A Newton-like algorithm and algebraic methods for Structured Low-Rank Approximation

Giorgio Ottaviani, Éric Schost
Pierre-Jean Spaenlehauer, Bernd Sturmfels

Inria Nancy Grand-Est/LORIA/CNRS, project CARAMEL

Séminaire SpecFun, 03/03/2014

Problem Statement

$p, q, r \in \mathbb{N}$

E a **linear/affine subspace** of $p \times q$ matrices with real entries

For $(M_{i,j})$ a $p \times q$ matrix, $\|M\|_F = \sqrt{\sum_{i,j} M_{i,j}^2}$,

$\langle M_1, M_2 \rangle = \mathrm{trace}(M_1 \cdot M_2^{\mathsf{T}})$

Problem Statement

$p, q, r \in \mathbb{N}$

E a **linear/affine subspace** of $p \times q$ matrices with real entries

For $(M_{i,j})$ a $p \times q$ matrix, $\|M\|_F = \sqrt{\sum_{i,j} M_{i,j}^2}$,

$\langle M_1, M_2 \rangle = \mathrm{trace}(M_1 \cdot M_2^\intercal)$

Structured Low-Rank Approximation

Given $M \in E$, compute **a matrix** $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\| M - \hat{M} \right\|_F$ is **small**.

Problem Statement

$p, q, r \in \mathbb{N}$

E a **linear/affine subspace** of $p \times q$ matrices with real entries

For $(M_{i,j})$ a $p \times q$ matrix, $\|M\|_F = \sqrt{\sum_{i,j} M_{i,j}^2}$,

$\langle M_1, M_2 \rangle = \mathrm{trace}(M_1 \cdot M_2^{\mathsf{T}})$

Structured Low-Rank Approximation

Given $M \in E$, compute **a matrix** $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\| M - \hat{M} \right\|_F$ is **small**.

"Behind every linear data modeling problem there is a (hidden) low-rank approximation problem: the model imposes relations on the data which render a matrix constructed from exact data rank deficient."

Markovsky, 08

- $E =$ **Sylvester matrices** $\rightsquigarrow$ univariate approximate GCD

$$\begin{bmatrix} a_3 & 0 & b_2 & 0 & 0 \\ a_2 & a_3 & b_1 & b_2 & 0 \\ a_1 & a_2 & b_0 & b_1 & b_2 \\ a_0 & a_1 & 0 & b_0 & b_1 \\ 0 & a_0 & 0 & 0 & b_0 \end{bmatrix}$$

Examples and applications

- $E =$ **Sylvester matrices** $\rightsquigarrow$ univariate approximate GCD
- $E =$ **Hankel matrices** $\rightsquigarrow$ denoising, signal processing

$$\begin{bmatrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \\ d & e & f & g & h \\ e & f & g & h & i \end{bmatrix}$$

Examples and applications

- E =**Sylvester matrices** $\rightsquigarrow$ univariate approximate GCD
- E =**Hankel matrices** $\rightsquigarrow$ denoising, signal processing
- E =affine **coordinate spaces** $\rightsquigarrow$ matrix completion

$$\begin{bmatrix} 3 & ? & ? & 5 & 5 \\ 1 & 2 & 3 & 2 & ? \\ 10 & 4 & ? & 9 & -4 \\ 6 & ? & 3 & 9 & 10 \\ ? & 5 & -2 & ? & 9 \end{bmatrix}$$

Examples and applications

- $E =$ **Sylvester matrices** $\rightsquigarrow$ univariate approximate GCD
- $E =$ **Hankel matrices** $\rightsquigarrow$ denoising, signal processing
- $E =$ affine **coordinate spaces** $\rightsquigarrow$ matrix completion
- $E =$ **Ruppert matrices** $\rightsquigarrow$ multivariate factorization

$$\begin{bmatrix} 0 & -2 & -a & 0 & -2b & -d \\ -1 & 0 & c & -b & 0 & e \\ a & 2c & 0 & d & 2e & 0 \\ 0 & 0 & 0 & 1 & a & c \\ 0 & 0 & 0 & -b & -d & -e \end{bmatrix}$$

$XY^2 + aXY + bY^2 + cX + dY + e \in \mathbb{C}[X, Y]$ factors $\Leftrightarrow \mathrm{rank} \leq 4$

Specification

Structured Low-Rank Approximation

Given $M \in E$, compute a matrix $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\| M - \hat{M} \right\|_F$ is "small".

Which notion of **"small"**, for which **distance**?

$\rightsquigarrow$ depends on **the application**

Specification

Structured Low-Rank Approximation

Given $M \in E$, compute a matrix $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\|M - \hat{M}\right\|_F$ is "small".

Which notion of **"small"**, for which **distance**?

$\rightsquigarrow$ depends on **the application**

Minimizing is hard

Specification

Structured Low-Rank Approximation

Given $M \in E$, compute a matrix $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\| M - \hat{M} \right\|_F$ is "small".

Which notion of **"small"**, for which **distance**?
$\rightsquigarrow$ depends on **the application**

Minimizing is hard

Option: **"not too far"** from the minimizer?
$\rightsquigarrow$ numerical algorithm, joint work with **Éric Schost**

Specification

Structured Low-Rank Approximation

Given $M \in E$, compute a matrix $\hat{M} \in E$ such that

- $\mathrm{Rank}(\hat{M}) \leq r$;
- $\left\| M - \hat{M} \right\|_F$ is "small".

Which notion of **"small"**, for which **distance**?
$\rightsquigarrow$ depends on **the application**

Minimizing is hard

Option: **"not too far"** from the minimizer?
$\rightsquigarrow$ numerical algorithm, joint work with **Éric Schost**

Symbolical minimization
joint work with **Giorgio Ottaviani** and **Bernd Sturmfels**
$\rightsquigarrow$ algebraic complexity of the problem
$\rightsquigarrow$ gives useful information for **numerical algorithms** (*e.g.* bounds on the number of **local minima**)

Main results (numerical algorithm)

$\mathscr{D}_r$: **manifold** of $p \times q$ matrices of rank r
E: **linear/affine subspace** of $p \times q$ matrices

Algorithm NewtonSLRA

`NewtonSLRA`: iterative algorithm with proven local quadratic convergence under mild **transversality assumptions**.

Main results (numerical algorithm)

$\mathscr{D}_r$: **manifold** of $p \times q$ matrices of rank r
E: **linear/affine subspace** of $p \times q$ matrices

Algorithm NewtonSLRA

`NewtonSLRA`: iterative algorithm with proven local quadratic convergence under mild **transversality assumptions**.

