DDE-Solver: a *Maple* package for Discrete Differential Equations

Hadrien Notarantonio[1]

[1]Inria, Université Paris-Saclay, France, hadrien.notarantonio@inria.fr.

Abstract

We introduce DDE-Solver, a Maple package designed for solving *Discrete Differential Equations* (DDEs). These equations are functional equations relating algebraically a formal power series $F(t, u)$ with polynomial coefficients in a "catalytic" variable u, with specializations of it with respect to the catalytic variable. Such equations appear in enumerative combinatorics, for instance in the enumeration of maps. Bousquet-Mélou and Jehanne showed in 2006 that when these equations are of a fixed point type in F, then F is an algebraic series. In the same paper, they proposed a systematic method for computing annihilating polynomials of these series. Bostan, Safey El Din and the author of this article recently designed new efficient algorithms for computing these witnesses of algebraicity. In this paper, we describe the first Maple package dedicated to the resolution of these DDEs. Moreover, we compare the timings of these new algorithms on DDEs from the literature.

1 Introduction

Context and motivations. Sequences of non-negative integers are ubiquitous in enumerative combinatorics. For instance, when studying bicolored maps (purple, white) such that the degree of each purple face is 3 and the degree of each white face is a multiple of 3, the introduction of the sequence $(c_n)_{n \in \mathbb{N}}$ of such maps with n purple faces yields the numbers $1, 1, 6, 54, 594, 7371, 99144, \cdots$. From such a sequence, many questions arise: does there exist a finite way of representing the infinite amount of data given by this sequence? Does there exist a close formula for $(c_n)_{n \in \mathbb{N}}$? What is the value of c_N, for $N \in \mathbb{Z}_{\geq 0}$ large (e.g. $N = 3 \cdot 10^6$), and how fast can it be computed? What is the asymptotic growth of c_n when $n \to \infty$?

In order to answer such questions, a common method is to introduce and study the properties of the generating function of the enumeration. For instance for the above maps enumeration, one would introduce $G(t) := \sum_{n \in \mathbb{N}} c_n t^n \in \mathbb{Q}[[t]]$. For sophisticated enumerations, it is usually hard to study directly the generating series. Refining the initial enumeration of interest usually yields to introducing a new variable called *catalytic*, which leads to consider a bivariate generating function, say $F(t, u)$ for u the catalytic variable. For well-chosen refinements, it is then possible to write a functional equation relating algebraically both $F(t, u)$ and $G(t)$.

For the above toy example of bicolored maps, one would typically refine the enumeration with the sequence $(c_{n,d})_{n,d \in \mathbb{N}}$ as number of 3-constellations having n purple faces and outer degree $3d$. Straightforwardly, one would also introduce its associated generating function $F(t, u) = \sum_{n,d \geq 0} c_{n,d} u^d t^n \in \mathbb{Q}[u][[t]]$. As $(\sum_{d=0}^{\infty} c_{n,d}) = c_n$, it holds that $F(t, 1) = G(t)$. An analysis on the construction of such maps [9, Fig. 7] yields the functional equation

$$F(t, u) = 1 + tuF(t, u)^3 + tu(2F(t, u) + F(t, 1))\frac{F(t, u) - F(t, 1)}{u - 1} \tag{1}$$

$$+ tu \frac{F(t, u) - F(t, 1) - (u - 1) \cdot \partial_u F(t, 1)}{(u - 1)^2}.$$

Note that for any positive integer k and for two polynomials f, Q, eq. (1) has the form

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u), \qquad (2)$$

where, for $a \in \mathbb{Q}$, we denote $\Delta_a : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,a)}{u-a} \in \mathbb{Q}[u][[t]]$ and Δ_a^ℓ the ℓth iteration of the *divided difference operator* Δ_a. Equations of the form (2) are called DDEs of *order* k. For instance, eq. (1) is a DDE of order $k = 2$, and the divided difference operator Δ_a is considered at the point $a = 1$. Observe that due to its fixed-point nature, there exists a unique solution in $\mathbb{Q}[u][[t]]$ to eq. (2). Moreover, the following result proved by Bousquet-Mélou and Jehanne in [9], reminiscient of Popescu's theorem [23, Thm. 1.4], implies that this solution is annihilated by a nonzero polynomial with coefficients in $\mathbb{Q}(t,u)$.

Theorem 1. ([9, Thm. 3]) *Let $\mathbb{K}$ be a field of characteristic 0 and consider two polynomials $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y_1, \ldots, y_k, t, u]$, where $k \in \mathbb{N} \setminus \{0\}$. Let $a \in \mathbb{K}$ and $\Delta_a : \mathbb{K}[u][[t]] \to \mathbb{K}[u][[t]]$ be the divided difference operator $\Delta_a F(t,u) := (F(t,u) - F(t,a))/(u-a)$. Let us denote by Δ_a^ℓ the operator obtained by iterating ℓ times Δ_a. Then, there exists a unique solution $F \in \mathbb{K}[u][[t]]$ to the functional equation*

$$F(t,u) = f(u) + t\, Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u), \qquad (3)$$

and moreover $F(t,u)$ is algebraic over $\mathbb{K}(t,u)$.

Applying Theorem 1 to our example of maps enumeration, there exists some polynomial $R \in \mathbb{Q}[t,z] \setminus \{0\}$ such that $R(t, G(t)) = 0$. Namely, $G(t) = 1 + t + 6t^2 + 54t^3 + 594t^4 + O(t^5)$ is annihilated by $R(t,z) = 81t^2 z^3 - 9t(9t-2)z^2 + (27t^2 - 66t + 1)z - 3t^2 + 47t - 1$. Moreover G is the unique root in z of R in $\mathbb{Q}[[t]]$. Thus R provides a finite amount of data that encode the sequence $(c_n)_{n \in \mathbb{N}}$. Also with R, it is possible to show[1] that $c_0 = 1$ and $c_n = \frac{6 \cdot 3^n}{(2n+2)(2n+1)} \binom{3n-1}{n-1}$, for all $n \geq 1$. Additionally one proves, using the polynomial R, that for all $n \in \mathbb{N} \setminus \{0,1\}$

$$(81n^2 + 81n + 18) \cdot c_n - (4n^2 + 14n + 12) \cdot c_{n+1} = 0,$$

with $c_0 = 1, c_1 = 1$. Using the above hypergeometric recurrence, computing $c_{3 \cdot 10^6}$ only takes 40 seconds in Maple by using the gfun package [24] of Salvy and Zimmermann. Now using the above closed formula, one deduces the asymptotic behavior

$$c_n \sim 3^{4n} 2^{-2n} (4\sqrt{3\pi} n^{-\frac{5}{2}} + O(n^{-\frac{7}{2}})), \text{ when } n \to \infty.$$

The oracle that provided us the polynomial $R(t,z) = 81t^2 z^3 - 9t(9t-2)z^2 + (27t^2 - 66t + 1)z - 3t^2 + 47t - 1$ hence allowed us to answer all the questions stated at the beginning of this section. We shall emphasize that it is not always possible to deduce closed-formulas for such sequences. However, linear recurrences are always computable since the generating functions are algebraic, thus solution of a linear differential equation with polynomial coefficients [25, Prop. 6.4.3, Thm. 6.4.6].

A central question is thus to *solve* equations like eq. (3) that is, to compute annihilating polynomials for their solutions. The problem considered in DDE-Solver is the following:

For F solution of a DDE of the form eq. (3), compute $R \in \mathbb{K}[t,z] \setminus \{0\}$ such that $R(t, F(t,a)) = 0$.

Previous algorithmic works   There exists a rich litterature regarding the effective resolution of DDEs. The articles [9, 6, 7] already contain a complete state-of-the-art, but we shall however give an overview of this litterature. Let us start with DDEs such that the polynomial Q in (3) has degree 2 in x. When $k = 1$, Brown introduced in [13] what is now

---

[1]Write the linear differential equation satisfied by $F(t,1)$, and solve the associated recurrence relation.

called the quadratic method. This method was generalized later by Bender and Canfield in [3] on a particular family of DDEs of arbitrary order, still with Q of degree 2 in x.

Also, the case where Q is linear in $x, y_1, \ldots, y_k$ appears in many walks enumeration problems (e.g. [9, §3.1]). Solving DDEs in this case is usually done by applying the kernel method, introduced by Knuth in [20] and coined by Banderier and Flajolet in [1]. Since the work of Bousquet-Mélou and Petkovšek [10], the linear case is considered as understood.

Regarding strategies based on the guess-and-prove paradigm (democratized in [22]), they have been notably studied by Zeilberger [26] in the case $k = 1$, and improved in this same case with Gessel in [19].

Bousquet-Mélou and Jehanne designed in [9] a general method which can be seen as a generalization of both the kernel and the quadratic methods. Their strategy consists in translating the resolution of a DDE into the resolution of a polynomial system admitting a solution with $F(t, a)$ as one of its coordinate. We refer to their algorithm as "duplication".

Recently, intensive studies have been undertaken in [6] (resp. in [7]) in the direction of effectivity in order to design efficient algorithms for solving DDEs of order $k = 1$ (resp. of any order). On the one hand, Bostan, Chyzak, Safey El Din and Notarantonio designed efficients algorithms based on effective algebraic geometry (we refer to these algorithms as "elimination" and "geometry") and on the other hand they used these first geometry-driven algorithms in a new *hybrid* guess-and-prove approach (refered as "hybrid").

In all these previous works, only four algorithms apply (under some technical assumptions studied in [7]) to DDEs of any order: the now standard "duplication" approach [9], and the recent "elimination", "geometry" and "hybrid" approaches [7].

Contributions   The present article contains three contributions. First, it describes the Maple package DDE-Solver, dedicated to solving DDEs of any order that satisfy the (harmless) assumptions explicited in [7]. More explicitely, DDE-Solver contains the function **annihilating_polynomial**. This function takes as input a DDE of the form eq. (3) and its order k, and outputs a nonzero polynomial annihilating the series $F(t, a)$. Also, some options can be specified, namely: the *algorithm* that shall be used amongs "duplication", "elimination", "geometry" and "hybrid"; and an option *variable* explained in section 5.

