Solving combinatorial equations via computer algebra

RTCA Topical Day: Elimination for Functional Equations, 11 December 2023

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Joint work with:

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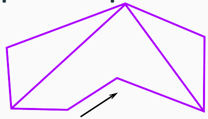
Which type of equations are we looking at?

rooted planar maps



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[Tutte '68]

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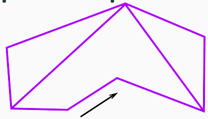

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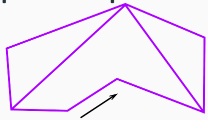

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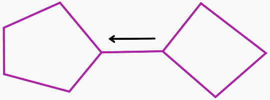

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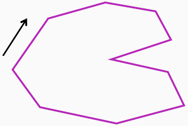

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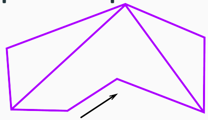

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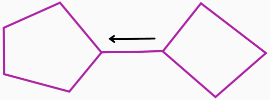

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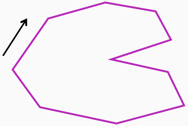

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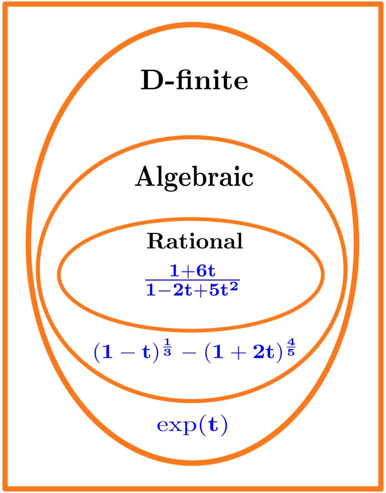
Solving functional equations



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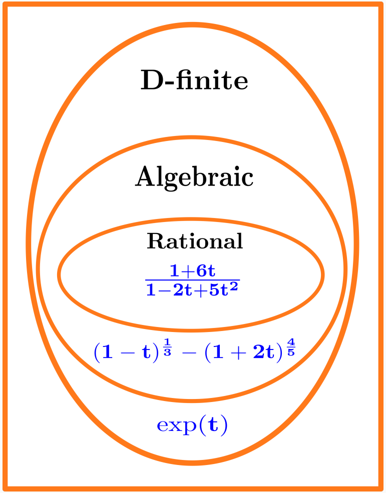

In this talk

Solving $=$ Classifying the initial series $F(t,1)$

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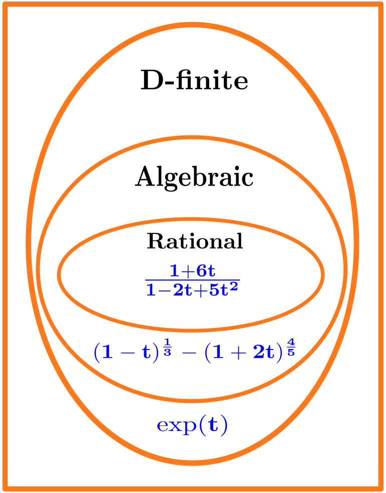

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Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

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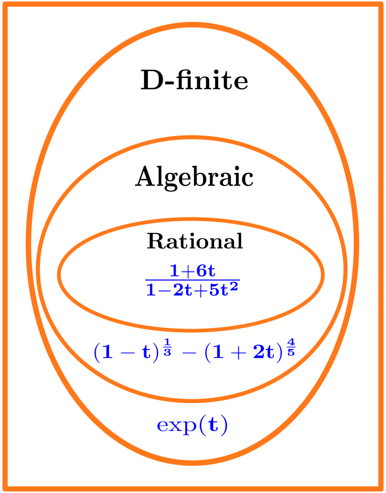

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From R:

- **(Recurrence)** $a_0 = 1$ and $(n+3)a_{n+1} - 6(2n+1)a_n = 0$,
- **(Closed-form)** $a_n = 2\frac{3^n(2n)!}{n(n+2)!}$,
- **(Asymptotics)** $a_n \sim 2\frac{12^n}{\sqrt{\pi n^5}}$, when $n \to +\infty$.

Content of the talk

Objectives

- **Introduce** so-called Discrete Differential Equations (DDEs),
- **Determine the nature** of the solutions of DDEs,
- Provide an **efficient algorithm** for computing a witness,
- Implementation in action $\rightsquigarrow$ Solving a problem **previously out of reach**.

Objects of interest: Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \ldots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \ldots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a *Discrete Differential Equation*, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^{\ell} \circ \Delta$.