More precisely: for any smooth point $\zeta \in \mathscr{D}_r \cap E$ where $\mathscr{D}_r$ and E **intersect transversely**, there exists a small neighborhood $U \supset \zeta$ such that for any input matrix $M_0 \in U$,

- the sequence of iterates $M_1, M_2, \ldots$ **converges quadratically** towards $M_\infty \in \mathscr{D}_r \cap E$, *i.e.* $\|M_i - M_\infty\| \leq (1/2)^{2^i - 1} \|M_0 - M_\infty\|$

Main results (numerical algorithm)

$\mathscr{D}_r$: **manifold** of $p \times q$ matrices of rank r
E: **linear/affine subspace** of $p \times q$ matrices

Algorithm NewtonSLRA

`NewtonSLRA`: iterative algorithm with proven local quadratic convergence under mild **transversality assumptions**.

More precisely: for any smooth point $\zeta \in \mathscr{D}_r \cap E$ where $\mathscr{D}_r$ and E **intersect transversely**, there exists a small neighborhood $U \supset \zeta$ such that for any input matrix $M_0 \in U$,

- the sequence of iterates $M_1, M_2, \dots$ **converges quadratically** towards $M_\infty \in \mathscr{D}_r \cap E$, *i.e.* $\|M_i - M_\infty\| \leq (1/2)^{2^i-1} \|M_0 - M_\infty\|$
- Let $\hat{M}$ be the **nearest solution**; then $\left\|M_\infty - \hat{M}\right\| = O(\mathrm{dist}(M_0, \mathscr{D}_r \cap E)^2)$.

Related problems

- $E \cap \mathscr{D}_r$ is **finite** $\rightsquigarrow$ **MinRank problem**.
  $\rightsquigarrow$ finite fields: Cryptology, Coding theory,...
  Bettale, Buss, Courtois, Frandsen, Gaborit, Goubin, Kipnis, Levy-dit-Vehel, Faugère, Perret, Ruatta, Safey, Shallit, Shamir, S., ...

Related problems

- $E \cap \mathscr{D}_r$ is **finite** $\rightsquigarrow$ **MinRank problem**.
  $\rightsquigarrow$ finite fields: Cryptology, Coding theory,...
  Bettale, Buss, Courtois, Frandsen, Gaborit, Goubin, Kipnis, Levy-dit-Vehel, Faugère, Perret, Ruatta, Safey, Shallit, Shamir, S., ...
- Low-rank approximation and **singular value decomposition**
  Demmel, Kahan, Golub, Van Loan,...
  Approximate SVD, **CUR decompositions**:
  Drineas, Halko, Martinsson, Tropp, Mahoney, Menon,...

Related problems

- $E \cap \mathscr{D}_r$ is **finite** $\rightsquigarrow$ **MinRank problem**.
  $\rightsquigarrow$ finite fields: Cryptology, Coding theory,...
  Bettale, Buss, Courtois, Frandsen, Gaborit, Goubin, Kipnis, Levy-dit-Vehel, Faugère, Perret, Ruatta, Safey, Shallit, Shamir, S., ...
- Low-rank approximation and **singular value decomposition**
  Demmel, Kahan, Golub, Van Loan,...
  Approximate SVD, CUR decompositions:
  Drineas, Halko, Martinsson, Tropp, Mahoney, Menon,...
- **Structured Total Least Norm**:
  Park, Rosen, Kaltofen/Zhi/Yang

Related problems

- $E \cap \mathscr{D}_r$ is **finite** $\rightsquigarrow$ **MinRank problem**.
  $\rightsquigarrow$ finite fields: Cryptology, Coding theory,...
  Bettale, Buss, Courtois, Frandsen, Gaborit, Goubin, Kipnis, Levy-dit-Vehel, Faugère, Perret, Ruatta, Safey, Shallit, Shamir, S., ...
- Low-rank approximation and **singular value decomposition**
  Demmel, Kahan, Golub, Van Loan,...
  Approximate SVD, **CUR decompositions**:
  Drineas, Halko, Martinsson, Tropp, Mahoney, Menon,...
- **Structured Total Least Norm**:
  Park, Rosen, Kaltofen/Zhi/Yang
- **Alternating projections**: *von Neumann, Cadzow*

Related problems

- $E \cap \mathscr{D}_r$ is **finite** $\rightsquigarrow$ **MinRank problem**.
  $\rightsquigarrow$ finite fields: Cryptology, Coding theory,...
  Bettale, Buss, Courtois, Frandsen, Gaborit, Goubin, Kipnis, Levy-dit-Vehel, Faugère, Perret, Ruatta, Safey, Shallit, Shamir, S., ...
- Low-rank approximation and **singular value decomposition**
  Demmel, Kahan, Golub, Van Loan,...
  Approximate SVD, **CUR decompositions**:
  Drineas, Halko, Martinsson, Tropp, Mahoney, Menon,...
- **Structured Total Least Norm**:
  Park, Rosen, Kaltofen/Zhi/Yang
- **Alternating projections**: *von Neumann, Cadzow*
- **Optimization**: *Chu/Funderlic/Plemmons*

Eckart-Young theorem

Let $M = U \cdot S \cdot V^\intercal$ be the **Singular Value Decomposition** of M, where $S = \mathrm{Diag}(\sigma_1, \ldots, \sigma_q)$ with $\sigma_1 \geq \cdots \geq \sigma_q$.

Set $\widehat{S} = \mathrm{Diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$.

Then $\widehat{M} = U \cdot \widehat{S} \cdot V^\intercal$ is the rank r matrix which **minimizes the Frobenius distance to M**.

Singular Value Decomposition

Eckart-Young theorem

Let $M = U \cdot S \cdot V^\intercal$ be the **Singular Value Decomposition** of M, where $S = \mathrm{Diag}(\sigma_1, \dots, \sigma_q)$ with $\sigma_1 \geq \dots \geq \sigma_q$.

Set $\widehat{S} = \mathrm{Diag}(\sigma_1, \dots, \sigma_r, 0, \dots, 0)$.

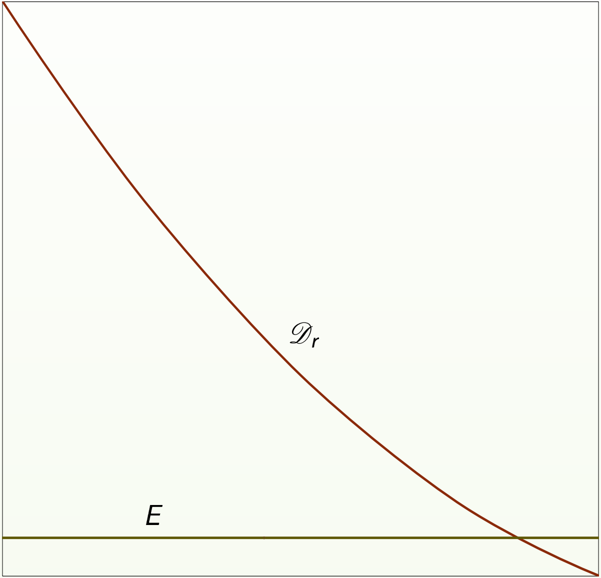
Then $\widehat{M} = U \cdot \widehat{S} \cdot V^\intercal$ is the rank r matrix which **minimizes the Frobenius distance to M**.