The second contribution is a careful practical comparison of these algorithms resulting in a table of timings.

The third contribution is the resolution with **annihilating_polynomial** of the DDE associated to the enumeration of 3-greedy Tamari intervals (resp. of 5-constellations) in 1 minute (resp. in 3 hours). It is the first time that these two equations are solved with an automatized approach. We however underline that these DDEs are solved without using effective elimination theory in [12] for greedy Tamari intervals and in [11] for constellations.

Structure of the paper   In section 2, we recall why solving a DDE is reduced [9, Sec. 2] to eliminating variables in a system of polynomial equations. Also, we introduce in section 2 different modelling of the underlying geometric problem. These modellings are useful later in the article. In section 3, we state some preliminaries on effective algebraic geometry that are used in section 4 when giving an overview on how the algorithms "duplication", "elimination", "geometry" and "hybrid" (resp. [7, §3, §5, §6, §4]) work. In section 5, we explain how to get started with DDE-Solver , and how to use the options of the function **annihilating_polynomial**. Finally in section 6, we provide some timings that illustrate the efficiency of **annihilating_polynomial** on various DDEs from the literature [9, 4, 8, 12].

Notations   We gather the notations used in this article. First, we always denote by $\mathbb{K}$ a field of characteristic 0, by $\overline{\mathbb{K}}$ its algebraic closure, by $\mathbb{Q}$ the field of rational numbers and by $\mathbb{F}_p$ the finite field with p elements. For a fixed positive integer k, we denote by $\underline{x}$ (resp. $\underline{u}$

and $\underline{z}$) the variables $x_1,\ldots,x_k$ (resp. $u_1,\ldots,u_k$ and $z_0,\ldots,z_{k-1}$). We denote by $\mathbb{K}[x_1,\ldots,x_k]$ the ring of polynomials in the variables $x_1,\ldots,x_k$ with coefficients in $\mathbb{K}$. For $P \in \mathbb{K}[\underline{x}]$, we denote by $V(P) \subset \overline{\mathbb{K}}^r$ its zero set. For any ideal $\mathcal{I} \subset \mathbb{K}[\underline{x}]$ and any set of polynomials $\mathcal{S} \subset \mathbb{K}[\underline{x}]$, we denote by $V(\mathcal{I}) \subset \overline{\mathbb{K}}^k$ (resp. by $V(\mathcal{S}) \subset \overline{\mathbb{K}}^k$) the zero set of $\mathcal{I}$ (resp. of $\mathcal{S}$). Still for $P \in \mathbb{K}[\underline{x}]$, we denote by $\mathrm{disc}_{x_j}(P)$ the discriminant [15, 16., §6, Ch.3] of P with respect to x_j and by $\mathrm{LC}_{x_j}(P)$ the leading coefficient of P with respect to x_j. We denote by $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]$ the ring $\bigcup_{d \geq 1} \overline{\mathbb{K}}[[t^{\frac{1}{d}}]]$ of Puiseux series in t with positive fractional exponents.

2 From DDEs to polynomial systems

Starting from a DDE of the form (3), we multiply it by the least power of $(u-a)$ such that we obtain a polynomial functional equation of the form

$$P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) = 0, \tag{4}$$

for some polynomial $P \in \mathbb{Q}[x,z_0,\ldots,z_{k-1},t,u]$. Taking the derivative of (4) with respect to u yields by the chain rule

$$\partial_u F(t,u) \quad \cdot \quad \partial_x P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) \tag{5}$$
$$+ \ \partial_u P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) \ = \ 0.$$

Consider the equation in u given by

$$\partial_x P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) = 0. \tag{6}$$

Observe that the solutions of (6) that belong to $\overline{\mathbb{Q}}[[t^{\frac{1}{\star}}]]$ are also solutions, by using (5), of the equation in u given by

$$\partial_u P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) = 0.$$

Thus any non constant solution in $\overline{\mathbb{Q}}[[t^{\frac{1}{\star}}]]$ of (6) is a solution in u of the system of constraints

$$\begin{cases} P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) & = 0, \\ \partial_x P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) & = 0, \quad u(u-a) \neq 0, \\ \partial_u P(F(t,u),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,u) & = 0. \end{cases} \tag{7}$$

We introduce below hypothesis which is necessary to assume in order to make the ongoing general strategy work. This assumption holds for the examples considered in this paper.

$\underline{\text{Global assumption:}}$ There exist k distinct solutions $U_1(t),\ldots,U_k(t) \in \overline{\mathbb{Q}}[[t^{\frac{1}{\star}}]] \setminus \overline{\mathbb{Q}}$ **(GA)**

to the system of equations (7).

In [7], the authors studied three geometric interpretations of (**GA**) that we recall below. The first one is introduced in [9], while the last two were introduced respectively in [7, §5, §6]. Also, it is a nontrivial consequence of the proof of [9, Thm. 3] that the series $\{\partial_u^{i-1}F(t,a), U_i(t), F(t,U_i(t))\}_{1 \leq i \leq k}$ considered in this paper are elements of $\overline{\mathbb{Q}(t)}$.

Duplication approach   This approach was introduced by Bousquet-Mélou and Jehanne in [9, Sec. 2] and works as follows. It results from (**GA**) that the following relations hold:

$$\forall \ 1 \leq i \leq k, \begin{cases} P(F(t,U_i(t)),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,U_i(t)) & = 0, \\ \partial_x P(F(t,U_i(t)),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,U_i(t)) & = 0, \\ \partial_u P(F(t,U_i(t)),F(t,a),\ldots,\partial_u^{k-1}F(t,a),t,U_i(t)) & = 0, \end{cases}$$

4

and

$$\prod_{1 \le i < j \le k} (U_i(t) - U_j(t)) \cdot \prod_{1 \le i \le k} U_i(t) \cdot (U_i(t) - a) \ne 0.$$

In terms of polynomial equations, the above relations are equivalent to saying that the *duplicated* polynomial system $\mathcal{S}_{\mathrm{dup}}$ defined by

$$\forall\, 1 \le i \le k, \begin{cases} P(x_i, z_0, \ldots, z_{k-1}, t, u_i) & = 0, \\ \partial_x P(x_i, z_0, \ldots, z_{k-1}, t, u_i) & = 0, \quad m \cdot \prod_{1 \le i < j \le k} (u_i - u_j) \cdot \prod_{1 \le i \le k} u_i \cdot (u_i - a) - 1 = 0, \quad (8) \\ \partial_u P(x_i, z_0, \ldots, z_{k-1}, t, u_i) & = 0, \end{cases}$$

admits the nontrivial solutions

$$x_i = F(t, U_i(t)), u_i = U_i(t), z_{i-1} = \partial_u^{i-1} F(t, a), \text{ for } 1 \le i \le k. \quad (9)$$

Note that (9) uniquely determines the value associated to the variable m. Also, the polynomial system $\mathcal{S}_{\mathrm{dup}}$ admits $3k + 1$ equations and unknowns (t is considered as a parameter); that is, the polynomials in $\mathcal{S}_{\mathrm{dup}}$ are seen as elements of the ring $\mathbb{Q}(t)[m, \underline{x}, \underline{z}, \underline{u}]$. We denote by $V(\mathcal{S}_{\mathrm{dup}})$ the solution set of $\mathcal{S}_{\mathrm{dup}}$ in $\overline{\mathbb{Q}(t)}^{k+3}$.

Elimination theory approach This second approach was introduced in [7, Sec. 5]. The idea is that it follows from (7) and (**GA**) that the system of polynomial constraints defined in $\overline{\mathbb{Q}(t)}[x, u]$ (t is again considered as a parameter) by:

$$\begin{cases} P(x, F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \\ \partial_x P(x, F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \quad u(u - a) \ne 0, \quad (10) \\ \partial_u P(x, F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \end{cases}$$

admits the k solutions $(x, u) = (F(t, U_i(t)), U_i(t)) \in \overline{\mathbb{Q}(t)}^2$, for $1 \le i \le k$. Note that these solutions have distinct u-coordinates. This observation can be reformulated geometrically in the following way. Consider the geometric projection $\pi : (x, u, \underline{z}) \in \overline{\mathbb{Q}(t)}^{k+2} \mapsto (\underline{z}) \in \overline{\mathbb{Q}(t)}^k$, and denote by $\mathcal{X} \subset \overline{\mathbb{Q}(t)}^{k+2}$ the solution set of the polynomial constraints (t is present but still considered as a parameter)

$$\begin{cases} P(x, z_0, \ldots, z_{k-1}, t, u) & = 0, \\ \partial_x P(x, z_0, \ldots, z_{k-1}, t, u) & = 0, \quad u(u - a) \ne 0. \quad (11) \\ \partial_u P(x, z_0, \ldots, z_{k-1}, t, u) & = 0, \end{cases}$$

For a point $\alpha \in \overline{\mathbb{Q}(t)}^k$, we denote by $\#_u(\mathcal{X}, \alpha)$ the number of u-coordinates in $\pi^{-1}(\alpha) \cap \mathcal{X}$ that are not constants and we define $\mathcal{F}_k(u, \mathcal{X}) := \{\alpha \in \overline{\mathbb{Q}(t)}^k \mid \#_u(\mathcal{X}, \alpha) \ge k\}$. Under assumption (**GA**), it holds that $(F(t, a), \ldots, \partial_u^{k-1} F(t, a)) \in \mathcal{F}_k(u, \mathcal{X})$.

Geometric approach This last approach was introduced in [7, Sec. 6]. The idea is that in addition to having (7), it follows from (**GA**) that the system of polynomial constraints defined in $\overline{\mathbb{Q}(t)}[x, z_1, u]$ (t is considered as a parameter) by:

$$\begin{cases} P(x, F(t, a), z_1, \partial_u^2 F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \\ \partial_x P(x, F(t, a), z_1, \partial_u^2 F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \quad u(u - a) \ne 0, \quad (12) \\ \partial_u P(x, F(t, a), z_1, \partial_u^2 F(t, a), \ldots, \partial_u^{k-1} F(t, a), t, u) & = 0, \end{cases}$$

5

admits the k solutions $(x, z_1, u) = (F(t, U_i(t)), \partial_u F(t, a), U_i(t)) \in \overline{\mathbb{Q}(t)}^{k+1}$, for $1 \leq i \leq k$. Note that these solutions are distinct. This observation can be reformulated in the following way.