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Bicolored planar maps: 3-constellations

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Theorem [Bousquet-Mélou, Jehanne '06]

The unique solution in $\mathbb{Q}[u][[t]]$ of **(DDE)** *is* algebraic *over $\mathbb{Q}(t,u)$.*

$\rightsquigarrow$ **Constructive proof $\Longrightarrow$ algorithm**

Bousquet-Mélou and Jehanne's algorithm

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$\rightarrow$ **Gröbner bases**

$$\mathcal{S}: \qquad \text{For } 1 \leq i \leq 2, \begin{cases} P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_x P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_u P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \end{cases} \qquad U_1 - U_2 \neq 0.$$

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Useful properties

- $\mathfrak{S}_2$ acts on $V(\mathcal{S})$ by **permuting** U_1, U_2,
- $\#V(\mathcal{S}) \leq$ **Bézout bound** associated with $\mathcal{S}$,
- Allows to forget $U_1 - U_2 \neq 0$ in the Bézout bound.

Quantitative estimates with A. Bostan and M. Safey El Din

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- Allows to forget $U_1 - U_2 \neq 0$ in the Bézout bound.

[Bostan, N., Safey El Din '23]

Under the above assumptions: $\qquad \delta := \deg(P)$

- There exists some nonzero polynomial $R \in \mathbb{Q}[z_0, t]$ whose partial degrees are upper bounded by $\delta^2(\delta-1)^4/2$, such that $R(F(t,1), t) = 0$.
- There exists an algorithm computing this R in $O_{\log}(\delta^{17})$ ops. in $\mathbb{Q}$.

(We proved a **general version** of this result)

Some preliminaries on Gröbner bases

$\mathcal{A} := \mathbb{Q}[x, \boldsymbol{y}]$ polynomial ring, where $\boldsymbol{y} = y_1, \ldots, y_s$.

Monomial orders

- $x^4 y_1^3 y_2^2 \succ_{lex} x^3 y_1^4 y_2^2$ for a **lexicographic order**,
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Properties

- Such bases always **exist** and **generate** $\mathcal{I}$,
- Computing Gröbner bases is **NP-hard**,
- Gröbner bases are a **powerful tool** in elimination theory.

New geometric modelling of the problem with A. Bostan and M. Safey El Din

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{0}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}. \end{cases}$$

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$$\mathcal{F}_2 := \{\alpha_{\underline{z}} \in \overline{\mathbb{Q}(t)}^2 \mid \# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq 2\}$$

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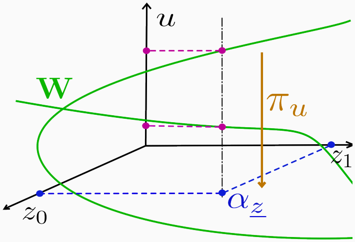

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[Extension theorem]

$\alpha \in \pi_u(V(G_u)) \implies \mathrm{LeadingCoeff}_u(\mathbf{g_2}) \neq 0$

Distinct solutions in $u \implies \mathrm{disc}_u(\mathbf{g_2}) \neq 0$ (**inequations**)

… yields an algorithm based on elimination theory (ISSAC'23)

Projecting $\Longrightarrow$ Elimination theorem

Lifting points of the projections $\Longrightarrow$ Extension theorem

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[Proposition] Let $g \in (\mathbb{Q}(t)[z_0, z_1])[u]$. Then g has **at least i distinct solutions** at $\alpha \in \overline{\mathbb{Q}(t)}^2$ **if and only if** the $(i \times i)$-minors of the **Hermite quadratic form** associated with g **do not vanish simultaneously** at α.

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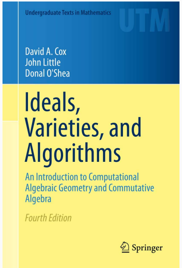

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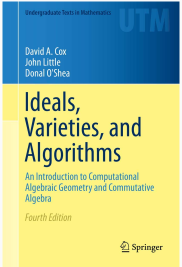

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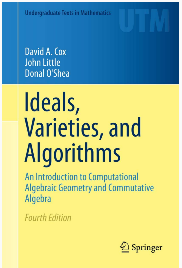

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[5—constellations $k = 4$]

Strategy	Timing	(d_{z_0}, d_t)
Duplication	> 5d	?
Elimination	**2d21h**	$(9, 3)$

Conclusion and perspectives

- **Decidability**: **geometry-driven algorithm** computing $R \in \mathbb{Q}[z,t] \setminus \{0\}$ s.t. $R(F(t,1),t) = 0$,
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- **Implementing** the algorithm in a *Maple* package?
  Available in 3 weeks!
- Work in progress with S. Yurkevich for systems of DDEs.
- More **nested** catalytic variables?
  (Work in progress with M. Bousquet-Mélou)