p-adic precision, differentials and the example of Gröbner bases.

SpecFun Seminar

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23 mars 2014

Motivation for p-adic algorithm

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3. Lift the factors into $\mathbb{Z}/p^k\mathbb{Z}[X]$ (with *Hensel's lemma*) ;
4. If p^k is big enough (*Mignotte's bound*), we can obtain a factorization over Q (up to the recombination of some factors).

p-adic algorithms : another example

Idea of Kedlaya's algorithm

Let C be an hyperelliptic curve of genus g over $\mathbb{F}_p$, defined by $y^2 = P(x)$ (with $deg(P) = 2g + 1$, squarefree). We would like to determine $|Jac(C, \mathbb{F}_p)|$.

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- We want to determine the action of F over A and $H^1_{MW}(C, A)$:

$$F(x) = x^p \mod p \qquad\qquad F(y) = y^p \mod p$$
$$P(F(x)) = F(y)^2$$

- With Weil's conjecture, $\chi_F \in \mathbb{Z}[T]$, and $|a_i| \leqslant 2^{2g} \sqrt{q}^i$.

Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=-l}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $l \in \mathbb{Z}$ and p a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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Definition

The **order**, or the **absolute precision** of $\sum_{i=k}^{d-1} a_i p^i + O(p^d)$ is d. Its **relative precision** corresponds to the number of its significant figures, and thus, is given by $d - \min\{i \in \mathbb{Z}, a_i \neq 0\}$.

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Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3, and its relative precision is $4 = 3 - (-1)$.

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The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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Remark

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1}p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,
$$\frac{1}{p^c y + O(p^d)} = y^{-1}p^{-c} + O(p^{d-2c})$$

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Theorem

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- *its coefficients are known up to $O(p^k)$.*
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- *its coefficients are known up to $O(p^k)$.*
- *$val(\Delta) < k$, with $\Delta = det((M_{i,j})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n})$.*

*Then the **loss of precision** to compute a row-echelon form of M is $\leq val(\Delta)$.*

Proof of the theorem

Gauss' method

$$M = \begin{bmatrix} m_{1,1} + O(p^k) & m_{1,2} + O(p^k) & \cdots\cdots & m_{1,m} + O(p^k) \\ m_{2,1} + O(p^k) & m_{2,2} + O(p^k) & \cdots\cdots & m_{2,m} + O(p^k) \end{bmatrix}$$

We assume that,

$$\det \left(\begin{bmatrix} m_{1,1} + O(p^k) & m_{1,2} + O(p^k) \\ m_{2,1} + O(p^k) & m_{2,2} + O(p^k) \end{bmatrix} \right) \neq O(p^k).$$

Proof of the theorem

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$$M \simeq \begin{bmatrix} p^{a_1} + O(p^k) & m_{1,2} + O(p^k) & \cdots\cdots & m_{1,m} + O(p^k) \\ \boxed{p^{a_2} + O(p^k)} & m_{2,2} + O(p^k) & \cdots & m_{2,m} + O(p^k) \end{bmatrix} \qquad L_2 \leftarrow L_2 - \frac{M_{2,1}^{(n-1)}}{M_{1,1}^{(n-1)}} L_1$$

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Indeed, $\boxed{M^{(n-1)}_{2,1} - \frac{M^{(n-1)}_{2,1}}{M^{(n-1)}_{1,1}} * M^{(n-1)}_{1,1} = 0}$ (formally).

In addition, $\frac{M^{(n-1)}_{2,1}}{M^{(n-1)}_{1,1}} = \frac{p^{a_2}+O(p^k)}{p^{a_1}+O(p^k)} = p^{a_2-a_1} + O(p^{k-a_1})$, therefore

$$\boxed{L_2 - \frac{M^{(n-1)}_{2,1}}{M^{(n-1)}_{1,1}} L_1 = L_2 + (p^{a_2-a_1} + O(p^{k-a_1}))L_1}.$$

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In the end, we get :

$$M = \begin{bmatrix} p^{a_1} + O(p^k) & m_{1,2} + O(p^k) & \cdots\cdots & m_{1,m} + O(p^k) \\ \boxed{0} & p^{a_2} + O(p^{k-a_1}) & \cdots & \boxed{m_{2,m} + O(p^{k-a_1})} \end{bmatrix}$$