Cadzow's algorithm (*Cadzow, 88, Lewis/Malick 08*):

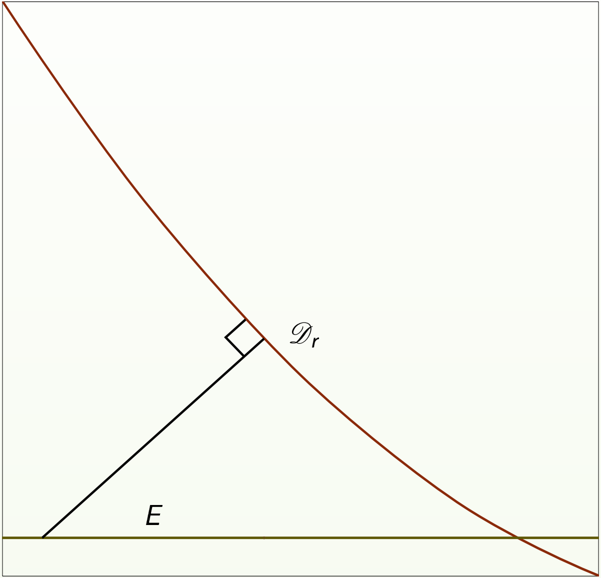
- **project** on $\mathscr{D}_r$ (the **manifold** of matrices of rank r) with **SVD**;
- **project** back on E.

Converges **linearly** towards a point in $\mathscr{D}_r \cap E$.
Does not converge to the **nearest solution**.

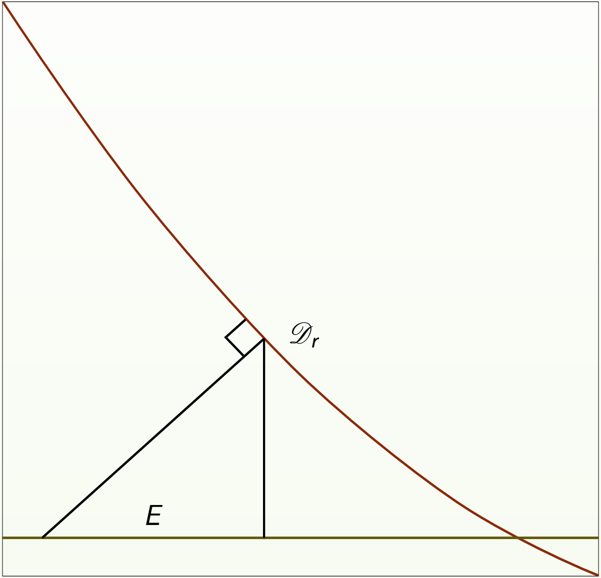
Cadzow's algorithm



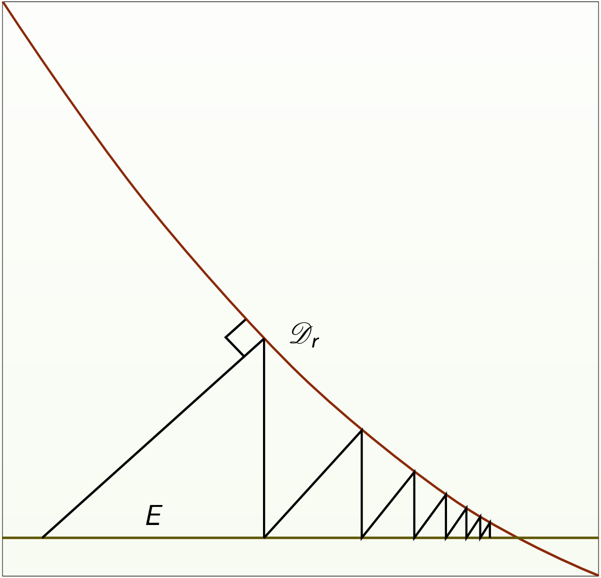
Cadzow's algorithm



Cadzow's algorithm



Cadzow's algorithm



Newton's method

Classical **Newton's method** for $f : \mathbb{R}^n \to \mathbb{R}^n$

$$N_f(x) = Df(x)^{-1}(f(x)).$$

Quadratic convergence when Df is locally invertible.

Newton's method

Classical **Newton's method** for $f : \mathbb{R}^n \to \mathbb{R}^n$

$$N_f(x) = Df(x)^{-1}(f(x)).$$

Quadratic convergence when Df is locally invertible.

Newton's method for **underdetermined systems**:
$f : \mathbb{R}^m \to \mathbb{R}^n$, $N_f(x) = Df(x)^{\dagger}(f(x))$.
$Df^{\dagger}$: **Moore-Penrose pseudo-inverse**.

Newton's method

Classical **Newton's method** for $f : \mathbb{R}^n \to \mathbb{R}^n$

$$N_f(x) = Df(x)^{-1}(f(x)).$$

Quadratic convergence when Df is locally invertible.

Newton's method for **underdetermined systems**:
$f : \mathbb{R}^m \to \mathbb{R}^n$, $N_f(x) = Df(x)^{\dagger}(f(x))$.
$Df^{\dagger}$: **Moore-Penrose pseudo-inverse**.

Quadratic convergence towards a point x_∞ such that $f(x_\infty) = 0$ when Df is locally surjective.

Classical **Newton's method** for $f : \mathbb{R}^n \to \mathbb{R}^n$

$$N_f(x) = Df(x)^{-1}(f(x)).$$

Quadratic convergence when Df is locally invertible.

Newton's method for **underdetermined systems**:
$f : \mathbb{R}^m \to \mathbb{R}^n$, $N_f(x) = Df(x)^{\dagger}(f(x))$.
$Df^{\dagger}$: **Moore-Penrose pseudo-inverse**.