Denote by $\check{z}_1$ the set of variables $z_0, z_2, \ldots, z_{k-1}$. Consider the geometric projection $\pi_{\check{z}_1}$: $(x, u, \underline{z}) \in \overline{\mathbb{Q}(t)}^{k+2} \mapsto (\check{z}_1) \in \overline{\mathbb{Q}(t)}^{k-1}$ and define the set

$$\mathcal{S}_k(\mathcal{X}) := \{\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{k-1}) \in \overline{\mathbb{Q}(t)}^k \mid \boldsymbol{\alpha} \in \pi(\mathcal{X}) \ \wedge \ \mathcal{X} \cap \pi_{\check{z}_1}^{-1}((\alpha_0, \alpha_2, \ldots, \alpha_{k-1})) \geq k\}.$$

Under assumption (**GA**), it holds that $(F(t, a), \ldots, \partial_u^{k-1} F(t, a)) \in \mathcal{S}_k(\mathcal{X})$.

Summary In section 2, we defined for each of the three approaches a solution set which contains a point whose z_0-coordinate is the series $F(t, a)$. The goal of section 4 is to explain how the algorithms studied in [7, §3, §5, §6] compute a polynomial characterization of these sets and deduce from this characterization an annihilating polynomial of $F(t, a)$.

3 Some preliminaries on polynomial tools

The ongoing section is devoted to the introduction of the polynomial tools used in section 4.

3.1 Elimination theory through Gröbner bases

For $x_1, \ldots, x_n$ some variables and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, we denote $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Also for $\mathcal{S} \subset \mathbb{K}[x_1, \ldots, x_n]$, we denote by $\langle \mathcal{S} \rangle$ the ideal generated by $\mathcal{S}$ in $\mathbb{K}[x_1, \ldots, x_n]$.

Monomial orders and Gröbner bases The idea of elimination monomial orderings is to attribute to a variable (or to a block of variables) that we want to eliminate a larger weight than the weights of the other variables. An example is the lexicographic order $>_{\text{lex}}$.

Definition 1. *([15, Def.3, §2, Ch.2]) Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}_{\geq 0}^n$. We say that $\boldsymbol{x}^{\boldsymbol{\alpha}} >_{lex} \boldsymbol{x}^{\boldsymbol{\beta}}$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive.*

> **Example 2.** *In $\mathbb{K}[x_1, x_2]$: $x_1^4 x_2^2 >_{lex} x_1^3 x_2^{10}$, $x_1^2 x_2 >_{lex} x_1$ and $x_1^2 >_{lex} x_2$.*

Another useful family of monomial orders that we will refer to in section 4 is the family of graded monomial orders. An example is the graded reverse lexicographic order $>_{\text{grevlex}}$.

Definition 2. *([15, Def.6, §2, Ch.2]) Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}_{\geq 0}^n$. We say that $\boldsymbol{x}^{\boldsymbol{\alpha}} >_{grevlex} \boldsymbol{x}^{\boldsymbol{\beta}}$ if $|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i$, or if $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.*

> **Example 3.** *In $\mathbb{K}[x_1, x_2, x_3]$: $x_1^5 x_2^7 x_3 >_{grevlex} x_1^4 x_2^2 x_3^3$, $x_1 x_2^4 x_3^2 >_{grevlex} x_1^3 x_2 x_3^3$*

We invite readers unfamiliar with monomial orders (see [15, Def.1,§2,Ch.2] for a general definition) to choose one of the two above monomial orders $>_{\text{lex}}$ or $>_{\text{grevlex}}$ for the next definition. Also, for $p \in \mathbb{K}[x_1, \ldots, x_n]$ a polynomial and $>$ a monomial order on $\mathbb{K}[x_1, \ldots, x_n]$, we denote by $\text{LT}_>(p)$ the leading term of p with respect to the monomial order $>$.

Definition 3. *([15, Def.5, §5, Ch.2]) Fix a monomial order $>$ on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. A finite subset $G = \{g_1, \ldots, g_s\}$ of an ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ different from $\{0\}$ is said to be a Gröbner basis of $\mathcal{I}$ for the order $>$ if $\langle \text{LT}_>(g_1), \ldots, \text{LT}_>(g_n) \rangle = \langle \{\text{LT}_>(g) \mid g \in \mathcal{I}\} \rangle$.*

A fundamental property [15, Cor.6, §5, Ch.2] of Gröbner bases is, with the notations of Definition 3, that such a basis G always exists, and that G generates $\mathcal{I}$, that is $\langle G \rangle = \mathcal{I}$.

Gröbner bases and projections. The results below justify the use of Gröbner bases as a dedicated theoretical and computational tool, allowing one to characterize projections as solution sets of conjunctions of polynomial equations and inequations.

Theorem 4 (Elimination theorem). *([15, Thm.2, §1, Ch.3]) Let $\mathcal{I} \subset \mathbb{K}[x_1,\ldots,x_n]$ be an ideal. Denote by G a Gröbner basis of $\mathcal{I}$ with respect to the order $\succ_{lex}$. Then for every $0 \le \ell \le n-1$, the set $G_\ell = G \cap \mathbb{K}[x_{\ell+1},\ldots,x_n]$ is a Gröbner basis of the ideal $\mathcal{I}_\ell := \mathcal{I} \cap \mathbb{K}[x_{\ell+1},\ldots,x_n]$.*

By Theorem 4, the set of polynomials G_ℓ finitely generates $\mathcal{I}_\ell$. Let us now introduce the geometric projection $\pi_\ell : (x_1,\ldots,x_n) \in \overline{\mathbb{K}}^n \mapsto (x_{\ell+1},\ldots,x_n) \in \overline{\mathbb{K}}^{n-\ell}$. Recall that a Zariski closed set in $\overline{\mathbb{K}}^n$ is defined as the solution set of some polynomial equations defined in $\mathbb{K}[x_1,\ldots,x_n]$.

Theorem 5 (Closure theorem). *([15, Thm.3, §2, Ch.3]) Let $\mathcal{I} \subset \mathbb{K}[x_1,\ldots,x_n]$ be an ideal and $V(\mathcal{I}) \subset \overline{\mathbb{K}}^n$ its zero set. Consider $\mathcal{I}_\ell := \mathcal{I} \cap \mathbb{K}[x_{\ell+1},\ldots,x_n]$. Then $V(\mathcal{I}_\ell)$ is the smallest Zariski closed set containing $\pi_\ell(V(\mathcal{I})) \subset \overline{\mathbb{K}}^{n-\ell}$.*

Theorems 4 and 5 imply that characterizing the Zariski closure of the projection of a zero set $V(\mathcal{I})$ onto some coordinate subspace is done by computing a Gröbner basis for $\succ_{lex}$.

Theorem 6 (Extension theorem). *([15, Thm.2, §5, Ch.3]) Let $G = \{g_1,\ldots,g_s\}$ be a Gröbner basis of $\mathcal{I} \subset \mathbb{K}[x_1,\ldots,x_n]$ for the order $\succ_{lex}$. For each $1 \le j \le s$, consider*

$$g_j = c_j(x_2,\ldots,x_n) \cdot x_1^{N_j} + (\text{terms in which } x_1 \text{ has degree} < N_j),$$

where $N_j \ge 0$ and $c_j \in \mathbb{K}[x_2,\ldots,x_n]$ is nonzero. Assume $\boldsymbol{\alpha} = (\alpha_2,\ldots,\alpha_n) \in V(\mathcal{I} \cap \mathbb{K}[x_2,\ldots,x_n])$ is a partial solution with the property that $\boldsymbol{\alpha} \notin V(c_1,\ldots,c_s)$. Then

$$\{f(x_1,\boldsymbol{\alpha}) \mid f \in \mathcal{I}\} = \{g_{j_0}(x_1,\boldsymbol{\alpha})\},$$

where $g_{j_0} \in G$ satisfies $c_{j_0}(\boldsymbol{\alpha}) \ne 0$ and g_{j_0} has minimal degree in x_1 among all elements g_j with $c_j(\boldsymbol{\alpha}) \ne 0$. Furthermore, if $g_{j_0}(\alpha_1,\boldsymbol{\alpha}) = 0$ for $\alpha_1 \in \overline{\mathbb{K}}$, then $(\alpha_1,\boldsymbol{\alpha}) \in V(\mathcal{I})$.

A consequence of Theorem 6 is that characterizing projections (and not their Zariski closure) can be done by considering disjunctions of conjunctions of equations and inequations.

3.2 Counting specific solutions

Let n,s be positive integers. We take the notation $\boldsymbol{x} = x_1,\ldots,x_n$ and $\boldsymbol{y} = y_1,\ldots,y_s$. Let $\mathcal{I} \subset \mathbb{K}[\boldsymbol{x}]$ be a radical ideal such that its zero set $V(\mathcal{I}) \subset \overline{\mathbb{K}}^n$ is finite. For $\boldsymbol{\alpha} \in V(\mathcal{I})$, we denote by $[x_1](\boldsymbol{\alpha})$ the x_1-coordinate of $\boldsymbol{\alpha}$. We introduce the projection $\pi_{x_1} : (\boldsymbol{x}) \in \overline{\mathbb{K}}^n \mapsto x_1 \in \overline{\mathbb{K}}$. Let $g \in \mathbb{K}[\boldsymbol{y}][z]$ and denote $\mathrm{LC}_z(g) \in \mathbb{K}[\boldsymbol{y}]$ the leading coefficients of g in z.

This current subsection introduces tools for answering the two problems below.

