Algorithms for solving fixed point equations of order 1

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Joint work with:
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Motivation: A non linear equation coming from combinatorics...

$\mathbb{K}$ effective field of characteristic 0. $\mathbb{K} = \mathbb{Q}, \mathbb{Q}(y), \ldots$

Starting point: F, solution in $\mathbb{K}[u][[t]]$ of the fixed point equation (FPE) of order 1

$$F(t,u) = 1 + tu\big(uF(t,u)^2 + F(t,u) + \Delta F(t,u)\big),$$

where Δ is the divided difference operator $\Delta F(t,u) := \frac{F(t,u)-F(t,1)}{u-1}$.

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Goals:

- Compute a polynomial $R \in \mathbb{K}[t,z] \setminus \{0\}$ such that $R(t, F(t,1)) = 0$.
- Estimate the size of R for any (FPE).
- Complexity estimates (ops. in $\mathbb{K}$) for the computation of R.

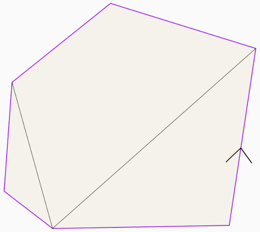
...associated to planar maps enumeration

Count

$c_n := \#\,\{\text{planar maps with } n \text{ edges}\}$

$\downarrow$ refinement

$c_{n,d} := \#\,\{\text{planar maps with } n \text{ edges},$
$d \text{ of them on the external face}\}$



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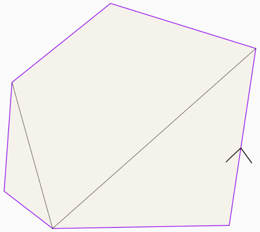

Solution in $\mathbb{K}[u][[t]]$

$G(t) := \sum_{n=0}^{\infty} c_n t^n$ generating function

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$F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^{n} c_{n,d} u^d t^n$ complete generating function

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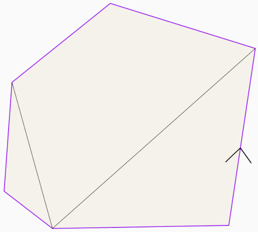

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(FPE) of order 1 [Tutte '68]

$$F(t,u) = 1 + tu^2 F(t,u)^2 + tu \frac{uF(t,u) - F(t,1)}{u-1}$$

Deletion-contraction of edges

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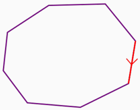

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State of the art

Theorem [Bousquet-Mélou, Jehanne '06] see also [Popescu '86]

Let $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y, t, u]$. Let $F(t, u)$ be the *unique* solution in $\mathbb{K}[u][[t]]$ of

$$F(t, u) = f(u) + tQ(F(t, u), \Delta F(t, u), t, u), \qquad \text{(FPE)}$$

where Δ is the divided difference operator $\Delta F := \frac{F(t,u) - F(t,1)}{u-1}$.

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[Tutte, Brown 60's], [Zeilberger '92]:	Guess-and-prove
[Gessel, Zeilberger '14]:	Guess-and-prove
[Brown '65]:	Quadratic method
[Knuth '68], [Banderier, Flajolet '02]:	Kernel method (linear case)
[Bousquet-Mélou, Jehanne '06]:	Polynomial elimination

Modelization: from (FPE) of order 1 to polynomial systems

[Bousquet-Mélou, Jehanne '06]:

Fixed Point Equation (FPE)

$\Downarrow$ numer

$$P(F(t,u), F(t,1), t, u) = 0$$

$\Downarrow$ ∂_u

$$\partial_u F(t,u) \cdot \partial_x P(F(t,u), F(t,1), t, u) + \partial_u P(F(t,u), F(t,1), t, u) = 0$$

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Example: planar maps

$$F(t,u) = 1 + tu^2 F(t,u)^2 + tu\frac{uF(t,u) - F(t,1)}{u-1} \qquad \text{(FPE)}$$

$$0 = (1 - F(t,u))(u-1) + tu^2(u-1)F(t,u)^2 + tu(uF(t,u) - F(t,1))$$

$$0 = \partial_u F(t,u) \cdot \left(1 - u + 2tu^2(u-1)F(t,u)^2 + tu^2\right) + \left(1 - F(t,u) + tu(3u-2)F(t,u)^2 + 2tu^2F(t,u) - tF(t,1)\right)$$

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solution $u = U(t) \in \mathbb{K}[[t]]$ of $\partial_x P(F(t,u), F(t,1), t, u) \ = \ 0$ $\rightarrow$

$$\begin{cases} P(x,z,t,u) = 0, \\ \partial_x P(x,z,t,u) = 0, \\ \partial_u P(x,z,t,u) = 0. \end{cases} \qquad (F(t,U(t)), F(t,1), U(t)) \text{ zero in } \mathbb{K}[[t]]^3$$

Our contributions

Inspired by [Bousquet-Mélou, Jehanne '06]

1. Geometric **refinements** of a method based on discriminants,
2. A new guess-and-prove method based on **geometry**,
3. A **complexity result** on the resolution of (FPE) of order 1.