Quadratic convergence towards a point x_∞ such that $f(x_\infty) = 0$ when Df is locally surjective.
If x_0 is the starting point of the iteration, let

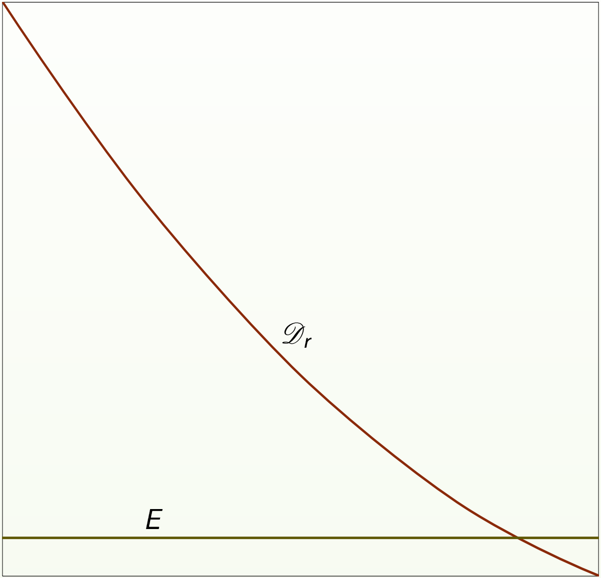
$$\hat{x} = \mathrm{argmin}_{f(y)=0} \|y - x_0\|.$$

Does not converge to the nearest solution $\hat{x}$, but

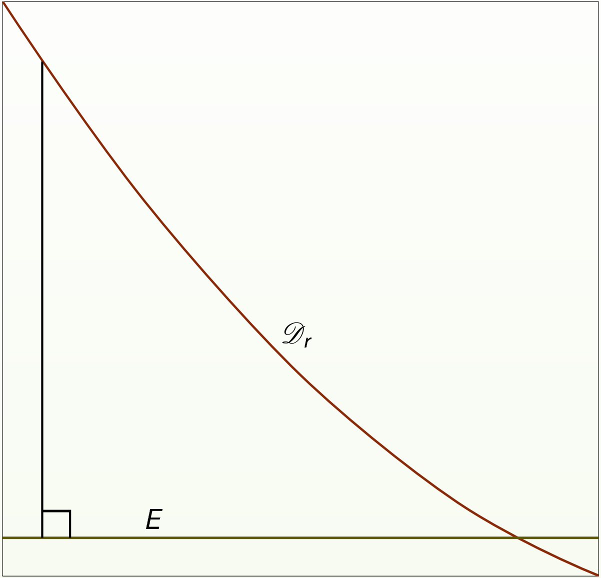
$$\|x_\infty - \hat{x}\| = O\left(\|x_0 - \hat{x}\|^2\right).$$

Ben-Israel 66, Allgower/Georg 90, Beyn 93, Shub/Smale 96, Dedieu/Shub 00, Dedieu/Kim 02, Dedieu 06