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$$val\left(\det\left(\begin{bmatrix} m_{1,1} + O(p^k) & m_{1,2} + O(p^k) \\ m_{2,1} + O(p^k) & m_{2,2} + O(p^k) \end{bmatrix}\right)\right) = a_1 + a_2, \text{ with } a_i > 0.$$

The loss in precision is upper-bounded by

$\boxed{val(det((M_{i,j})_{1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant 2}))}$

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The Macaulay matrix

Notations

From now on, k is a field, $n, s \in \mathbb{N}$, and $R = k[X_1, \dots, X_n]$. We denote by R_d the homogeneous polynomials of degree d of R.
Let ω be a monomial order on R.

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Proposition (D. Lazard 83)

For an homogeneous ideal $I = (f_1, \ldots, f_s) \subset R$ ($f_1, \ldots, f_s$ being homogeneous), $d \in \mathbb{N}$, $I \cap R_d =< x^\alpha f_i, \, |\alpha| + deg(f_i) = d >$, as k-vector spaces .

The Macaulay matrix

Definition (Macaulay's matrix)

We denote by $Mac_d(f_1, \dots, f_s)$ the matrix :

$$\begin{array}{c c}
 & x^{d_1} > \quad \dots \quad > \quad \dots \quad > x^{d_{\binom{n+d-1}{n-1}}} \\
\begin{array}{c} x^{\alpha_{1,1}} f_1 \\ \vdots \\ x^{\alpha_{1,\binom{n+d-d_1-1}{n-1}}} f_1 \\ x^{\alpha_{2,1}} f_2 \\ \vdots \\ x^{\alpha_{s,\binom{n+d-d_s-1}{n-1}}} f_s \end{array} & \left[\begin{array}{c} \\ \\ x^{\alpha} f_i \quad \text{written} \quad \text{in the basis of the} \quad x^{d_i} \\ \\ \\ \end{array} \right]
\end{array}.$$

Its rows $x^\alpha f_i$ are written in the basis $x^{d_1}, \dots, x^{d_{\binom{n+d-1}{n-1}}}$, with $|\alpha| + \deg(f_i) = d$. Also, $x^{\alpha_{i,j}} < x^{\alpha_{i,j+1}}$.

An algorithm

The idea of the Matrix-F5 algorithm

The idea is to successively row-echelon the matrices $Mac_d(f_1, \dots, f_i)$ iteratively with d and i.
If you know the profile of $Mac_d(f_1, \dots, f_i)$, then you know what are the leading terms in $LT((f_1, \dots, f_i)_d)$ and so, you can remove useless rows in $Mac_{d'}(f_1, \dots, f_{i'})$ with $d' > d$ and $i' > i$.

An algorithm

The Matrix-F5 algorithm

Algorithm 1 Matrix-F5 algorithm

Let $F = (f_1, \ldots, f_s) \in R^s$, of degree $d_1, \ldots, d_s$, and $D \in \mathbb{N}$.
$G \leftarrow F$
for $d \in [\![0, D]\!]$ **do**
  for $i \in [\![1, s]\!]$ **do**
    Build $Mac_d f_1, \ldots, f_i$;
    Remove the rows $x^\alpha f_i$ such that x^α is the leading term of a row of $\widetilde{Mac_{d-d_i,i-1}}$;
    Compute the row-echelon form $\widetilde{Mac_{d,i}}$;
    Add to G the rows with a new leading monomial.
  end for
end for

The position of the leading terms ideals

Problem with testing nullity

A major issue can happen when dealing with finite-precision numbers : not being able to decide whether there is no non-zero pivot on a column or whether the precision is not enough.

Being able to compute the leading terms ideals

$$\begin{bmatrix} 1+O(p^k) & 1+O(p^k) & 1+O(p^k) & 0 \\ 1+O(p^k) & 1+O(p^k) & 0 & 1+O(p^k) \end{bmatrix} \qquad L_2 \leftarrow L_2 - \frac{M_{2,1}}{M_{1,1}} L_1$$

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What is the leading term for the second row ?

Moreno-Socias conjecture

Definition (weakly-w-ideal)

I is said to be a weakly-w-ideal if :

- for all x^α a leading monomial according to w of the reduced Gröbner basis of I,
- for all x^β such that $|\alpha| = |\beta|$ and $x^\beta > x^\alpha$,

we have $x^\beta \in LM(I)$.