Attention is paid to

- assumptions,
- degree bounds on the output,
- complexity estimates,
- potential for generalization.

Input: $P :=$ numerator(FPE),
Goal: $\langle P, \partial_x P, \partial_u P \rangle \cap \mathbb{K}[t, z]$.

Algebraic elimination via iterated discriminants

$\text{disc}_x(P) = \text{Res}_x(P, \partial_x P)$ the discriminant of P in x.

Theorem [Bousquet-Mélou, Jehanne '06]

Suppose $\deg_x(P) \geq 2$ and $u = U(t) \in \mathbb{K}[[t]]$ is a root of

$$\partial_x P(F(t,u), F(t,1), t, u).$$

Then $u = U(t)$ is a **double root** of $\text{disc}_x(P)(F(t,1), t, u)$.

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$$P := (1-x)(u-1) + tu^2(u-1)x^2 + tu(ux - z)$$

gives $\mathrm{disc}_u(\mathrm{disc}_x P)$ equal to

$$-256t^4 \cdot (27t^2z^2 - 18tz + 16t + z - 1) \cdot (tz-1)^2$$

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$$P := 97t^3u^2 + (-73u^4 - 56u^2x^2 + 87u^2x - 62x^2 + 124xz - 62z^2)t - xu^2 + u^2$$

gives $\mathrm{disc}_x P$ equal to

$$-16352t^2u^6 + \left(21728t^4 - 10535t^2 + 50t + 1\right)u^4 + 248t\left(97t^3 - 56t\,z^2 + 87tz - z + 1\right)u^2$$

which has a double root at $u = 0$.

Contribution 1: ensuring non-nullity of double discriminant

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Suppose $\delta = \deg(P)$

- **(H0)** $\deg_x(P) \geq 2$,
- **(H1)** $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- **(R)** the zero set $V(P) \subset \overline{\mathbb{K}}^4$ is smooth outside $V(u-1) \subset \overline{\mathbb{K}}^4$.

Set $D_0 := \mathrm{disc}_x P$, $D_1 :=$ **SqFreePart**(D_0) and $D_2 := \mathrm{disc}_u D_1$.

Then

- $R :=$ **SqFreePart**(D_2) is **non-zero** in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.

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- $R :=$ $\mathbf{SqFreePart}(D_2)$ is **non-zero** in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.
- R has total size $16\delta^8$ with degree in each variable at most $4\delta^4$,
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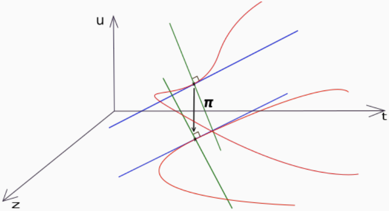

$D_1 := \mathbf{SqFreePart}(\mathrm{disc}_x(P))$ satisfies $\partial_u D_1(U(t), F(t,1), t) = 0$.

$$\begin{cases} (\partial_u D_1 \ \ \partial_z D_1 \ \ \partial_t D_1) \cdot (u \ \ z \ \ t)^T = 0, \\ (\partial_z D_1 \ \ \partial_t D_1) \cdot (z \ \ t)^T = 0 \end{cases}$$

Contribution 1 (cont'd): using geometry arguments to refine the complexity

$P \in \mathbb{K}[x, z, t, u]$ and $\delta := \deg(P)$.

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Suppose

- **(H1)** $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- $\langle P, \partial_x P, \partial_u P \rangle : (u-1)^\infty \subset \mathbb{K}(t)[x, z, u]$ is radical and 0-dimensional over $\mathbb{K}(t)$.

Then one can compute $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$

- with degree in each variable at most δ^3 and total size δ^6,
- in $O_{\log}(L\delta^6 + \delta^{7.89}) \subset O_{\log}(\delta^{10})$ ops. in $\mathbb{K}$,
  where L = cost of evaluating P at $(x, z, t, u) \in \mathbb{K}^4$.

- Geometric resolution: [Giusti, Lecerf, Salvy '01]

Contribution 2: Guess-and-prove based on geometry

Input: $P(F(t,u), F(t,1), t, u) = 0$, $\delta := \deg(P)$.

Output: $R \in \mathbb{K}[t,z] \setminus \{0\}$ annihilating $F_1 = F(t,1)$, i.e. $R(t, F_1) = 0$.