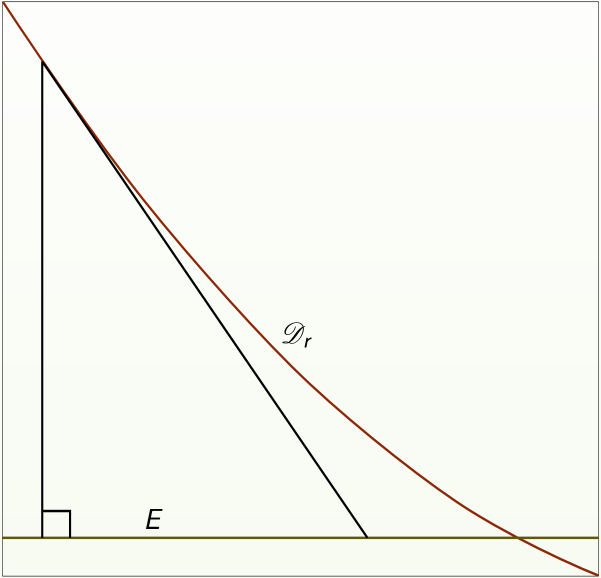
Newton's method



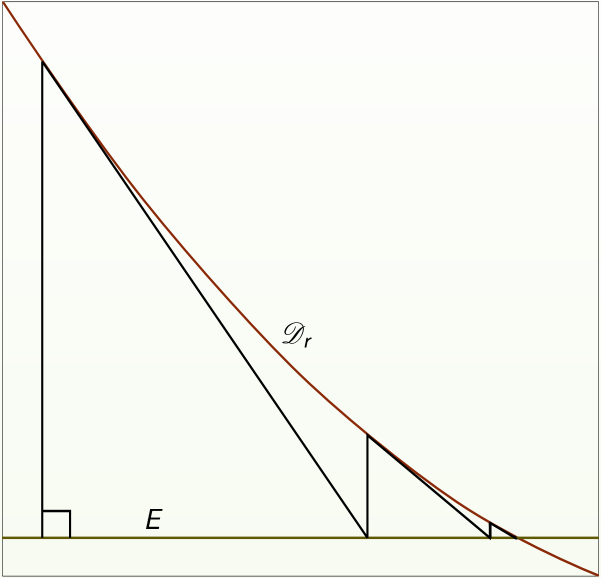
Newton's method



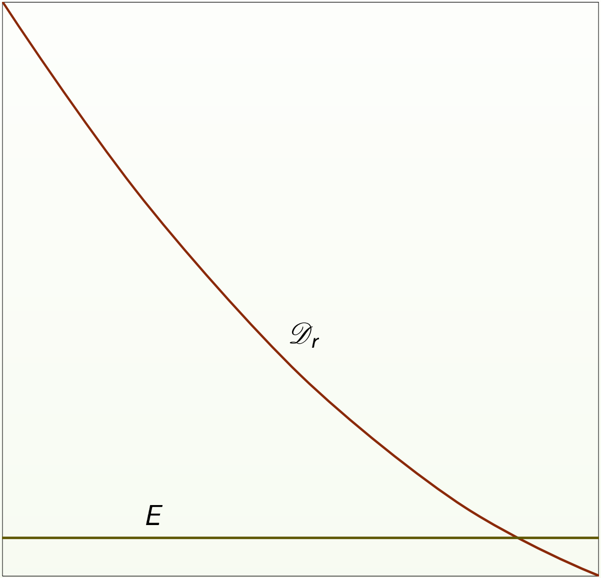
Newton's method



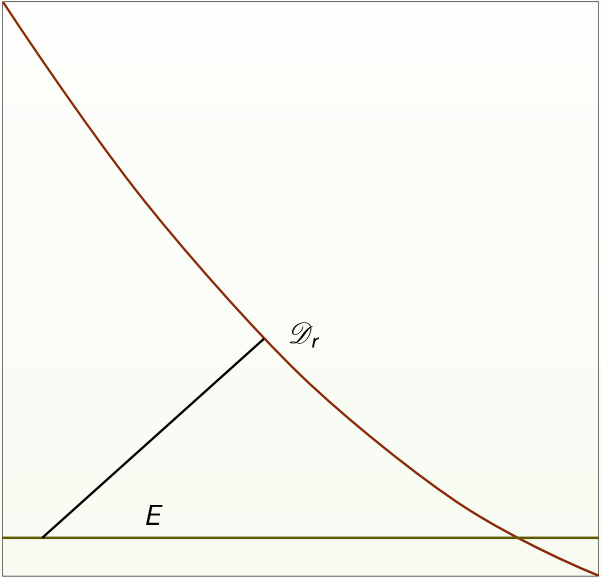
Newton's method



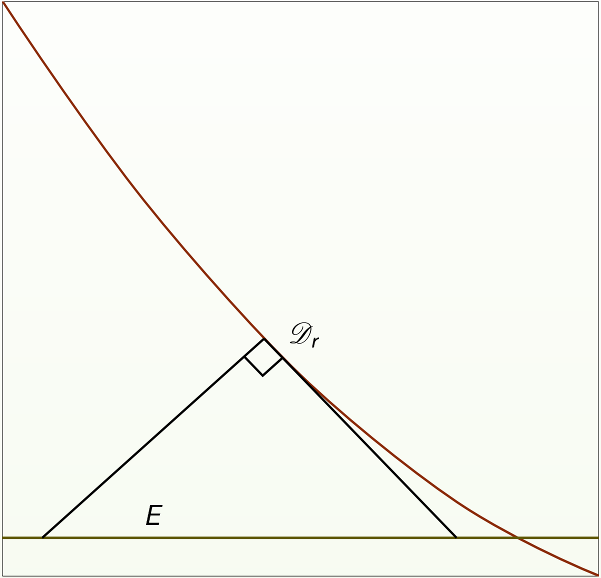
NewtonSLRA



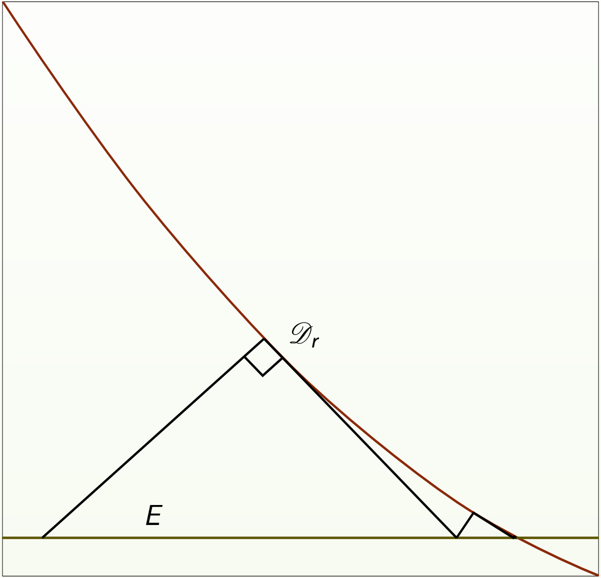
NewtonSLRA



NewtonSLRA



NewtonSLRA



Normal space of determinantal varieties

$\overline{\mathscr{D}_r}$: **algebraic variety** of matrices of rank **at most** r.
$\rightsquigarrow$ well-studied in algebraic geometry/commutative algebra
Bruns, Conca, Eisenbud, Herzog, Lascoux, Room, Sturmfels,...

Normal space of determinantal varieties

$\overline{\mathscr{D}_r}$: **algebraic variety** of matrices of rank **at most** r.
$\rightsquigarrow$ well-studied in algebraic geometry/commutative algebra
Bruns, Conca, Eisenbud, Herzog, Lascoux, Room, Sturmfels,...

Classical theorem

Let M be $p \times q$ matrix of rank r.
Then the **normal space** to $\mathscr{D}_r$ at M is

$$\mathrm{Ker}(M^\intercal) \otimes \mathrm{Ker}(M).$$

Bases of the kernels of M and $M^\intercal$ can be read off from the Singular Value Decomposition of M.

NewtonSLRA

procedure NewtonSLRA($M \in E$, $(E_1, \ldots, E_d)$ an orthonormal basis of E, $r \in \mathbb{N}$)

  $(U, S, V) \leftarrow \mathrm{SVD}(M)$

  $S_r \leftarrow r \times r$ top-left submatrix of S

  $U_r \leftarrow$ first r columns of U

  $V_r \leftarrow$ first r columns of V

  $\widetilde{M} \leftarrow U_r \cdot S_r \cdot V_r^{\mathsf{T}}$

  $\widetilde{u_1}, \ldots, \widetilde{u_{p-r}} \leftarrow$ last $p - r$ columns of U

  $\widetilde{v_1}, \ldots, \widetilde{v_{q-r}} \leftarrow$ last $q - r$ columns of V

  for $i \in \{1, \ldots, p-r\}, j \in \{1, \ldots, q-r\}$ **do**

    $N_{(i-1)(q-r)+j} \leftarrow \widetilde{u_i} \cdot \widetilde{v_j}^{\mathsf{T}}$

  end for

  $A \leftarrow (\langle N_i, E_j \rangle)_{i,j}$

  $b \leftarrow (\langle N_i, \widetilde{M} - M \rangle)_i$

  return $M + \begin{bmatrix} E_1 & \ldots & E_d \end{bmatrix} \cdot A^{\dagger} \cdot b$

end procedure

Rate of convergence

Quadratic convergence

For any **smooth** point $\zeta \in \mathscr{D}_r \cap E$ where $\mathscr{D}_r$ and E intersect **transversely**, there exists a small neighborhood $U \supset \zeta$ such that for any input matrix $M_0 \in U$,

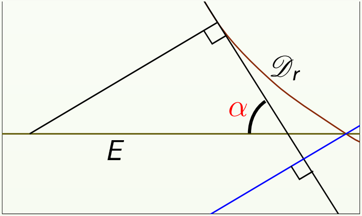
- the sequence of iterates $M_1, M_2, \ldots$ **converges quadratically** towards a matrix $M_\infty \in \mathscr{D}_r \cap E$, *i.e.* $\|M_i - M_\infty\| \leq (1/2)^{2^i-1} \|M_0 - M_\infty\|$
- Let $\hat{M}$ be the **nearest solution**; then $\left\|M_\infty - \hat{M}\right\| = O(\mathrm{dist}(M_0, \mathscr{D}_r \cap E)^2)$.

Rate of convergence

Quadratic convergence

For any **smooth** point $\zeta \in \mathscr{D}_r \cap E$ where $\mathscr{D}_r$ and E intersect **transversely**, there exists a small neighborhood $U \supset \zeta$ such that for any input matrix $M_0 \in U$,

- the sequence of iterates $M_1, M_2, \ldots$ **converges quadratically** towards a matrix $M_\infty \in \mathscr{D}_r \cap E$, *i.e.* $\|M_i - M_\infty\| \leq (1/2)^{2^i - 1} \|M_0 - M_\infty\|$
- Let $\hat{M}$ be the **nearest solution**; then $\left\| M_\infty - \hat{M} \right\| = O(\mathrm{dist}(M_0, \mathscr{D}_r \cap E)^2)$.