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Conjecture (Moreno-Socias)

If k is an infinite field, $s \in \mathbb{N}$, $d_1, \dots, d_s \in \mathbb{N}$, then there is a non-empty Zariski-open subset U in $R_{d_1} \times \cdots \times R_{d_s}$ such that for all $(f_1, \dots, f_s) \in U$, $I = (f_1, \dots, f_s)$ is a weakly-grevlex ideal.

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Remark

If the conjecture holds, then regular sequences generating a weakly grevlex ideal are generic.

An algorithm suited for weakly-w-ideal

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We assume :

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Then, we can proceed :

- *At first, we proceed like in the normal F5M algorithm ;*

An algorithm suited for weakly-w-ideal

Proposition ("weak" F5M algorithm)

We assume :

- $(f_1, \ldots, f_s)$ *is a regular,*
- *the* $< f_1, \ldots, f_l >$ *are weakly-w-ideals,*
- *precision on the f_i's is enough.*

Then, we can proceed :

- *At first, we proceed like in the normal F5M algorithm ;*
- *But, as soon as a column with no non-zero pivot is encountered,* ***we halt*** *the row-echelon computation. Instead, we replace the non-reduced rows by (already reduced) multiples of the rows of* $\widetilde{Mac}_{d-1,i}$*, so as to get a matrix under row-echelon form.*

3 quadrics in 6 variables

An example

With 3 generic quadrics in 6 variables, what we get after reducing the Macauly matrix in degree 3 is the following :

$$\left[\begin{array}{c|c} \text{a 9x9 invertible block} & \text{(loss in precision : determinant of the 9x9 matrix)} \\ \hline 0 & \text{9 rows, multiples of rows of the matrix in degree 2} \end{array}\right].$$

About strongly stable ideals

Strongly stable ideal is not enough

In $\mathbb{Q}_p[x, y, z]$,let us take $f_1 = x^3 + xy^2$, $f_2 = x^2 y$, and $f_3 = x^2 z$. They generate a strongly stable initial ideal regarding to grevlex.

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Yet, one can not recover the initial ideal from approximations of $f_1, f_2, f_1 + f_3$.

$$x^3 > x^2y > xy^2 > y^3 > x^2z > \dots$$

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To sum up in one result

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We assume :

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Remark

Moreno-Socias conjecture implies that **Structure** is generic for grevlex.

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Step-by-step analysis is not optimal.

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- If we apply Gaussian elimination, the average precision on L is $O(p^{n-\frac{2d}{p-1}})$.
- If we study Cramer-style formulae, the intrinsic precision determined for L is $O(p^{n-2\log_p(d)})$.

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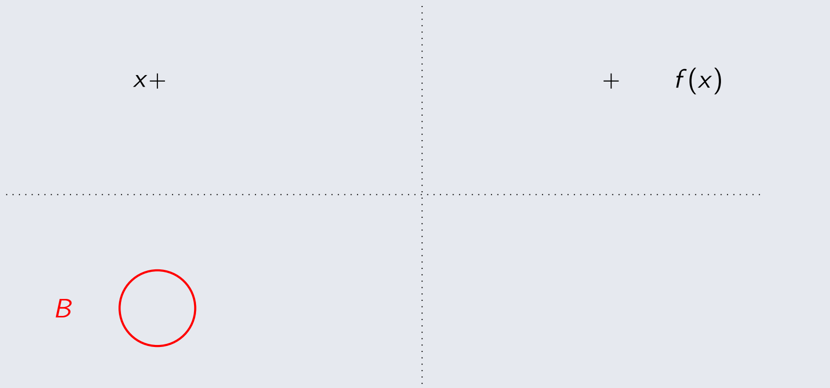
Geometrical meaning