Problem 7. *Let ℓ be a positive integer. Characterize with polynomial equations defined in $\mathbb{K}[\boldsymbol{x}]$ the set $\{\boldsymbol{\beta} \in V(\mathcal{I}) \mid \pi_{x_1}^{-1}([x_1](\boldsymbol{\beta})) \cap V(\mathcal{I}) \ge \ell\}$.*

Problem 8. *Let $1 \le \ell \le \deg_z(g)$. Characterize with polynomial inequations defined in $\mathbb{K}[\boldsymbol{y}]$ the points $\boldsymbol{\beta} \in \overline{\mathbb{K}}^s \setminus V(\mathrm{LC}_x(g))$ such that $g(\boldsymbol{y} = \boldsymbol{\beta}, z)$ has at least ℓ distinct roots.*

Answer to Problem 7 Denote $\mathcal{A} := \mathbb{K}[\boldsymbol{x}]/\mathcal{I}$. It results from the finiteness of $V(\mathcal{I})$ that $\mathcal{A}$ has finite dimension as a $\mathbb{K}$-vector space [15, Thm.6, §3, Ch.5]. Define $m_{x_1} : f \in \mathcal{A} \mapsto x_1 \cdot f \in \mathcal{A}$ to be the *multiplication map by x_1 in $\mathcal{A}$*, and denote by $\chi_{x_1} \in \mathbb{K}[T]$ its characteristic polynomial. The following results from the radicality of $\mathcal{I}$ and from [16, Prop.2.7, §2, Ch.4].

Fact 9. *We have the equality $\chi_{x_1} = \prod_{\alpha \in V(\mathcal{I})}(T - [x_1](\alpha))$.*

As a consequence of Fact 9, we obtain the following set equality:

$$\{\beta \in V(\mathcal{I}) \mid \pi_{x_1}^{-1}([x_1](\beta)) \cap V(\mathcal{I}) \geq \ell\} \tag{13}$$
$$= \{\beta \in \overline{\mathbb{K}}^n \mid \beta \in V(\mathcal{I}) \wedge \chi_{x_1}([x_1](\beta)) = 0 \wedge \cdots \wedge \partial_T^{\ell-1}\chi_{x_1}([x_1](\beta)) = 0\}.$$

Equality (13) provides a conjunction of polynomial equations that answers Problem 7.

Example 10. *Assume $G = \{g_1, g_2\} \subset \mathbb{K}[x_1, x_2]$ generates $\mathcal{I}$ and $\ell = 2$. The conjunction of polynomial equations in $\mathbb{K}[x_1, x_2]$ is $g_1(x_1, x_2) = 0 \wedge g_2(x_1, x_2) = 0 \wedge \chi_{x_1}(x_1) = 0 \wedge (\partial_T \chi_{x_1})(x_1) = 0$.*

Answer to Problem 8   Define the *Hermite quadratic form* associated with g by

$$H_g : (f, h) \in (\mathbb{K}(\mathbf{y})[z]/\langle g \rangle)^2 \mapsto \text{Trace}(m_{f \cdot h}) \in \mathbb{K}(\mathbf{y}),$$

where Trace$(\cdot)$ is the trace operator and $m_{f \cdot h}$ is the multiplication map by $f \cdot h$ in $\mathbb{K}(\mathbf{y})[z]/\langle g \rangle$. Also, denote by $M_{H_g} \in \mathbb{K}(\mathbf{y})^{\deg_z(g) \times \deg_z(g)}$ the matrix of H_g in the basis $\{1, z, \ldots, z^{\deg_z(g)-1}\}$.

The following fact is an immediate consequence of [16, Thm.5.2, §5, Ch.2] and of the observation that, by performing euclidian divisions by g in $\mathbb{K}(\mathbf{y})[z]$, the denominator of the image of z^i in $\mathbb{K}(\mathbf{y})[z]/\langle g \rangle$ can only be a power of $\text{LC}_z(g)$.

Fact 11. *The denominators in the matrix M_{H_g} are powers of $\text{LC}_z(g)$. Moreover if $1 \leq \ell \leq \deg_z(g)$, then the points $\beta \in \overline{\mathbb{K}}^s \setminus V(\text{LC}_z(g))$ at which $g(\mathbf{y} = \beta, z)$ admits less than ℓ distinct solutions are precisely the points $\beta \in \overline{\mathbb{K}}^s \setminus V(\text{LC}_z(g))$ at which the $\ell \times \ell$-minors of M_{H_g} all vanish.*

Fact 11 implies that the conjunction of polynomial inequations answering Problem 8 is given by the non vanishing of $\text{LC}_z(g)$ and by the non vanishing of the $\ell \times \ell$-minors of M_{H_g}.

Example 12. *Assume that the $\ell \times \ell$-minors of M_{H_g} are $m_1, m_2, m_3, m_4 \in \mathbb{K}(\mathbf{y})$. The conjunction of inequations in $\mathbb{K}(\mathbf{y})$ is*

$$\text{LC}_z(g) \neq 0 \wedge m_1(\mathbf{y}) \neq 0 \wedge m_2(\mathbf{y}) \neq 0 \wedge m_3(\mathbf{y}) \neq 0 \wedge m_4(\mathbf{y}) \neq 0.$$

3.3   Change of monomial ordering

For n generic elements of $\mathbb{K}[x_1, \ldots, x_n]$ of degree d, computing a Gröbner basis of $\mathcal{I}$ for the order $>_{\text{grevlex}}$ has an arithmetic cost which is in $d^{O(n)}$ [2, Prop.1].

Let $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal whose solution set is finite. The following fact mentions an algorithm called FGLM [17] that on input a Gröbner basis of $\mathcal{I}$ for the order $>_{\text{grevlex}}$ outputs a Gröbner basis of $\mathcal{I}$ for the order $>_{\text{lex}}$. The complexity of this algorithm is, under some genericity assumptions, in $\tilde{O}(D^\omega)^2$ (see [21]), where D is the cardinality of $V(\mathcal{I})$ ($\mathcal{I}$ is assumed radical). By the Bézout bound, and with the notations of the above paragraph, we have the bound $D \leq d^n$, this makes the following fact useful for computing in $d^{O(n)}$ a Gröbner bases of $\mathcal{I}$ for a $>_{\text{lex}}$ order.

---

^{2}We denote by ω the constant of multiplication matrix that is, multiplying two matrices in $\mathbb{K}^{n \times n}$ can be done by using $\tilde{O}(n^\omega)$ arithmetic operations in $\mathbb{K}$, where $\tilde{O}(\cdot)$ is a $O(\cdot)$ that hides polylogarithmic factors.

Fact 13. *On input an ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ whose solution set $V(\mathcal{I}) \subset \overline{\mathbb{K}}^n$ is finite, there exists an algorithm that computes a Gröbner basis $G_{>_{grevlex}}$ of $\mathcal{I}$ for the order $>_{grevlex}$ and that turns it into a Gröbner basis $G_{>_{lex}}$ of $\mathcal{I}$ for the order $>_{lex}$.*

We mention that with the notations as above and without using the algorithm FGLM [17] mentioned in Fact 13, the computation of a Gröbner basis of $\mathcal{I}$ for the order $>_{lex}$ has an arithmetic cost bounded by $C_2 d^{C_3 n^3}$ [14], for $C_2, C_3 \in \mathbb{Z}_{\geq 0}$.

4 Function implemented in DDE-Solver

The function implemented in DDE-Solver is:

<div align="center">

annihilating_polynomial

</div>

We explain in this section the algorithms on which this function, and its options, rely.

The function **annihilating_polynomial** computes an annihilating polynomial of $F(t, a)$. Four algorithms can be applied: "duplication", "elimination", "geometry", "hybrid". These algorithms are studied respectively in [7, §3, §5, §6, §4]. As the algorithm "hybrid" relies on the three other algorithms, we first describe "duplication", "elimination" and "geometry".

In section 2, we defined solution sets that contain a point whose z_0-coordinate is $F(t, a)$:

- "duplication": $V(\mathcal{S}_{\mathsf{dup}})$ contains the point defined by (9),
- "elimination": $\mathcal{F}_k(u, \mathcal{X})$ contains the point $(F(t, a), \ldots, \partial_u^{k-1} F(t, a))$,
- "geometric": $\mathcal{S}_k(\mathcal{X})$ contains the point $(F(t, a), \ldots, \partial_u^{k-1} F(t, a))$.

We introduce a finiteness assumption that holds for the examples studied in this article.

<div align="center">

<u>Finiteness assumption:</u> The sets $V(\mathcal{S}_{\mathsf{dup}})$, $\mathcal{F}_k(u, \mathcal{X})$ and $\mathcal{S}_k(\mathcal{X})$ are finite. **(FA)**

</div>

General strategy   Under (**FA**), the general spirit of the algorithms from [7, §3, §5, §6] is:

- (i) To compute a disjunction of polynomial equations whose solution set is the set of interest,
- (ii) To eliminate all variables except (z_0, t) from the polynomial characterization of step (i).

We apply the above steps to the three approaches from section 2.

4.1 Option "duplication"

Step (i) of the general strategy: The polynomial system $\mathcal{S}_{\mathsf{dup}}$ characterizes $V(\mathcal{S}_{\mathsf{dup}})$.

Step (ii) of the general strategy: Formally, we denote by $\mathcal{I}_{\mathsf{dup}} \subset \mathbb{Q}(t)[m, \underline{x}, \underline{z}, \underline{u}]$ the ideal generated by the polynomials in $\mathcal{S}_{\mathsf{dup}}$. By assumption (**FA**), it is possible to apply to the ideal $\mathcal{I}_{\mathsf{dup}}$ the algorithm from Fact 13. This algorithm outputs, for a proper choice of variable ordering, a Gröbner basis $G_{>_{lex}}$ of $\mathcal{I}_{\mathsf{dup}}$ that contains an element of $\mathbb{Q}(t)[z_0]$. Applying Theorem 5 to the projection onto the z_0-coordinate space together with Theorem 4 yields, under (**FA**), that this polynomial is nonzero. Denote by $R \in \mathbb{Q}[t, z_0]$ its numerator. By [7, Prop. 2], we have $R(t, F(t, a)) = 0$.