Sketch of proof:

- lower bound for α;
- Taylor approximation of $\Pi_{\mathscr{D}_r}$;
- manage corrective terms when $\dim(\mathscr{D}_r \cap E) > 0$.

Main features of NewtonSLRA

- Combines the **generality of alternating projections** and the **quadratic convergence of Newton's method**.
- Computationally most intensive step: computing the SVD (polynomial in p, q at fixed precision).
- Algorithm for SLRA with **proven quadratic rate of convergence**.

Application to approximate GCD

Approximate GCD

Let $m, n, d \in \mathbb{N}$, $f, g \in \mathbb{R}[x]$ with $\deg(f) = m, \deg(g) = n$.
Find $f^*, g^* \in \mathbb{R}[x]$, $\deg(f^*) = m$, $\deg(g^*) = n$ such that

$$\deg(\mathrm{GCD}(f^*, g^*)) \geq d$$

and (f^*, g^*) are close to (f, g).

Application to approximate GCD

Approximate GCD

Let $m, n, d \in \mathbb{N}$, $f, g \in \mathbb{R}[x]$ with $\deg(f) = m, \deg(g) = n$.
Find $f^*, g^* \in \mathbb{R}[x]$, $\deg(f^*) = m$, $\deg(g^*) = n$ such that

$$\deg(\mathrm{GCD}(f^*, g^*)) \geq d$$

and (f^*, g^*) are close to (f, g).

- **Euclidean distance** on the pairs (f, g):

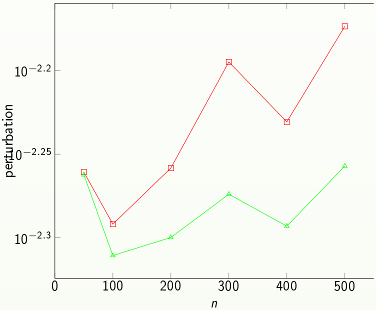
$$\|(\sum_{i=0}^{m} f_i x^i, \sum_{j=0}^{n} g_j x^j)\|^2 = \sum_{i=0}^{m} f_i^2 + \sum_{j=0}^{n} g_j^2.$$

- What does **"close"** mean
  $\rightsquigarrow$ quasi-GCD, *Schönhage 85*
  $\rightsquigarrow$ ε-GCD, *Emiris/Galligo/Lombardi 97, Zeng/Dayton 04, Bini/Boito 06-09*
  $\rightsquigarrow$ nearest pair for a given norm, *Karmarkar/Lakshman 98, Kaltofen/Zhi/Yang 05-08, Terui 09*

Comparison with GPGCD, *Terui, ISSAC'09*.

	sizes of iteration steps	
iteration	NewtonSLRA	GPGCD
1	0.9e-1	0.9e-1
2	0.5e-3	0.5e-3
3	0.6e-8	0.2e-5
4	0.1e-17	0.8e-8
5	0.1e-36	0.4e-10

$n = m = 25,\ d = 10.$



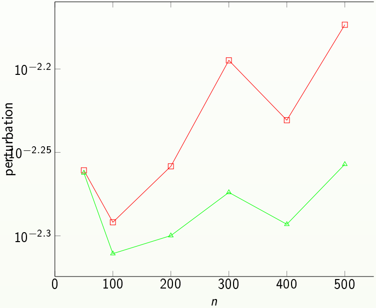
$n = m = 2d.$

Experimental results

Comparison with `GPGCD`, *Terui, ISSAC'09*.

	sizes of iteration steps	
iteration	`NewtonSLRA`	`GPGCD`
1	0.9e-1	0.9e-1
2	0.5e-3	0.5e-3
3	0.6e-8	0.2e-5
4	0.1e-17	0.8e-8
5	0.1e-36	0.4e-10

$n = m = 25,\ d = 10.$



$n = m = 2d.$

Fast convergence towards $\mathscr{D}_r \cap E$

$\rightsquigarrow$ starting point for a certified **Gauss-Newton iteration**

Auroux/Chèze/Masmoudi/Yakoubsohn 06

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & ? & ? \\ 1 & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & 7 & ? \\ 1 & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & 7 & ? \\ 1 & ? & 9 & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & 7 & ? \\ 1 & ? & 9 & ? \\ ? & ? & ? & 7 \end{bmatrix}$$

Uncover m **entries** at random.

Matrix Completion

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & 7 & ? \\ 1 & ? & 9 & ? \\ ? & ? & ? & 7 \end{bmatrix}$$

Uncover m **entries** at random.
How many entries do we need? How to reconstruct the matrix?

Unknown matrix of **rank** r:

$$\begin{bmatrix} ? & 4 & ? & ? \\ ? & ? & 7 & ? \\ 1 & ? & 9 & ? \\ ? & ? & ? & 7 \end{bmatrix}$$

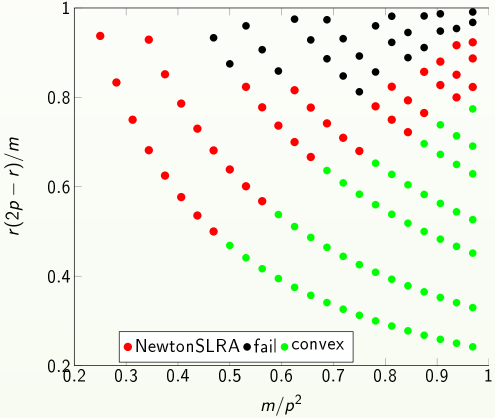
Uncover m **entries** at random.
How many entries do we need? How to reconstruct the matrix?

- Algebraic structure, *Merle/Giusti, '81*
- Alternating minimization, *Jain, Netrapalli, Sanghavi, 12*
- Riemannian optimization, *Absil/Amodei/Meyer 12, Vandereycken 12*
- Convex relaxation, *Candes, Tao, Plan, Recht, 09-13*

Overdetermined SLRA problems

Transversality assumption do not hold $\rightsquigarrow$ no quadratic convergence.

Square matrix of size $p = 40$



An algebraic approach to SLRA

The Euclidean distance degree
Draisma/Horobet/Ottaviani/Sturmfels/Thomas 13

$V \in \mathbb{C}^n$ an algebraic variety, $\mathbf{u} \in \mathbb{C}^n$ a generic point. The **EDdegree** of V is the number of **complex critical points** of the function

$$\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2$$

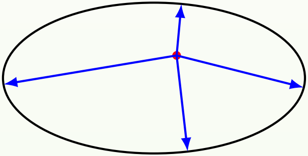
on the smooth locus of V.

