

Algorithms for solving fixed point equations of order 1

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Joint work with:

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The Inria logo is written in a red, cursive script font.

Motivation: A non linear equation coming from combinatorics...

\mathbb{K} field of characteristic 0.

$\mathbb{K} = \mathbb{Q}, \mathbb{C}, \dots$

Starting point: F , solution in $\mathbb{K}[u][[t]]$ of the fixed point equation (FPE) of order 1

$$F(t, u) = 1 + tu(uF(t, u)^2 + F(t, u) + \Delta F(t, u)),$$

where Δ is the divided difference operator $\Delta F(t, u) := \frac{F(t, u) - F(t, 1)}{u - 1}$.

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Goals:

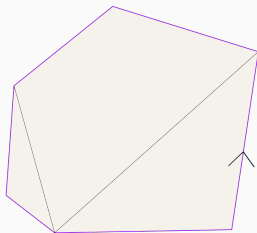
- Compute a polynomial $R \in \mathbb{K}[t, z] \setminus \{0\}$ such that $R(t, F(t, 1)) = 0$.
- Estimate the size of R for any (FPE).
- Complexity estimates (ops. in \mathbb{K}) for the computation of R .

Count

$$c_n := \# \{ \text{planar maps with } n \text{ edges} \}$$

↓ refinement

$$c_{n,d} := \# \{ \text{planar maps with } n \text{ edges,} \\ \text{ } d \text{ of them on the external face} \}$$

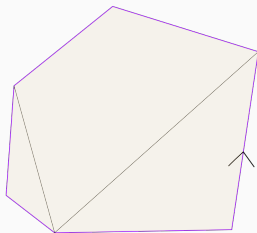


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$$G(t) := \sum_{n=0}^{\infty} c_n t^n \quad \text{generating function}$$

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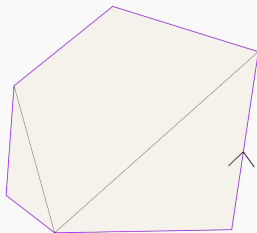
$$F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^n c_{n,d} u^d t^n \\ \text{complete generating function}$$

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complete generating function

(FPE) of order 1 [Tutte '68]

$$F(t, u) = 1 + t u^2 F(t, u)^2 \\ + t u \frac{u F(t, u) - F(t, 1)}{u - 1}$$

Theorem [Bousquet-Mélou, Jehanne '06]

Let $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y, t, u]$. Let $F(t, u)$ be the *unique* solution in $\mathbb{K}[u][[t]]$ of

$$F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u), \quad (\text{FPE})$$

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Then F is **algebraic** over $\mathbb{K}(t, u)$.

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Systematic methods

[Tutte, Brown 60's], [Zeilberger '92]:

Guess-and-prove

[Brown '65]:

Quadratic method

[Popescu '86]:

Algebraicity result

[Bousquet-Mélou, Jehanne '06]:

Polynomial elimination

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Algorithms

[Knuth '68], [Banderier, Flajolet '02]:

Kernel method (linear case)

[Bousquet-Mélou, Jehanne '06]:

Polynomial elimination

[Gessel, Zeilberger '14]:

Guess-and-prove

Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]: Take $P \in \mathbb{K}[x, z, t, u]$ the "numerator" of (FPE)

(FPE) of order 1

→

solution $u = U(t) \in \mathbb{K}[[t]]$ of
 $\partial_x P(F(t, u), F(t, 1), t, u) = 0$

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$$\begin{cases} P(x, z, t, u) = 0, \\ \partial_x P(x, z, t, u) = 0, \\ \partial_u P(x, z, t, u) = 0. \end{cases}$$

$(F(t, U(t)), F(t, 1), U(t))$
zero in $\mathbb{K}[[t]]^3$

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$$\begin{aligned} 0 = (1 - F(t, u))(u - 1) + tu^2(u - 1)F(t, u)^2 \\ + tu(uF(t, u) - F(t, 1)) \end{aligned}$$

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$$0 = \partial_u F(t, u) \cdot (1 - u + 2tu^2(u - 1)F(t, u)^2 + tu^2) \\ + (1 - F(t, u) + tu(3u - 2)F(t, u)^2 + 2tu^2F(t, u) - tF(t, 1))$$

1. Geometric **refinements** of a method based on discriminants,
2. A new guess-and-prove method based on **geometry**,
3. A **complexity result** on the resolution of (FPE) of order 1.

Attention is paid to

- assumptions,
- degree bounds on the output,
- complexity estimates,
- potential for generalization.

Input: $P := \text{numerator}(\text{FPE})$,
Goal: $\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z]$.

$\text{disc}_x(P) = \text{Res}_x(P, \partial_x P)$ the discriminant of P in x .

Theorem [Bousquet-Mélou, Jehanne '06]

Suppose $\deg_x(P) \geq 2$ and $u = U(t) \in \mathbb{K}[[t]]$ is a root of

$$\partial_x P(F(t, u), F(t, 1), t, u).$$

Then $u = U(t)$ is a **double root** of $\text{disc}_x(P)(F(t, 1), t, u)$.