Interpretation



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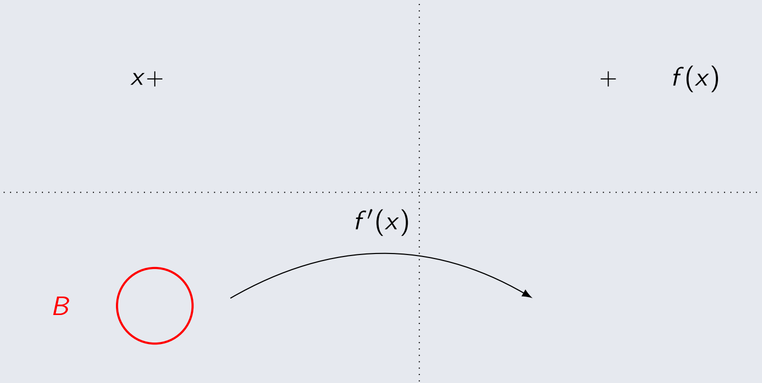

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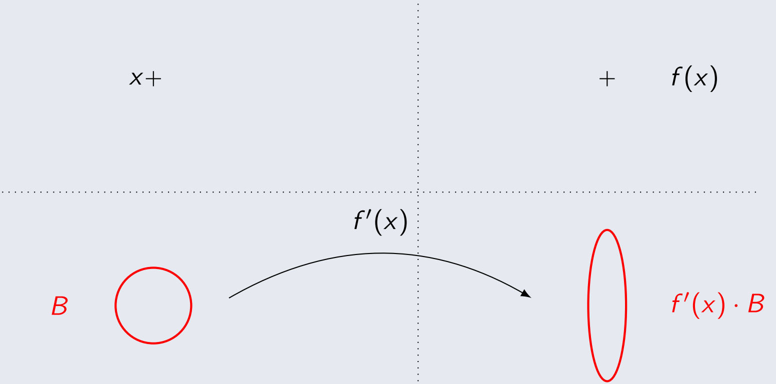

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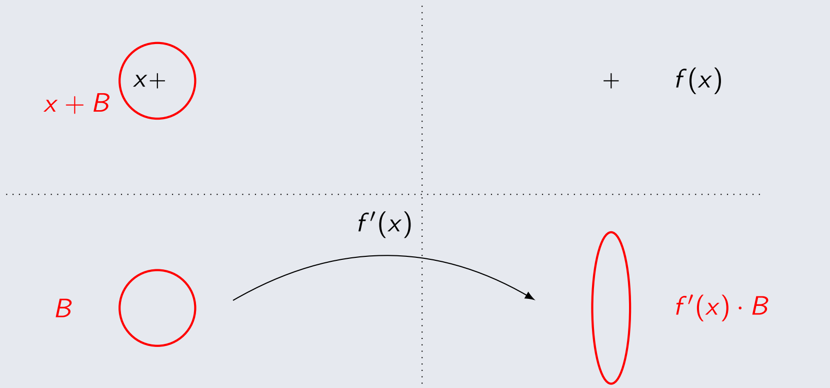

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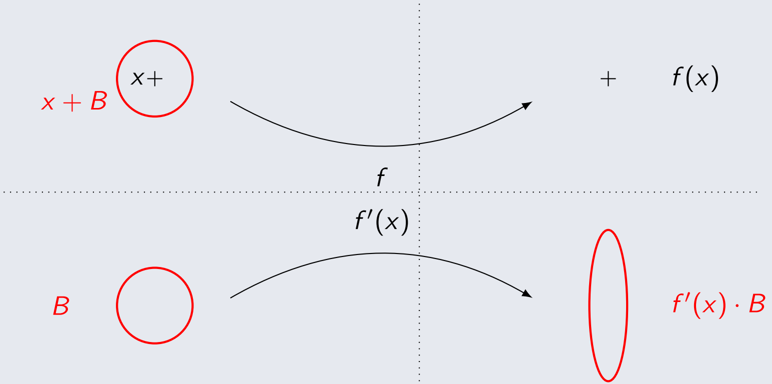

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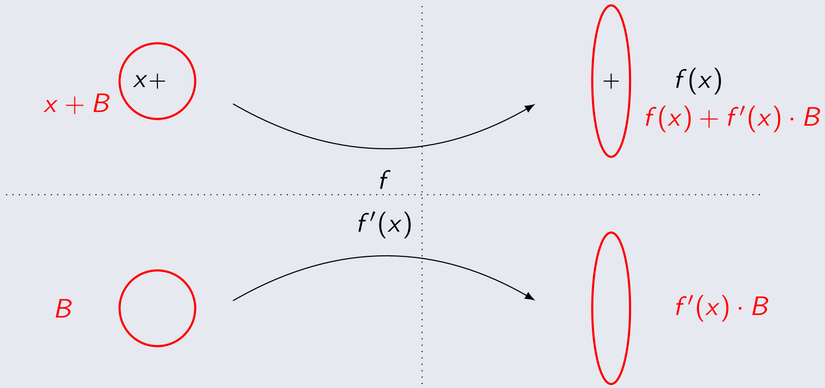