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geometry

(1) Functional equation

↓

(2) Polynomial system

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(3) Bounds

- $\deg_t(R) \leq b_t$,
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guess-and-prove

(4) Expand F_1

↓

(5) Compute $R \in \mathbb{K}[t,z]$ s.t. $R(t, F_1) = O(t^{\sim b_t b_z})$

↓

(6) Certify that $R(t, F_1) = 0$

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tools

- Newton iteration

↓

- Algebraic approximants "seriestoalgeq"

↓

- Multiplicity lemma: $R(t, F_1) = O(t^{\sim 2 b_t b_z})$ implies $R(t, F_1) = 0$

Complexity result / degree bounds for geometry-driven guess-and-prove

$\theta \in [2, 3]$ a feasible exponent of matrix multiplication

Theorem [Bostan, Chyzak, N., Safey El Din '22]

Define $A_u := (F(t,u), F(t,1), u)$ and assume that

- there exists $u = U(t) \in \mathbb{K}[[t]] \setminus \{1\}$ solution of $\partial_x P(F(t,u), F(t,1), t, u) = 0$,
- the Jacobian of $(P, \partial_x P, \partial_u P)$ w.r.t $\{x, z, u\}$ is invertible at $A_{U(t)} \in \mathbb{K}[[t]]^3$.

Then, the geometry-driven guess-and-prove computes $R \in \mathbb{K}[t,z] \setminus \{0\}$

- such that $R(t, F(t,1)) = 0$,
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- having its partial degrees bounded by δ^3 and total size δ^6,
- in $O_{\log}(\delta^{10.14})$ arithmetic operations in $\mathbb{K}$.
  $O_{\log}(L\delta^6 + \delta^{3\theta+3})$ ops. in $\mathbb{K}$, where L = cost for evaluating P at $(x,z,t,u) \in \mathbb{K}^4$.

Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din '22]

There exists $R \in \mathbb{K}[t, z] \setminus \{0\}$ annihilating $F(t, 1)$ of total arithmetic size δ^6.
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Sketch of the proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne '06]

$$P, \delta, \quad \langle P, \partial_x P, \partial_u P\rangle \cap \mathbb{K}[t, z], \quad \mathcal{J} \text{ ideal of } \mathbb{K}(t)[x, z, u]$$

$$\rightarrow$$

$$P_\epsilon, \delta_\epsilon = O(\delta), \quad \langle P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon\rangle \cap \mathbb{K}[t, \epsilon, z], \quad \mathcal{J}_\epsilon \text{ ideal of } \mathbb{K}(t, \epsilon)[x, z, u] \text{ radical, 0-dimensional}$$

Contribution 3: a polynomial time complexity for solving a (FPE) of order 1

Theorem [Bostan, Chyzak, N., Safey El Din '22]

There exists $R \in \mathbb{K}[t,z] \setminus \{0\}$ annihilating $F(t,1)$ of total arithmetic size δ^6.
Moreover, one can compute R in $O_{\log}(\delta^{14})$ arithmetic operations in $\mathbb{K}$.

Sketch of the proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne '06]

$P, \delta,$ $\langle P, \partial_x P, \partial_u P\rangle \cap \mathbb{K}[t,z]$ $\mathcal{J}$ ideal of $\mathbb{K}(t)[x,z,u]$	$\rightarrow$	$P_\epsilon, \delta_\epsilon = O(\delta),$ $\langle P_\epsilon, \partial_x P_\epsilon, \partial_u P_\epsilon\rangle \cap \mathbb{K}[t,\epsilon,z]$ $\mathcal{J}_\epsilon$ ideal of $\mathbb{K}(t,\epsilon)[x,z,u]$ radical, 0-dimensional
(FPE)	$\rightarrow$	(FPE) $+\epsilon\sqrt{t}\Delta F$

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Sketch of proof:

- Symbolic homotopy [Bousquet-Mélou, Jehanne '06]
  $\rightarrow \mathcal{J}_\epsilon \subset \mathbb{K}(t, \epsilon)[x, z, u]$ radical, 0-dimensional
- "Stickelberger's theorem" [Stickelberger 1897], [Cox '20]
  $\rightarrow$ take R char. pol. of a linear map m_z defined over $\mathbb{K}(t, \epsilon)[x, z, u]/\mathcal{J}_\epsilon$
- Parametric geometric resolution [Schost '03]
  $O_{\log}(L_\epsilon \delta_\epsilon^9)$ ops. in $\mathbb{K}$, with $L_\epsilon = O(\delta L)$ $\rightarrow z = \frac{V(t,\epsilon,\lambda)}{\partial_\lambda W(t,\epsilon,\lambda)}, W(t, \epsilon, \lambda) = 0.$
- Bivariate resultants [Villard '18], [Hyun, Neiger, Schost '19]
  $O_{\log}(\delta_\epsilon^{10.89})$ ops. in $\mathbb{K}$ $\rightarrow R = \mathrm{Res}_\lambda(z - E(t, \epsilon, \lambda), W(t, \epsilon, \lambda)).$