Hence, $F(t, 1)$ is a root of $\text{disc}_u(\text{disc}_x(P))$.

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$$P := (1-x)(u-1) + tu^2(u-1)x^2 \\ + tu(ux-z)$$

gives $\text{disc}_u(\text{disc}_x P)$ equal to

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$$P := 97t^3u^2 + (-73u^4 - 56u^2x^2 + 87u^2x - 62x^2 + 124xz - 62z^2)t - xu^2 + u^2$$

gives $\text{disc}_x P$ equal to

$$-16352t^2u^6 \\ + (21728t^4 - 10535t^2 + 50t + 1)u^4 \\ + 248t(97t^3 - 56tz^2 + 87tz - z + 1)u^2$$

which has a double root at $u = 0$.

Contribution 1: ensuring non-nullity of double discriminant

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Suppose

$$\delta = \deg(P)$$

- **(H0)** $\deg_x(P) \geq 2$,
- **(H1)** $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- **(R)** the zero set $V(P) \subset \overline{\mathbb{K}}^4$ is smooth outside $V(u - 1) \subset \overline{\mathbb{K}}^4$.

Set $D_0 := \text{disc}_x P$, $D_1 := \text{SqFreePart}(D_0)$ and $D_2 := \text{disc}_u D_1$.

Then

- $R := \text{SqFreePart}(D_2)$ is **non-zero** in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.

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- $R := \text{SqFreePart}(D_2)$ is **non-zero** in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.
- R has total size $16\delta^8$ with degree in each variable at most $4\delta^4$,
- R can be computed in $O_{\log}(\delta^{10})$ ops. in \mathbb{K} .

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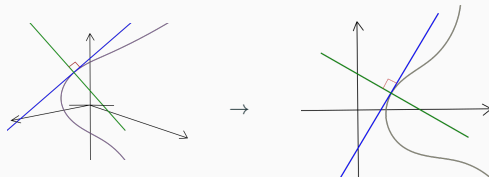
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$D_1 := \text{SqFreePart}(\text{disc}_x(P))$ satisfies
 $\partial_u D_1(U(t), F(t, 1), t) = 0$.

$$\begin{cases} (\partial_u D_1 \ \partial_z D_1 \ \partial_t D_1) \cdot (u \ z \ t)^T = 0, \\ (\partial_z D_1 \ \partial_t D_1) \cdot (z \ t)^T = 0 \end{cases}$$

Contribution 1 (cont'd): using geometry arguments to refine the complexity

$P \in \mathbb{K}[x, z, t, u]$ and $\delta := \deg(P)$.

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Suppose

- **(H1)** $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- $\langle P, \partial_x P, \partial_u P \rangle : (u - 1)^\infty \subset \mathbb{K}(t)[x, z, u]$ is radical and 0-dimensional over $\mathbb{K}(t)$.

Then one can compute $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$

- with degree in each variable at most δ^3 and total size δ^6 ,
- in $O_{\log}(L\delta^6 + \delta^{7.89}) \subset O_{\log}(\delta^{10})$ ops. in \mathbb{K} ,
where $L = \text{cost of evaluating } P \text{ at } (x, z, t, u) \in \mathbb{K}^4$.

Contribution 2: Guess-and-prove based on geometry

Input: $P(F(t, u), F(t, 1), t, u) = 0$, $\delta := \deg(P)$.

Output: $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F_1 = F(t, 1)$, i.e. $R(t, F_1) = 0$.

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geometry

(1) Functional
equation



(2) Polynomial
system



(3) Bounds

- $\deg_t(R) \leq b_t$,
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guess-and-prove

(4) Expand F_1



(5) Compute $R \in \mathbb{K}[t, z]$ s.t.
 $R(t, F_1) = O(t^{\sim b_t b_z})$



(6) Certify that $R(t, F_1) = 0$

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tools

- Newton iteration



- Algebraic approximants
"seriestoalgeq"



- Multiplicity lemma:
 $R(t, F_1) = O(t^{\sim 2b_t b_z})$
implies $R(t, F_1) = 0$

$\theta \in [2, 3]$ a feasible exponent of matrix multiplication

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Define $A_u := (F(t, u), F(t, 1), u)$ and assume that

- there exists $u = U(t) \in \mathbb{K}[[t]] \setminus \{1\}$ solution of $\partial_x P(F(t, u), F(t, 1), t, u) = 0$,
- the Jacobian of $(P, \partial_x P, \partial_u P)$ w.r.t $\{x, z, u\}$ is invertible at $A_{U(t)} \in \mathbb{K}[[t]]^3$.

Then the geometry-driven guess-and-prove computes $R \in \mathbb{K}[t, z] \setminus \{0\}$

- such that $R(t, F(t, 1)) = 0$,
- having its partial degrees bounded by δ^3 and total size δ^6 ,
- in $O_{\log}(\delta^{10.14})$ arithmetic operations in \mathbb{K} .