We refer the reader to Annex 7.1 for the resolution of 3-constellations via this method.

4.2 Option "elimination"

Recall from Section 2 that $\mathcal{F}_k(u,\mathcal{X}) = \{\boldsymbol{\alpha} \in \overline{\mathbb{Q}(t)}^k \mid \#_u(\mathcal{X},\boldsymbol{\alpha}) \geq k\}$. The algorithm from [7, §5] requires technical assumptions that we do not explicit for sake of simplicity. Under these technical assumptions and (**FA**), the algorithm designed in [7, §5] works as follows.

Step (i) of the general strategy:

1. We compute using Theorems 4 and 6 successive disjunctions of conjunctions of polynomial equations and inequations defined in $\mathbb{K}(t)[u,\underline{z}]$ whose solution sets are successively the projections onto the $(u,\underline{z})$-coordinate space, then onto the $(\underline{z})$-coordinate space.

   The union of the polynomial constraints in these disjunctions have the form $\{I,E\}$, where I is a set of inequations defined in $\mathbb{Q}(t)[u,\underline{z}]$ and E is a set of equations defined in $\mathbb{Q}(t)[u,\underline{z}]$. Without loss of generality, we assume that the polynomials in E are a Gröbner basis of the ideal they generate, for the order $\succ_{\text{lex}}$ with u greater than $\underline{z}$.

2. We apply Theorem 6 to E: at $\underline{z} = \boldsymbol{\alpha} \in \overline{\mathbb{Q}(t)}^k$ fixed, the cardinality condition in the definition of $\mathcal{F}_k(u,\mathcal{X})$ is equivalent to studying for a polynomial g_{j_0} in E:

   - if g_{j_0} has its leading coefficient in u that does not vanish at $\underline{z} = \boldsymbol{\alpha}$ and, by Theorem 6, if g_{j_0} is the polynomial in E of minimal degree in u that satisfies this property: checking this minimality condition yields a disjunction of polynomial equations and inequations defined in $\mathbb{Q}(t)[\underline{z}]$,

   - for such a g_{j_0}, it remains to add a conjunction of polynomial conditions in $\mathbb{Q}(t)[u,\underline{z}]$ so that $g_{j_0}(u,\underline{z} = \boldsymbol{\alpha})$ has at least k distinct solutions:
     If $\deg_u(g_{j_0}) < k$, adds the vanishing of all the coefficients (in u) of g_{j_0} to E; else adds the conjunction of inequations given by the non vanishing of the $k \times k$-minors of $M_{H_{g_{j_0}}}$ from Fact 11.

3. Steps **1** and **2** above computed a disjunction of polynomial equations and inequations defined in $\mathbb{Q}(t)[u,\underline{z}]$ whose solution set in $\overline{\mathbb{Q}(t)}^{k+1}$ is $\mathcal{F}_k(u,\mathcal{X})$. As by (**FA**) this set is finite, this polynomial characterization can be turned into a disjunction of conjunction of polynomial equations by using Rabinowitsch trick[3] in order to remove the solution set given by the inequations: the introduced variables are eliminated using Theorem 4. We denote by $\mathcal{D}_{\text{elim}}$ the so-obtained disjunction.

Step (ii) of the general strategy: We perform this step in the same spirit as what we did in section 4.1. By (**FA**), the solution set of $\mathcal{D}_{\text{elim}}$ is finite. We can thus apply the algorithm from Fact 13 to each disjunction in $\mathcal{D}_{\text{elim}}$. Each application of this algorithm allows to compute a nonzero polynomial of $\mathbb{Q}(t)[z_0]$: this polynomial might be equal to 1 when there is no solution to the studied conjunction. We consider the numerators of all these polynomials and denote by $R \in \mathbb{Q}[t,z_0]$ their product. By [7, Prop. 5.5], we have $R(t,F(t,a)) = 0$.

We refer the reader to Annex 7.2 for the resolution of 3-constellations via this method.

4.3 Option "geometry"

Recall from Section 2 that

$$\mathcal{S}_k(\mathcal{X}) := \{\boldsymbol{\alpha} = (\alpha_0,\ldots,\alpha_{k-1}) \in \overline{\mathbb{Q}(t)}^k \mid \boldsymbol{\alpha} \in \pi(\mathcal{X}) \ \wedge \ \mathcal{X} \cap \pi_{z_1}^{-1}((\alpha_0,\alpha_2,\ldots,\alpha_{k-1})) \geq k\}.$$

---

[3]Instead of considering an inequation $Q \neq 0$ (for a given polynomial $Q \in \mathbb{K}[x_1,\ldots,x_n]$), we introduce an extra variable m and consider the equation $m \cdot Q - 1 = 0$ defined in $\mathbb{K}[m,x_1,\ldots,x_n]$.

The algorithm from [7, §6] requires many additional technical assumptions (see the beginning of [7, §6]). We do not make them explicit for sake of simplicity. For the rest of this section, we denote by $>_{\text{bgrevlex}}$ the block monomial order over $\mathbb{Q}(t)[m,x,u,\underline{z}]$ defined as follows:

- We use the monomial order $>_{\text{grevlex}}$ on each of the two blocks $\{m,x,u,z_1\}$ and $\{\check{z}_1\}$,
- Two monomials are compared w.r.t. the variables m,x,u,z_1. In case of equality, they are compared w.r.t. the variables $\check{z}_1$.

> **Example 14.** *For $k = 3$:*
> $$x^3 u z_1^2 m >_{\text{bgrevlex}} z_0^{20} z_2^3, \quad x^3 u z_1^2 m z_2 >_{\text{bgrevlex}} x^3 u z_1^2 m z_0 \quad \text{and} \quad z_0^4 >_{\text{bgrevlex}} z_0^2 z_2.$$

Under these assumptions and (**FA**), the algorithm designed in [7, §6] works as follows.

Step (i) of the general strategy:

1. Compute a Gröbner basis G of the ideal $\mathcal{I} := \langle P, \partial_x P, \partial_u P, m \cdot u(u-a) - 1 \rangle \subset \mathbb{Q}(t)[m,x,u,\underline{z}]$ for the order $>_{\text{bgrevlex}}$.

2. Compute, using normal form computations modulo $\mathcal{I}$[4], the matrix M_{z_1} of the multiplication map m_{z_1} in the quotient ring $\mathbb{Q}(t,\check{z}_1)[m,x,u,z_1]/j(\mathcal{I})$, where $j(\mathcal{I})$ is the image of the injective map $j : \mathbb{Q}(t)[m,x,u,\underline{z}] \to \mathbb{Q}(t,\check{z}_1)[m,x,u,z_1]$.

3. Compute the characteristic polynomial $\chi_{z_1} \in \mathbb{Q}(t,\check{z}_1)[T]$. Using the assumptions of [7, §6], we can replace χ_{z_1} by its numerator with respect to the variables $\check{z}_1$, so that we now have $\chi_{z_1} \in \mathbb{Q}(t)[\check{z}_1,T]$.

4. We denote by $\mathcal{D}_{\text{geom}}$ the set of equations given by the vanishing of the polynomials in G and of the new conditions $(\chi_{z_1})|_{T=z_1} = 0 \wedge \ldots \wedge (\partial_T^{k-1} \chi_{z_1})|_{T=z_1} = 0$.

Step (ii) of the general strategy: We perform this step in the same spirit as we did in section 4.1. By (**FA**), the solution set of $\mathcal{D}_{\text{geom}}$ is finite. We thus apply the algorithm from Fact 13 to the ideal generated by the polynomials associated with the equations in $\mathcal{D}_{\text{geom}}$. The application of this algorithm allows to compute a nonzero polynomial of $\mathbb{Q}(t)[z_0]$. We consider the numerator of this polynomial and denote it by $R \in \mathbb{Q}[t,z_0]$. By [7, Prop. 6.4], we have $R(t,F(t,a)) = 0$.

We refer the reader to Annex 7.3 for the resolution of 3-constellations via this method.

4.4 Option "hybrid"

The *hybrid guess-and-prove* algorithm [7, §4] works as follows:

1. Compute bounds b_{z_0}, b_t such that $\deg_t(R) \leq b_t$ and $\deg_{z_0}(R) \leq b_{z_0}$, for some nonzero polynomial $R \in \mathbb{Q}[t,z_0]$ annihilating $F(t,a)$ (we detail this step in the next section),

2. Compute the truncated series $F(t,a) \mod t^{2b_t b_{z_0}+1}$,

3. Guesses a polynomial $M \in \mathbb{Q}[t,z_0]$ such that $M(t,F(t,a)) = O(t^{(b_t+1)(b_{z_0}+1)-1})$,

4. Checks that $M(t,F(t,a)) = O(t^{2b_t b_{z_0}+1})$.

When step **4** is satisfied, applying [7, Prop.5] yields $M(t,F(t,a)) = 0$.

We refer the reader to Annex 7.4 for the resolution of 3-constellations via this method.

---

[4]A normal form computation modulo an ideal is the multivariate generalization of an euclidean division by a polynomial.

4.5 Details of implementations

As we target practical efficiency, the implementations of the options "duplication", "elimination" and "geometry" incorporate the following key improvements from computer algebra. Recall that the output of these three methods is a nonzero $R \in \mathbb{Q}[t, z_0]$ such that $R(t, F(t, a)) = 0$.

For reducing the computations from $\mathbb{Q}(t)$ to $\mathbb{Q}$, we perform evaluation-interpolation on t.

Also, we rely on fast multi-modular arithmetic. This consists in applying multiple times a given algorithm with the base field $\mathbb{Q}$ replaced by successive distinct prime fields $\mathbb{F}_p$. These computations output various images mod p of the same $R \in \mathbb{Q}[t, z_0]$. From there, one lifts the modular coefficients over $\mathbb{Q}$ by applying the Chinese Remainder Theorem (CRT) together with rational numbers reconstruction [18, §5.10].