Conclusion and future works

Conclusion

- Refinement of an existing method based on discriminants
- Design of a new guess-and-prove algorithm based on geometric bounds
- A general complexity result for solving (FPE) of order 1

Future works

- Improve the previous complexity estimates
- Implement and compare the algorithms
- Study the case of higher order equations

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Contribution 1: ensuring non-nullity of double discriminant

Theorem [Bostan, Chyzak, N., Safey El Din '22]

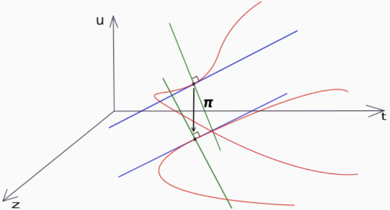
Suppose $\delta = \deg(P)$

- **(H0)** $\deg_x(P) \geq 2$,
- **(H1)** $\deg_u(\partial_x P(x, z, 0, u)) \geq 1$ and $\partial_x P(F(t, c), F(t, 1), t, c) \neq 0$ for all $c \in \mathbb{K}$,
- **(R)** the zero set $V(P) \subset \overline{\mathbb{K}}^4$ is smooth outside $V(u-1) \subset \overline{\mathbb{K}}^4$.

Set $D_0 := \mathrm{disc}_x P$, $D_1 :=$ **SqFreePart**(D_0) and $D_2 := \mathrm{disc}_u D_1$.

Then

- $R :=$ **SqFreePart**(D_2) is **non-zero** in $\mathbb{K}[z, t]$ and satisfies $R(F(t, 1), t) = 0$.
- R has total size $16\delta^8$ with degree in each variable at most $4\delta^4$,
- R can be computed in $O_{\log}(\delta^{10})$ ops. in $\mathbb{K}$.



$D_1 :=$ **SqFreePart**$(\mathrm{disc}_x(P))$ satisfies $\partial_u D_1(U(t), F(t, 1), t) = 0$.

$$\begin{cases} (\partial_u D_1 \;\; \partial_z D_1 \;\; \partial_t D_1) \cdot (u \;\; z \;\; t)^T = 0, \\ (\partial_z D_1 \;\; \partial_t D_1) \cdot (z \;\; t)^T = 0 \end{cases}$$

Example where (H1) is not satisfied

Example

Consider the functional equation

$$F(t,u) = 1 + t((u-1)F(t,u)^2 + F(t,u) - F(t,1)). \quad (1)$$

Here $P = 1 - x + t((u-1)x^2 + x - z)$.
Therefore, $\partial_x P(x,z,0,u) = 1$, hence assumption **(H1)** is not satisfied.

Algorithm DD of page 16:

1. $\text{disc}_x P = 4t^2uz - 4t^2z + t^2 - 4tu + 2t + 1$,
2. $\text{disc}_u(\text{disc}_x(P)) = 1$.

The output is $R = 1$, which is obviously wrong.

In fact, the unique solution $F(t,u)$ of (1) in $\mathbb{Q}[u][[t]]$ satisfies $F(t,1) = 1$, and is a root of $R := t(u-1)x^2 + (t-1)x + 1 - t$.

Recap

Generic case

Page	Contribution	Hypothesis	Total size	Complexity	Relative exponent
16	DD	**(H0)**, **(H1)**, **(R)**	δ^8	$O_{\log}(\delta^{10})$	$\frac{10}{8} = 1.25$
9	Geom	**(H1)**, radical, 0-dim	δ^6	$O_{\log}(L\delta^6 + \delta^{7.89})$	$\frac{10}{6} = 1.6$
11	G&P	**(H1)**, Jac$\neq 0$	δ^6	$O_{\log}(L\delta^6 + \delta^{3\theta+3})$	$\frac{10.14}{6} = 1.69$
13	General	None	δ^6	$O_{\log}(\delta^{14})$	$\frac{14}{6} \sim 2.33$

Sparse case

Page	Contribution	Hypothesis	Total size	Complexity	Relative exponent
16	DD	**(H0)**, **(H1)**, **(R)**	δ^8	?	?
9	Geom.	**(H1)**, radical, 0-dim	δ^6	$O_{\log}(\delta^{7.89})$	$\frac{7.89}{6} = 1.315$
11	G&P	**(H1)**, Jac$\neq 0$	δ^6	$O_{\log}(\delta^{3\theta+3})$	$\frac{\theta+1}{2} \sim 1.69 \rightarrow \frac{\theta}{2} \sim 1.19$
13	General	None	δ^6	$O_{\log}(\delta^{10.89})$	$\frac{10.89}{6} \sim 1.815$