The Euclidean distance degree
Draisma/Horobet/Ottaviani/Sturmfels/Thomas 13

$V \in \mathbb{C}^n$ an algebraic variety, $\mathbf{u} \in \mathbb{C}^n$ a generic point. The **EDdegree** of V is the number of **complex critical points** of the function

$$\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2$$

on the smooth locus of V.



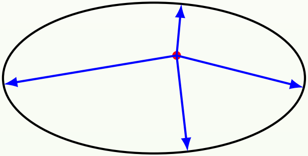
$\mathrm{EDdegree}(\text{ellipse}) = 4.$

The Euclidean distance degree
Draisma/Horobet/Ottaviani/Sturmfels/Thomas 13

$V \in \mathbb{C}^n$ an algebraic variety, $\mathbf{u} \in \mathbb{C}^n$ a generic point. The **EDdegree** of V is the number of **complex critical points** of the function

$$\lambda_1(x_1 - u_1)^2 + \cdots + \lambda_n(x_n - u_n)^2$$

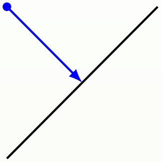
on the smooth locus of V.



$\mathrm{EDdegree}(\text{ellipse}) = 4.$

Nearest solution of SLRA:

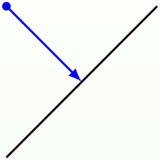
critical point of the distance function on a **linear section of a determinantal variety** $\mathscr{D}_r \cap E$.



Line

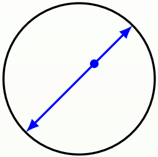
EDdegree$= 1$

The EDdegree as a complexity measure



Line

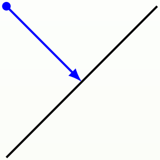
EDdegree$= 1$



Circle

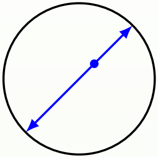
EDdegree$= 2$

The EDdegree as a complexity measure



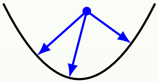
Line

EDdegree$=1$



Circle

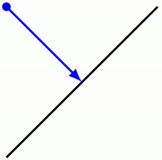
EDdegree$=2$



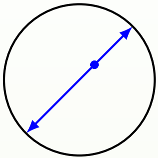
Parabola

EDdegree$=3$

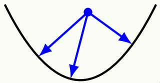
The EDdegree as a complexity measure



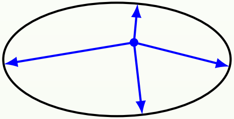
Line
EDdegree$=1$



Circle
EDdegree$=2$

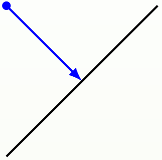


Parabola
EDdegree$=3$

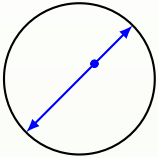


Ellipse
EDdegree$=4$

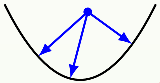
The EDdegree as a complexity measure



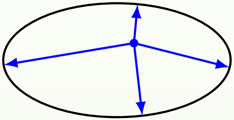
Line
EDdegree$= 1$



Circle
EDdegree$= 2$



Parabola
EDdegree$= 3$



Ellipse
EDdegree$= 4$

Strong **experimental correlation** between timings (**symbolic solving** with Gröbner bases) and **EDdegree**.

Problems

Structured Low-Rank Approximation:
$\rightsquigarrow$ family of computationally hard problems

with (relatively) **low algebraic degree**!

Timings of Gröbner basis software (FGb, Magma):
$\rightsquigarrow$ related to the EDdegree

Problems

Structured Low-Rank Approximation:
$\rightsquigarrow$ family of computationally hard problems

with (relatively) **low algebraic degree**!

Timings of Gröbner basis software (FGb, Magma):
$\rightsquigarrow$ related to the EDdegree

Goals:

- find efficient formulations as polynomial systems;
- Algebraic geometry techniques for estimating the EDdegree;
- Certification of numerical methods?

A general bound

critical points of $\lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2$ on V_{smooth}

A general bound

critical points of $\lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2$ on V_{smooth}

Ottaviani/S./Sturmfels '13

Let $\mathcal{L}$ be a **generic codimension** s linear space of $p \times q$ matrices, and V be the variety of **rank-deficient matrices** in $\mathcal{L}$. The **generic EDdegree** of V equals

$$\delta_0 + \cdots + \delta_{pq-2-s}.$$

where

$$\begin{aligned} \delta_\ell &= \sum_{k=\ell}^{p+q-2} (-1)^{p+q-k} \binom{k+1}{\ell+1} v_k \\ v_k &= \left[s^{p-1} t^{q-1}\right] (1+s)^p (1+t)^q (t+s)^k. \end{aligned}$$

A general bound

critical points of $\lambda_{1,1}(x_{1,1} - u_{1,1})^2 + \cdots + \lambda_{p,q}(x_{p,q} - u_{p,q})^2$ on V_{smooth}

Ottaviani/S./Sturmfels '13

Let $\mathcal{L}$ be a **generic codimension** s linear space of $p \times q$ matrices, and V be the variety of **rank-deficient matrices** in $\mathcal{L}$. The **generic EDdegree** of V equals

$$\delta_0 + \cdots + \delta_{pq-2-s}.$$

where

$$\begin{aligned} \delta_\ell &= \sum_{k=\ell}^{p+q-2} (-1)^{p+q-k} \binom{k+1}{\ell+1} v_k \\ v_k &= \left[s^{p-1} t^{q-1}\right] (1+s)^p (1+t)^q (t+s)^k. \end{aligned}$$

+ conjectured formula for the **Frobenius norm** $(\lambda_1 = \ldots = \lambda_{p,q} = 1)$.

$$
\begin{array}{rcl}
f(X) & = & f_m X^m + \cdots + f_1 X + f_0 \\
g(X) & = & g_n X^n + \cdots + g_1 X + f_0 \\
\|(f,g)\|^2 & = & \alpha_m f_m^2 + \cdots + \alpha_0 f_0^2 + \beta_n g_n^2 + \cdots + \beta_0 g_0^2
\end{array}
$$

$V \subset \mathbb{C}^{m+n+2}$: vanishing locus of the **resultant**.

$$\begin{aligned}
f(X) &= f_m X^m + \cdots + f_1 X + f_0 \\
g(X) &= g_n X^n + \cdots + g_1 X + f_0 \\
\|(f,g)\|^2 &= \alpha_m f_m^2 + \cdots + \alpha_0 f_0^2 + \beta_n g_n^2 + \cdots + \beta_0 g_0^2
\end{aligned}$$

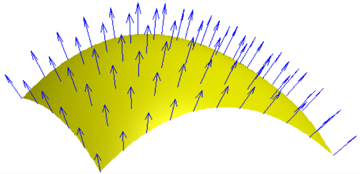
$V \subset \mathbb{C}^{m+n+2}$: vanishing locus of the **resultant**.