Implementation of the option "hybrid": Computing the bounds (b_t, b_{z_0}) in step **1** is done using the option "elimination" of section 4.2. More precisely, computing b_{z_0} is done by specializing t at a random value θ of some random prime field $\mathbb{F}_p$ and performing the option "elimination" with base field $\mathbb{F}_p$: it outputs $R(\theta, z_0)$ and yields $b_{z_0} = \deg_{z_0}(R(\theta, z_0))$ (a similar computation is done for computing b_t). The generation of terms is done using a divide and conquer approach which computes the first terms of $F(t, u) \mod t^{2b_t b_{z_0} + 1}$ before specializing them to $u = a$. Finally, the guessing is done by computing Hermite-Padé approximants. For this, we use the function seriestoalgeq of the Maple package gfun [24].

5 Getting started

5.1 Installation and customization

Files provided: DDE-Solver is available in the ".mla" format. If one wishes to customize the package, the Maple scripts "ddesolver.mpl" and "build.mpl" can be downloaded on the dedicated github webpage. Used as follows, they allow to modify the package by modifying "ddesolver.mpl" and to generate the corresponding version of "ddesolver.mla" by executing "build.mpl".

Customization of DDE-Solver: Replacing "PATH/TO" in the Maple file "build.mpl" by the relevant path to the file where "ddesolver.mpl" and "build.mpl" are located allows one, by executing "build.mpl", to generate a new version of "ddesolver.mla".

Loading DDE-Solver: The Maple variable libname shall be set so that "ddesolver.mla" is located in a visible place.

> libname := "/home/notarantonio/ddesolver/lib", libname:

Once libname has been correctly set up, one executes in Maple

> with(ddesolver);

in order to load and use the package.

5.2 Using DDE-Solver

5.2.1 Input/Output syntax

The arguments of **annihilating_polynomial** are (P, k, var), where $P \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u]$ is the polynomial in (4), k is the order of the DDE and var is $[x, z_0, \dots, z_{k-1}, t, u]$ (in that order). The output of **annihilating_polynomial**(P, k, var) is some nonzero polynomial $R \in \mathbb{Q}[t, z_0]$ such that $R(t, F(t, a)) = 0$.

Example 15. *Consider the DDE [9, Eq. 29] of the enumeration of 3-constellations:*

$$F(t, u) = 1 + tuF(t, u)^3 + tu(2F(t, u) + F(t, 1))\frac{F(t, u) - F(t, 1)}{u - 1} \tag{14}$$
$$+ tu\frac{F(t, u) - F(t, 1) - (u - 1)\partial_u F(t, 1)}{(u - 1)^2}.$$

Multiplying (14) by $(u - 1)^2$ yields

$$0 = (u - 1)^2(1 - F(t, u) + tuF(t, u)^3) + tu(u - 1)(2F(t, u) + F(t, 1))(F(t, u) - F(t, 1))$$
$$+ tu(F(t, u) - F(t, 1) - (u - 1)\partial_u F(t, 1)).$$

Thus $P := (u-1)^2(1 - x + tux^3) + tu(u-1)(2x + z_0)(x - z_0) + tu(x - z_0 - (u-1)z_1)$ and $k := 2$. We continue the analysis with Maple

> $P := (u - 1)^2(1 - x + tux^3) + tu(u - 1)(2x + z_0)(x - z_0) + tu(x - z_0 - (u - 1)z_1);$
> $with(ddesolver):$
> $annihilating\_polynomial(P, 2, [x, z_0, z_1, t, u]);$
> $(16tz_0^2 - 8tz_0 + t - 16)(81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$

Thus $R := (16tz_0^2 - 8tz_0 + t - 16)(81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$ is an annihilating polynomial of $F(t, a)$. A direct analysis on the solutions of $R(t, z_0)$ that are finite at $t = 0$ shows that the second factor of R is the minimal polynomial of $F(t, 1)$.

Options It is possible to benefit in practice from two options: the choice of the *algorithm*, and the choice of the *variable* (either t or z_0) on which we perform evaluation–interpolation.

Choice of the algorithm Four algorithms are implemented: "duplication", "elimination", "geometry" (only when $k = 2$) and "hybrid". By default, the algorithm used is "elimination".
Choice of the variable The two choices are t and z_0. The default choice is t.
The choice of the algorithm and of the variable on which we perform evaluation–interpolation can be made by executing **annihilating_polynomial**$(P, k, \text{var}, \text{algorithm}, \text{variable})$. Note that these two options must be either not specified at all, or specified in the same call.

Example 16. *We continue Example 15 with* algorithm = *"geometry" and* variable = z_0.

> $annihilating\_polynomial(P, 2, [x, z_0, z_1, t, u], "geometry", z_0);$
> $(16tz_0^2 - 8tz_0 + t - 16)(81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$

We refer to the next section for a discussion on the choice of the parameter variable.

6 Examples

6.1 Impact of the option variable

The motivation of this study is the following. Assume that we perform evaluation–interpolation on z_0. The chosen algorithm (e.g. "elimination" from section 4.1) will be run a first time with t specialized (in order to get the degree in z_0 of the output $R \in \mathbb{Q}[t, z_0]$), then it will be run multiple times at z_0 specialized (in order to evaluate–interpolate R w.r.t z_0). Hence if the algorithm is faster at t specialized than at z_0 specialized (and if the partial degrees of the output R are "close"), then the choice of the evaluation–interpolation variable has a significant impact on the timings. This is what happens below.

Example 17. *Consider the DDE of the enumeration of 3-Tamari lattices [8, Prop. 8]:*

$$F(t,u) = u + tuF(t,u)\frac{F(t,u)\frac{\frac{F(t,u)-F(t,1)}{u-1}-F(t,1)\partial_u F(t,1)}{u-1} - \frac{F(t,1)^2\partial_u^2 F(t,1)}{2} - F(t,1)\partial_u F(t,1)^2}{u-1}, \quad (15)$$

Multiplying (15) by $(u-1)^3$ yields the functional equation $P(F(t,u), F(t,1), \partial_u F(t,1), t, u) = 0$, where $P := -tu^3 xz_0^2 z_2 - 2tu^3 xz_0 z_1^2 - 2tu^2 x^2 z_0 z_1 + 2tu^2 xz_0^2 z_2 + 4tu^2 xz_0 z_1^2 + 2tux^4 - 2tux^3 z_0 + 2tux^2 z_0 z_1 - tuxz_0^2 z_2 - 2tuxz_0 z_1^2 + 2u^4 - 2u^3 x - 6u^3 + 6u^2 x + 6u^2 - 6ux - 2u + 2x \in \mathbb{Q}[x, z_0, z_1, z_2, t, u]$. We thus have in Maple

> $P :=$
> $-tu^3 xz_0^2 z_2 - 2tu^3 xz_0 z_1^2 - 2tu^2 x^2 z_0 z_1 + 2tu^2 xz_0^2 z_2 + 4tu^2 xz_0 z_1^2 + 2tux^4 - 2tux^3 z_0 +$
> $2tux^2 z_0 z_1 - tuxz_0^2 z_2 - 2tuxz_0 z_1^2 + 2u^4 - 2u^3 x - 6u^3 + 6u^2 x + 6u^2 - 6ux - 2u + 2x :$
> $with(ddesolver):$
> $time(annihilating\_polynomial(P, 2, [x, z_0, z_1, z_2, t, u]));$
> $\hspace{4cm} \text{# variable} = t \text{ (implicitly chosen)}$
>
> $\hspace{3cm} 2089.599$
> $time(annihilating\_polynomial(P, 2, [x, z_0, z_1, z_2, t, u], "elimination", z_0));$
> $\hspace{6cm} \text{# variable} = z_0$
>
> $\hspace{3cm} 141.897$
> $R := annihilating\_polynomial(P, 2, [x, z_0, z_1, z_2, t, u], "elimination", z_0);$
>
> $R := t^5 z_0^{16} + 135t^4 z_0^{13} + 1024t^4 z_0^{12} + 7290t^3 z_0^{10} - 1762560t^3 z_0^9 + 393216t^3 z_0^8 +$
> $196830t^2 z_0^7 + 111694464t^2 z_0^6 + 580976640t^2 z_0^5 + t(67108864t + 2657205)z_0^4 -$
> $661978656tz_0^3 + 4721836032tz_0^2 + (-8371830784t + 14348907)z_0 +$
> $4294967296t - 14348907$

6.2 Practical results of the function annihilating_polynomial

We provide below a table gathering timings obtained after using the function

annihilating_polynomial.

Each column is associated with a DDE taken in the literature [4, 9, 8, 12]. For each of these DDEs, we precise: its order k, the variable on which we perform evaluation–interpolation, the algorithm which is used in the call to **annihilating_polynomial**, and finally the bi-degree of the output polynomial $R \in \mathbb{Q}[t, z_0]$.

14

Data	[1]		[2]		[3]		[4]		[5]		[6]		[7]		[8]	
k	2		2		2		2		3		3		3		4	
eval–interp variable	z_0	t	z_0	t	z_0	t	z_0	t	z_0	t	z_0	t	z_0	t	z_0	t
"duplication"	0.5s	0.7s	1.6s	2.3s	41m*	10m*	2.7s	1.5s	2m45s	2m50s	13h	27h	∞	∞	∞	∞
"elimination"	**0.2s**	0.3s	**0.4s**	**0.4s**	1h20m*	4m*	1.7s	1.2s	1m10s	**45s**	**2m20s**	35m	1m	7m30s	2d19h	2d
"geometry"	2s	2.2s	1.6s	0.9s	54m*	**2m***	0.8s	**0.7s**	×	×	×	×	×	×	×	×
"hybrid"	50s		2m		∞		1.4s		18s		1h42m		**34s**		**2h41m**	
$(\deg_t(R), \deg_{z_0}(R))$	(4, 9)		(2, 3)		(132, 6)		(3, 5)		(3, 7)		(5, 16)		(2, 4)		(3, 9)	

Table 1: Practical results[5][6] of **annihilating–polynomial** on DDEs from the literature.