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Introduction to the Somos-4 sequence

Definition

We define the **Somos-4** sequence by recursion, with :

$$x_0, x_1, x_2, x_3 \in \mathbb{Z}_p^\times,$$

$$x_{n+4} = \frac{x_{n+1}x_{n+3} + x_{n+2}^2}{x_n}.$$

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Proposition

For all n, $x_n \in \mathbb{Z}_p$, i.e. $v_p(x_n) \geq 0$.

The Laurent phenomenon

Remark

If x_0, x_1, x_2, x_3 are known up to $O(p^m)$, then because of the division by x_n, a naive step-by-step analysis show that x_{n+4} is known up to $O(p^{m-\sum_{k=0}^{n} v_p(x_k)})$.

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Theorem (Fomin, Zelevinsky)

Let P_n be the rational fraction defined by the recursion formula defining Somos-4 :

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If m is big enough,

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Corollary

There is no intrinsic loss of precision : x_n is determined up to $O(p^m)$.

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*Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ be a **differentiable** mapping.*
*Let $x \in \mathbb{Q}_p^n$. We assume that $f'(x)$ is **surjective**.*
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Remark

This allows more models of precision, like

$$(x, y) = (1 + O(p^{10}), 1 + O(p)).$$

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As a consequence,

$$\frac{1}{k!} f^{(k)}(x) \cdot (p^m \mathbb{Z}_p)^k \subset p^m \mathbb{Z}_p.$$

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Computation in SOMOS-4

Loss in precision in SOMOS-4 with Sage ?!

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An explanation

The **gain** in precision in x_8 is invisible.

Lifting techniques

Methods comparison

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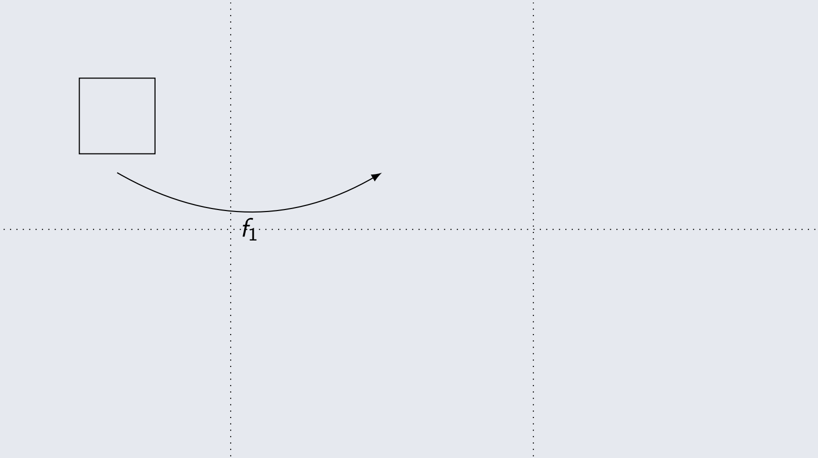

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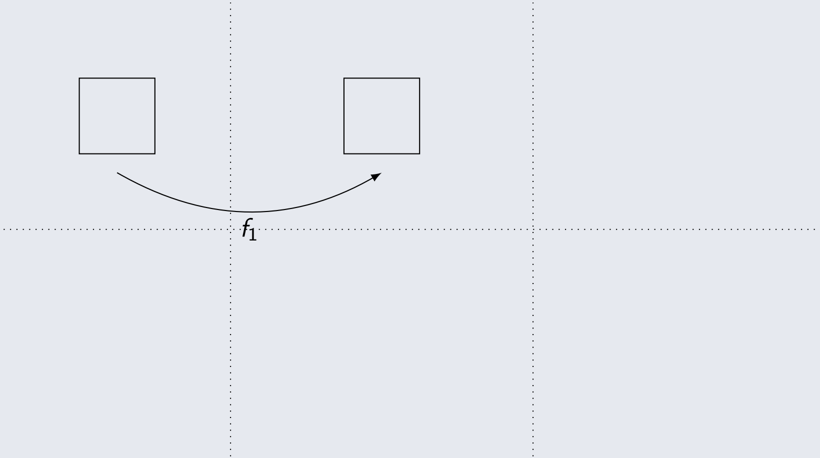

