^2.$$

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After the swap, $D_{i-1} = \prod_{k=1}^{i-1} ||f_k^*||^2$ decreases by a factor $\frac{3}{4}$ at least. Besides, $D_j = \text{vol} \left(\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_j \right)$, so D_j (for $j \neq i$) remains constant. It follows that

 $\Delta = D_1 \cdots D_r$

is a strictly positive integer which decreases by $\frac{3}{4}$ after a swap.

Lattice reduction | Lattice reduction

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Arithmetic complexity
 The number of swaps is bounded by log(Δ)/log(³/₄).

The initial value of log Δ is bounded by n² log A.

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• Binary complexity Hard to do it right... Current best (of a variant of LLL) is $O(n^{5+\epsilon} \log(A)^{1+\epsilon})$.

Applications of lattice reduction

- Cryptography
- Experimental mathematics
 - Disproof of Mertens' conjecture (Odlyzko, te Riele 1985)
 - Guessing recurrence relations with little data (Kauers, Koutschan 2022)
- Integer linear programming
- ...

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