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- having its partial degrees bounded by δ^3 and total size δ^6 ,
- in $O_{\log}(\delta^{10.14})$ arithmetic operations in \mathbb{K} .
 $O_{\log}(L\delta^6 + \delta^{3\theta+3})$ ops. in \mathbb{K} , where $L = \text{cost for evaluating } P \text{ at } (x, z, t, u) \in \mathbb{K}^4$.

Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din '22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size δ^6 .
Moreover, one can compute R in $O_{\log}(\delta^{14})$ arithmetic operations in \mathbb{K} .

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Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne '06]

$$\begin{aligned} &P, \delta, \\ &\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z] \\ &\mathcal{J} \text{ ideal of } \mathbb{K}(t)[x, z, u] \end{aligned}$$

→

$$\begin{aligned} &P_\epsilon, \delta_\epsilon = O(\delta), \\ &\langle P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon \rangle \cap \mathbb{K}[t, \epsilon, z] \\ &\mathcal{J}_\epsilon \text{ ideal of } \mathbb{K}(t, \epsilon)[x, z, u] \\ &\text{radical, 0-dimensional} \end{aligned}$$

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(FPE)

→

(FPE) $+ \epsilon \sqrt{t} \Delta F$

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Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne '06]
 $\rightarrow \mathcal{J}_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u]$ radical, 0-dimensional
- “Stickelberger’s theorem” [Stickelberger 1897], [Cox '20]
 \rightarrow take R char. pol. of a linear map m_z defined over $\mathbb{K}(t, \epsilon)[x, z, u]/\mathcal{J}_\epsilon$
- Parametric geometric resolution [Schost '03]
 $O_{\log}(L_\epsilon \delta_\epsilon^9)$ ops. in \mathbb{K} , with $L_\epsilon = O(\delta L)$ $\rightarrow z = \frac{V(t, \epsilon, \lambda)}{\partial_\lambda W(t, \epsilon, \lambda)}, W(t, \epsilon, \lambda) = 0$.
- Bivariate resultants [Villard '18], [Hyun, Neiger, Schost '19]
 $O_{\log}(\delta_\epsilon^{10.89})$ ops. in \mathbb{K} $\rightarrow R = \text{Res}_\lambda(z - E(t, \epsilon, \lambda), W(t, \epsilon, \lambda))$.

Conclusion

- Refinement of an existing method based on discriminants,
- Design of a new guess-and-prove algorithm based on geometric bounds,
- A general complexity result for solving (FPE) of order 1.

Future works

- Improve the previous complexity estimates,
- Implement and compare the algorithms,
- Study the case of higher order equations.

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Example where (H1) is not satisfied

Example

Consider the functional equation

$$F(t, u) = 1 + t((u - 1)F(t, u)^2 + F(t, u) - F(t, 1)). \quad (1)$$

Here $P = 1 - x + t((u - 1)x^2 + x - z)$.

Therefore, $\partial_x P(x, z, 0, u) = 1$, hence assumption **(H1)** is not satisfied.

Algorithm DD of page 7:

1. $\text{disc}_x P = 4t^2uz - 4t^2z + t^2 - 4tu + 2t + 1$,
2. $\text{disc}_u(\text{disc}_x(P)) = 1$.

The output is $R = 1$, which is obviously wrong.

In fact, the unique solution $F(t, u)$ of (1) in $\mathbb{Q}[u][[t]]$ satisfies $F(t, 1) = 1$, and is a root of $R := t(u - 1)x^2 + (t - 1)x + 1 - t$.

Generic case

Page	Contribution	Hypothesis	Total size	Complexity	Relative exponent
7	DD	(H0), (H1), (R)	δ^8	$O_{\log}(\delta^{10})$	$\frac{10}{8} = 1.25$
8	Geom	(H1) , radical, 0-dim	δ^6	$O_{\log}(L\delta^6 + \delta^{7.89})$	$\frac{10}{6} = 1.6$
10	G&P	(H1) , Jac $\neq 0$	δ^6	$O_{\log}(L\delta^6 + \delta^{3\theta+3})$	$\frac{10.14}{6} = 1.69$
12	General	None	δ^6	$O_{\log}(\delta^{14})$	$\frac{14}{6} \sim 2.33$

Sparse case

Page	Contribution	Hypothesis	Total size	Complexity	Relative exponent
7	DD	(H0), (H1), (R)	δ^8	?	?
8	Geom.	(H1) , radical, 0-dim	δ^6	$O_{\log}(\delta^{7.89})$	$\frac{7.89}{6} = 1.315$
10	G&P	(H1) , Jac $\neq 0$	δ^6	$O_{\log}(\delta^{3\theta+3})$	$\frac{\theta+1}{2} \sim 1.69 \rightarrow \frac{\theta}{2} \sim 1.19$
12	General	None	δ^6	$O_{\log}(\delta^{10.89})$	$\frac{10.89}{6} \sim 1.815$