- [1]: Enumeration of 2-Tamari lattices, [8, Prop. 8] for $m = 3$,
- [2]: Enumeration of 2-greedy Tamari intervals, [12, Prop. 3.1] for $m = 2$,
- [3]: Enumeration of non-separable near-triangulations in which all intern vertices have degree at least 5, [4, Prop. 4.3, Eq. 22],
- [4]: Enumeration of 3-constellations, [9, Equation 29],
- [5]: Enumeration of 4-constellations, [9, Prop. 12] for $m = 4$,
- [6]: Enumeration of 3-Tamari lattices, [8, Prop. 8] for $m = 4$,
- [7]: Enumeration of 3-greedy Tamari intervals, [12, Prop. 3.1] for $m = 3$,
- [8]: Enumeration of 5-constellations. [9, Prop. 12] for $m = 5$.

- ∞: The computations did not finish within 5 days,
- ×: The algorithm is not implemented for $k > 2$,
- *: We added the condition $m' \cdot (1 + t^3)(1 - t^3)t - 1 = 0$ to the input polynomial constraints.

From Table 1, we strengthen the message of section 6.1 and draw some conclusions:

- *Choice of the algorithm* The best algorithm to choose depends in an important way on the studied DDE. Even among DDEs of the same family (e.g. DDEs [4], [5] and [8]) the answer is not clear: the first two are solved faster with the option algorithm ="elimination" while the last one is solved faster with the option algorithm= "hybrid". The efficiency of this last option algorithm ="hybrid" comes from the fact that the output polynomial has small bidegree $(3, 9)$, thus allowing a fast computation of the first terms of $F(t, 1)$; and at the same time, that the computations are hard with the other algorithms.

  This impossibility to predict which algorithm is more efficient for an input DDE justifies that we let the user choose the algorithm that shall be applied.

- *Choice of the variable* As in section 6.1, the table shows that the choice of the variable (z_0 or t) on which we perform evaluation–interpolation is important. Let us pick one example to illustrate it. For the DDE [3], performing evaluation–interpolation over t with algorithm ="geometry" takes 2 minutes while it takes 54 minutes when performing evaluation–interpolation over z_0. With the notations of section 4.3, the reason is that at z_0 specialized to some random $\theta \in \mathbb{F}_p$, the main computational cost (in addition to the Gröbner basis computation) is to compute the discriminant w.r.t. T of $(\chi_{z_1})|_{z_0=\theta} \in \mathbb{Q}[t, T]$, whose partial degrees are 6 in T and 423 in t. In the same time at t specialized to some random $\nu \in \mathbb{F}_p$, the polynomial $(\chi_{z_1})|_{t=\nu} \in \mathbb{Q}[z_0, T]$ has degree 15 in z_0 and 6 in T: it yields a faster computation of its discriminant w.r.t. T. As all these intermediate computational data are hard to predict from the input DDE, we let the user choose on which variable we shall perform evaluation–interpolation.

- *Solving DDEs previously out of reach!* The DDEs [7], [8] are solved via theoretical arguments in (resp.) [12, 11]. Here, it is the first time that they are solved in an automatized way!

7 Appendix

7.1 Solving 3-constellations using section 4.1

The DDE [9, Eq. 29] associated with the enumeration of 3-constellations is given by

$$F(t, u) = 1 + tuF(t, u)^3 + tu(2F(t, u) + F(t, 1)) \frac{F(t, u) - F(t, 1)}{u - 1} \tag{16}$$
$$+ tu \frac{F(t, u) - F(t, 1) - (u - 1)\partial_u F(t, 1)}{(u - 1)^2}$$

Multiplying (16) by $(u - 1)^2$ yields $P(F(t, u), F(t, 1), \partial_u F(t, 1), t, u) = 0$, where

$$P := (u - 1)^2(1 - x + tux^3) + tu(u - 1)(2x + z_0)(x - z_0) + tu(x - z_0 - (u - 1)z_1) \in \mathbb{Q}[x, z_0, z_1, t, u].$$

Define the polynomial system

$$\mathcal{S}_{\text{dup}} := (P(x_1, z_0, z_1, t, u_1) = 0, \partial_x P(x_1, z_0, z_1, t, u_1) = 0, \partial_u P(x_1, z_0, z_1, t, u_1) = 0,$$
$$P(x_2, z_0, z_1, t, u_2) = 0, \partial_x P(x_2, z_0, z_1, t, u_2) = 0, \partial_u P(x_2, z_0, z_1, t, u_2) = 0,$$
$$m \cdot (u_1 - u_2) \cdot (u_1 - 1) \cdot (u_2 - 1) \cdot u_1 \cdot u_2 - 1 = 0),$$

and set $\mathcal{I}_{\text{dup}} \subset \mathbb{Q}(t)[m, x_1, x_2, z_0, z_1, u_1, u_2]$ to be the ideal generated by the polynomials in $\mathcal{S}_{\text{dup}}$.

The algorithm described in section 4.1:

16

1. Computes a generator $R \in \mathcal{I}_{\mathsf{dup}} \cap \mathbb{Q}(t)[z_0]$ by applying Fact 13 and finds

$$R = (81t^2 z_0^3 - 81t^2 z_0^2 + 27t^2 z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)(16tz_0^2 - 8tz_0 + t - 16).$$

7.2 Solving 3-constellations using section 4.2

Define $\mathcal{S} := (P, \partial_x P, \partial_u P, m \cdot u(u-1) - 1)$, for P as in section 7.1. Denote by $\mathcal{X} \subset \overline{\mathbb{Q}(t)}^5$ the solution set of the conditions $P = 0 \wedge \partial_x P = 0 \wedge \partial_u P = 0 \wedge u(u-1) \neq 0$ in $\overline{\mathbb{Q}(t)}^4$.

Goal: Characterize $\mathcal{F}_k(u, \mathcal{X}) := \{\alpha \in \overline{\mathbb{Q}(t)}^2 \mid \#_u(\mathcal{X}, \alpha) \geq 2\}$ as the solution set of some polynomial constraints and deduce a nonzero $R \in \mathbb{Q}[t, z_0]$ such that $R(t, F(t, 1)) = 0$.

The algorithm described in section 4.2:

1. Characterizes the projection of $\mathcal{X}$ onto the (u, z_0, z_1)-coordinate space:

   > $G := \text{Groebner[Basis]}(\mathcal{S}, \text{lexdeg}([m, x], [u, z_0, z_1, t]))$:
   > $E := \text{Groebner[Basis]}(\text{remove(has}, G, \{m, x\}), \text{lexdeg}([u], [z_0, z_1, t]))$:
   > $Leading := \text{select(has}, G, \{m, x\})$:

   The first element of $Leading$ has the form $1534(u-1) \cdot x + (\text{polynomial in } u, z_0, z_1, t)$. As $\mathcal{X} \cap \{u = 1\} = \emptyset$, it results from the extension theorem that the zero set of E in $\overline{\mathbb{Q}(t)}^3$ is precisely the projection of $\mathcal{X}$ onto the (u, z_0, z_1)-coordinate space.

2. Identifies algebraic conditions in E describing the cardinality of the fiber in $\mathcal{F}_k(u, \mathcal{X})$:

   > **# Dealing with the polynomials of degree < 2 in u in E:**
   > $NewConditions := []$;
   > for i to nops(E) do
   >   if degree($E[i], u$) < 2 then
   > $NewConditions := [\text{op}(NewConditions), \text{coeffs}(E[i], u)]$:
   >   fi:
   > od:
   > $E := \text{Groebner[Basis]}([\text{op}(E), \text{op}(NewConditions)], \text{lexdeg}([u], [z_0, z_1, t]))$:
   >
   > **# Dealing with the polynomials of degree ≥ 2 in u in E:**
   > $\text{map}(p \rightarrow \text{degree}(p, u), E)$; # prints the degrees in u of the polynomials in E
   >
   > $$[0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3]$$
   >
   > # In order to have $\#_u(\mathcal{X}, \alpha) \geq 2$ for a given $\alpha \in \overline{\mathbb{Q}(t)}^2$,
   > # we shall exclude the solution set of $\text{disc}_u(E[10])$:
   > $sat := \text{discrim}(E[10], u)$:

   Denote by $c_{10}(z_0, z_1) \in \mathbb{Q}(t)[z_0, z_1]$ the coefficients of u^2 in $E[10]$. As $V(E) \cap \{c_{10} = 0\} = \emptyset$, the algorithm deduces from Theorem 6 that $\mathcal{F}_k(u, \mathcal{X})$ is the solution set of the polynomial equations given by E and of the inequation $sat \neq 0$.

3. Turns the inequation $sat \neq 0$ into the equation $m \cdot sat - 1 = 0$ by introducing an extra variable m. Finally, it deduces a nonzero polynomial $R \in \mathbb{Q}[t, z_0]$ annihilating $F(t, 1)$.

> $H := \text{Groebner}[\text{Basis}]([\text{op}(E), m \cdot sat - 1], \text{lexdeg}([m, u, z_1], [z_0, t]))$:
> $\text{op}(\text{remove}(\text{has}, H, \{m, u, z_1\}))$;
>
> $(16tz_0^2 - 8tz_0 + t - 16)(81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1)$

Thus

$$R = (16tz_0^2 - 8tz_0 + t - 16)(81t^2z_0^3 - 81t^2z_0^2 + 27t^2z_0 + 18tz_0^2 - 3t^2 - 66tz_0 + 47t + z_0 - 1).$$

7.3   Solving 3-constellations using section 4.3

Denote by $P \in \mathbb{Q}[x, z_0, z_1, t, u]$ the polynomial in section 7.1 and by $\mathcal{S}$ the set of polynomials $(P, \partial_x P, \partial_u P, m \cdot tu(u-1) - 1) \subset \mathbb{Q}(t)[m, x, z_0, z_1, u]$. Moreover, recall that $\mathcal{X} \subset \overline{\mathbb{Q}(t)}^4$ is the solution set of the constraints $P = 0 \wedge \partial_x P = 0 \wedge \partial_u P = 0 \wedge u(u-1) \neq 0$ and that

$$\mathcal{S}_k(\mathcal{X}) := \{\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{k-1}) \in \overline{\mathbb{Q}(t)}^k \mid \boldsymbol{\alpha} \in \pi(\mathcal{X}) \ \wedge \ \mathcal{X} \cap \pi_{z_1}^{-1}((\alpha_0, \alpha_2, \dots, \alpha_{k-1})) \geq k\}.$$

Goal: Characterize $\mathcal{S}_k(\mathcal{X})$ as the solution set of some polynomial constraints and deduce a nonzero $R \in \mathbb{Q}[t, z_0]$ such that $R(t, F(t, 1)) = 0$.