Ottaviani/S./Sturmfels '13

The **generic EDdegree** of V equals $4(m+n)-2$.

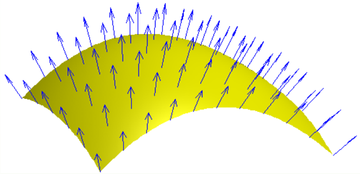
For all weights (α, β), the number of **locally nearest pairs** (f', g') with a non trivial GCD is bounded by $4(m+n)-2$.

In the case of the **rotation invariant quadratic form**, we conjecture that the **ED degree** equals $2\max(n,m)$.



Let $X \subset \mathbb{C}^n$ be an affine cone (the vanishing locus of homogeneous polynomials). The *conormal variety* $\mathcal{N}_X \subset \mathbb{C}^n \times \mathbb{C}^n$ is defined as

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\text{smooth}}, \mathbf{v} \in N_{\mathbf{x}}X\}}.$$



Let $X \subset \mathbb{C}^n$ be an affine cone (the vanishing locus of homogeneous polynomials). The *conormal variety* $\mathcal{N}_X \subset \mathbb{C}^n \times \mathbb{C}^n$ is defined as

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\text{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

$$\mathbf{x} \in X_{\text{smooth}} \text{ critical point of } \sum \lambda_i (x_i - u_i)^2$$
$$\Updownarrow$$
$$\nabla \sum \lambda_i (x_i - u_i)^2 \in N_x X$$
$$\Updownarrow$$
$$\begin{bmatrix} 2\lambda_1 x_1 - v_1 \\ \vdots \\ 2\lambda_n x_n - v_n \end{bmatrix} = \begin{bmatrix} 2\lambda_1 u_1 \\ \vdots \\ 2\lambda_n u_n \end{bmatrix} \text{ for } (\mathbf{x}, \mathbf{v}) \in \mathcal{N}_X$$

Duality

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. Equality holds when the diagonal of the **conormal variety** is empty.

Duality:

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\text{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. Equality holds when the diagonal of the **conormal variety** is empty.

Duality:

$$\mathcal{N}_X = \overline{\{(\mathbf{x},\mathbf{v}) : \mathbf{x} \in X_{\mathrm{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

$$\begin{array}{rccc} \pi_2 : & \mathcal{N}_X & \to & \mathbb{C}^n \\ & (\mathbf{x},\mathbf{v}) & \mapsto & \mathbf{v} \end{array} \qquad\qquad X^* = \mathrm{Im}(\pi_2)$$

Rank r matrices are dual to corank r matrices.

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. Equality holds when the diagonal of the **conormal variety** is empty.

Duality:

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\mathrm{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

$$\begin{array}{rrcl} \pi_2 : & \mathcal{N}_X & \to & \mathbb{C}^n \\ & (\mathbf{x}, \mathbf{v}) & \mapsto & \mathbf{v} \end{array} \qquad\qquad X^* = \mathrm{Im}(\pi_2)$$

Rank-deficient matrices are dual to rank 1 matrices

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. Equality holds when the diagonal of the **conormal variety** is empty.

Duality:

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\mathrm{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

$$\begin{array}{rccc} \pi_2 : & \mathcal{N}_X & \to & \mathbb{C}^n \\ & (\mathbf{x}, \mathbf{v}) & \mapsto & \mathbf{v} \end{array} \qquad X^* = \mathrm{Im}(\pi_2)$$

Rank-deficient matrices are dual to rank 1 matrices
$\rightsquigarrow$ **Segre varieties**.

Proposition (Draisma/Horobet/Ottaviani/Sturmfels/Thomas)

The **EDdegree** of a projective variety is bounded by the **sum of the degrees of its polar classes**. Equality holds when the diagonal of the **conormal variety** is empty.

Duality:

$$\mathcal{N}_X = \overline{\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \in X_{\mathrm{smooth}}, \mathbf{v} \in N_{\mathbf{x}} X\}}.$$

$$\begin{array}{rccc} \pi_2 : & \mathcal{N}_X & \to & \mathbb{C}^n \\ & (\mathbf{x}, \mathbf{v}) & \mapsto & \mathbf{v} \end{array} \qquad X^* = \mathrm{Im}(\pi_2)$$

Rank-deficient matrices are dual to rank 1 matrices
$\rightsquigarrow$ **Segre varieties**.

Other applications:

- low-rank approximation of tensors
- low-rank approximation of Hankel matrices

Open questions

What about the **number** of real critical points/local minima?

Open questions

What about the **number** of real critical points/local minima?

Open even for **unstructured** weighted low-rank approximation!

Question: is the number of local minima of rank 1 (resp. corank 1) approximation bounded by $\min(p, q)$ (Rey'13)?

Open questions

What about the **number** of real critical points/local minima?

Open even for **unstructured** weighted low-rank approximation!

Question: is the number of local minima of rank 1 (resp. corank 1) approximation bounded by $\min(p,q)$ (Rey'13)?

Ottaviani/S./Sturmfels'13: **negative answer**

$$U = \begin{bmatrix} -59 & 11 & 59 \\ 11 & 59 & -59 \\ 59 & -59 & 11 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 1 & 9 \\ 1 & 9 & 6 \end{bmatrix}$$

Rank 1 approximation of U has 7 local minima. $\mathrm{EDdegree} = 39$, number of real critical points: 19.

Can we find **more real critical points/local minima**?

Conclusions and perspectives

Linear sections of determinantal varieties

rich **structure** with a lot of facets
(numeric/symbolic, finite fields/characteristic 0, real solutions)
which appears in many applications.

Perspectives

- Replacing the SVD by other **rank approximation techniques** to speed up the computations
- Model of **noise**
- Impact of the choice of the **distance**
- **Certification** of `NewtonSLRA` *a la Dedieu*: α,γ theorems?
- **Algebraic** properties of special linear subspaces.

Conclusions and perspectives

Linear sections of determinantal varieties

rich **structure** with a lot of facets
(numeric/symbolic, finite fields/characteristic 0, real solutions)
which appears in many applications.

Perspectives

- Replacing the SVD by other **rank approximation techniques** to speed up the computations
- Model of **noise**
- Impact of the choice of the **distance**
- **Certification** of `NewtonSLRA` *a la Dedieu*: α,γ theorems?
- **Algebraic** properties of special linear subspaces.

Thank you!