The algorithm described in section 4.3:

1. Computes a Gröbner basis of the ideal generated by $\mathcal{S}$ for the order $\succ_{\text{bgrevlex}}$:

> $G := \text{Groebner}[\text{Basis}](\mathcal{S}, \text{lexdeg}([m, x, u, z_1], [t, z_0]))$:

2. Computes the matrix M_{z_1} of the multiplication map $m_{z_1} : f \mapsto z_1 \cdot f$ in the quotient ring $\mathbb{Q}(t, z_0)[m, x, u, z_1]/\langle j(\mathcal{S}) \rangle$, where j is the usual inclusion map of $\mathbb{Q}(t)[m, x, z_0, z_1]$ in $\mathbb{Q}(t, z_0)[m, x, u, z_1]$:

> $L_1, L_2 := \text{Groebner}[\text{NormalSet}](\text{subs}(t = rand(), z_0 = rand(), G), \text{tdeg}(m, x, u, z_1))$:
> $M := \text{Groebner}[\text{MultiplicationMatrix}](z_1, L_1, L_2, G, \text{tdeg}(m, x, u, z_1))$:

3. Computes the characteristic polynomial of M_{z_1} and defines χ_{z_1} to be its numerator:

> $\xi_{z_1} := \text{LinearAlgebra}[\text{CharacteristicPolynomial}](M, T)$:
> $\chi_{z_1} := \text{factor}(\text{numer}(\xi))$;
>
> $(\mathbf{T}t + tz_0^2 - 1) \cdot (729\mathbf{T}^4t^4 + 108t^3(27tz_0^2 - 9tz_0 + 2t - 12)\mathbf{T}^3 + 2t(2187t^3z_0^4 - 1350t^3z_0^3 + 459t^3z_0^2 - 72t^3z_0 - 2376t^2z_0^2 + 8t^3 + 1728t^2z_0 - 492t^2 - 36tz_0 + 276t - 8)\mathbf{T}^2 + 4t(729t^3z_0^6 - 621t^3z_0^5 + 261t^3z_0^4 - 55t^3z_0^3 - 1404t^2z_0^4 + 6t^3z_0^3 + 1536t^2z_0^3 - 560t^2z_0^2 - 36tz_0^3 + 68t^2z_0 + 640tz_0^2 - 8t^2 - 916tz_0 - 8z_0^2 + 328t + 8z_0)\mathbf{T} + 16 + 358t^4z_0^6 - 84t^4z_0^5 + 9t^4z_0^4 - 1160t^3z_0^4 - 3224t^2z_0^3 + 1320t^2z_0^2 - 128t^2z_0 - 720tz_0^2 + 729t^4z_0^8 - 756t^4z_0^7 - 2160t^3z_0^6 + 2752t^3z_0^5 - 72t^2z_0^5 + 2104t^2z_0^4 - 16tz_0^4 + 16t^2 - 32z_0 - 544t + 96tz_0^3 - 24t^3z_0^2 + 208t^3z_0^3 + 1184tz_0 + 16z_0^2)$

4. Thus $\mathcal{S}_k(\mathcal{X})$ is the solution set of the constraints $P = 0 \wedge \partial_x P = 0 \wedge \partial_u P = 0 \wedge u(u-1) \neq 0 \wedge \text{disc}_T(\chi_{z_1}) = 0$. An annihilating polynomial of $F(t, 1)$ is given by the discriminant $R \in \mathbb{Q}[t, z_0]$ of χ_{z_1} with respect to T, that is

$$R = 12230590464t^{14} \cdot (16tz_0^2 - 8tz_0 + t - 16)^2 \cdot (1 + (3z_0 - 1)^2t^2 + (-6z_0 - 14)t)^3$$
$$\cdot (tz_0 + 1)^{12} \cdot (81t^2z_0^3 - 9t(9t - 2)z_0^2 + (27t^2 - 66t + 1)z_0 - 3t^2 + 47t - 1)^2.$$

18

7.4 Solving 3-constellations using section 4.4

Consider (16) and the polynomial $P \in \mathbb{Q}[x, z_0, z_1, t, u]$ of section 7.1. Define the set of polynomials $\mathcal{S} := (P, \partial_x P, \partial_u P, m \cdot u(u-1) - 1) \in \mathbb{Q}(t)[m, x, z_0, z_1, u]$.

Goal: Compute a nonzero polynomial $R \in \mathbb{Q}[t, z_0]$ such that $R(t, F(t, 1)) = 0$.

The algorithm described in section 4.4:

1. Applies twice the strategy described in section 7.2 to P. It draws at random a prime number p and some $\theta \in \mathbb{F}_p$, say $p = 12301$ and $\theta = 1328$, and applies section 7.2 with $\mathcal{S}$ replaced by $(\mathcal{S})|_{t=\theta}$ and $\mathbb{Q}$ replaced by $\mathbb{F}_p$. It outputs

$$7957(z_0 + 11829)(z_0^3 + 4863z_0^2 + 8711z_0 + 3012)(z_0 + 6622)$$

   The degree being 5, it sets $b_{z_0} = 5$. Similarly, it obtains $b_t = 3$.

2. Computes the 32 first terms of $F(t, 1)$:

$$F_1 = 1 + t + 6t^2 + 54t^3 + 594t^4 + \cdots + 9130759946511565848406513262326325946t^{31} \bmod t^{32}$$

3. Uses the function seriestoalgeq of the Maple package gfun [24]:

   > collect(subs($T(t) = z_0$, gfun[seriestoalgeq](series(F_1, t, degree(F_1, t)), $T(t)$))[1], z_0, factor);
   > $$81t^2z_0^3 - 9t(9t - 2)z_0^2 + (27t^2 - 66t + 1)z_0 - 3t^2 + 47t - 1$$

   The above polynomial, denoted M, is a candidate for annihilating $F(t, 1)$.

4. Proves the guessed polynomial with the below lines

   > series(subs($z_0 = F_1, M$), t, 31); # Compute $M(t, F(t, 1) \bmod t^{32})$
   > $$O(t^{31})$$

   Thus
   $$R = 81t^2z_0^3 - 9t(9t - 2)z_0^2 + (27t^2 - 66t + 1)z_0 - 3t^2 + 47t - 1.$$

8 Acknowledgement

References

[1] C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. volume 281, pages 37–80. 2002. Selected papers in honour of Maurice Nivat.

[2] M. Bardet, J.-C. Faugère, and B. Salvy. On the complexity of the F_5 Gröbner basis algorithm. *J. Symbolic Comput.*, 70:49–70, 2015.

[3] E. A. Bender and E. R. Canfield. The number of degree-restricted rooted maps on the sphere. *SIAM J. Discrete Math.*, 7(1):9–15, 1994.

[4] O. Bernardi. On triangulations with high vertex degree. *Ann. Comb.*, 12(1):17–44, 2008.

[5] J. Berthomieu, C. Eder, and M. Safey El Din. `msolve`: a library for solving polynomial systems. In *ISSAC'21*, pages 51–58. ACM, New York, [2021] ©2021.

[6] A. Bostan, F. Chyzak, H. Notarantonio, and M. S. El Din. Algorithms for discrete differential equations of order 1. In *ISSAC '22—Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, pages 101–110. ACM, New York, [2022] ©2022.

[7] A. Bostan, H. Notarantonio, and M. Safey El Din. Fast algorithms for discrete differential equations. In *Proceedings of the International Symposium on Symbolic & Algebraic Computation (ISSAC 2023)*, pages 80–89. ACM, New York, [2023] ©2023.

[8] M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the *m*-Tamari lattices. *Electron. J. Combin.*, 18(2):Paper 31, 26, 2011.

[9] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623–672, 2006.

[10] M. Bousquet-Mélou and M. Petkovšek. Linear recurrences with constant coefficients: the multivariate case. volume 225, pages 51–75. 2000. Formal power series and algebraic combinatorics (Toronto, ON, 1998).

[11] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[12] M. Bousquet-Mélou and F. Chapoton. Intervals in the greedy tamari posets, 2023.

[13] W. G. Brown. On the existence of square roots in certain rings of power series. *Math. Ann.*, 158:82–89, 1965.

[14] L. Caniglia, A. Galligo, and J. Heintz. Some new effectivity bounds in computational geometry. In *International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, 1988.

[15] D. A. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.

[16] D. A. Cox, J. B. Little, and D. O'Shea. *Using Algebraic Geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, first edition, 1998.

[17] J. C. Faugère, P. Gianni, D. Lazard, and T. Mora. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *J. Symbolic Comput.*, 16(4):329–344, 1993.

[18] J. von zur Gathen and J. Gerhard. *Modern computer algebra*. Cambridge Univ. Press, third edition, 2013.

[19] I. M. Gessel and D. Zeilberger. An empirical method for solving (rigorously!) algebraic functional equations of the form f(p(x,t), p(x,1),x,t)=0, 2014.

[20] D. E. Knuth. *The art of computer programming*, volume 1. Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1968.

[21] V. Neiger and E. Schost. Computing syzygies in finite dimension using fast linear algebra. *J. Complexity*, 60:101502, 30, 2020.

[22] G. Polya. *How to Solve It. A New Aspect of Mathematical Method*. Princeton University Press, Princeton, N. J., 1948.

[23] D. Popescu. General Néron desingularization and approximation. *Nagoya Math. J.*, 104:85–115, 1986.

[24] B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[25] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[26] D. Zeilberger. A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length *n* is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete Math.*, 102(1):85–93, 1992.