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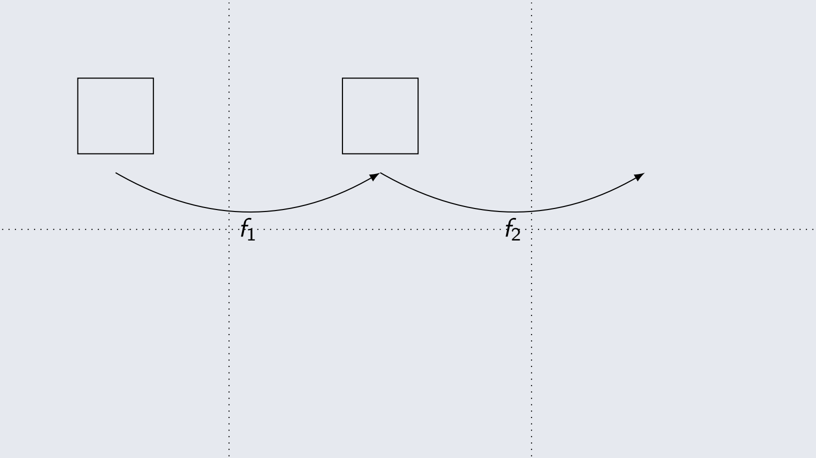

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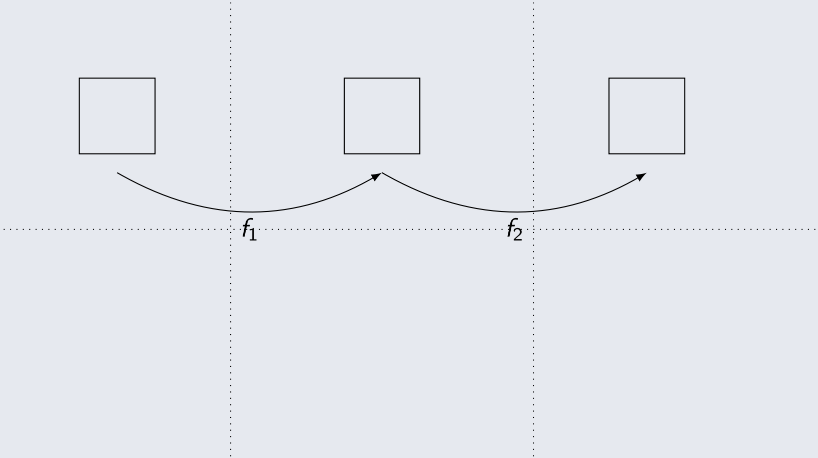

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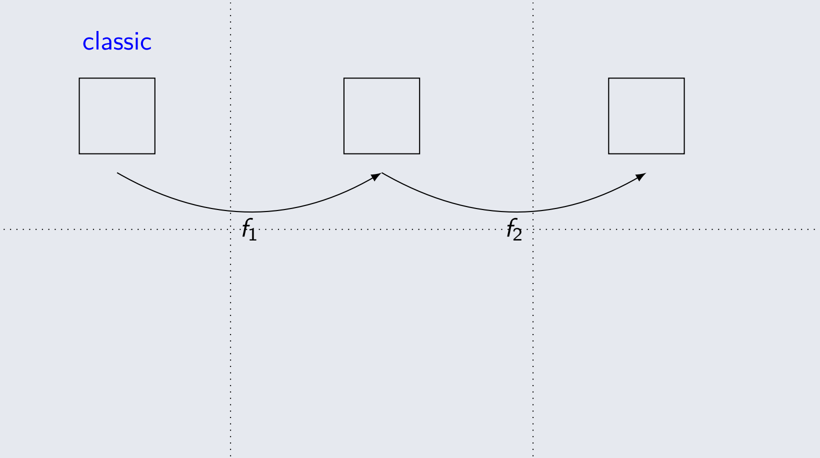

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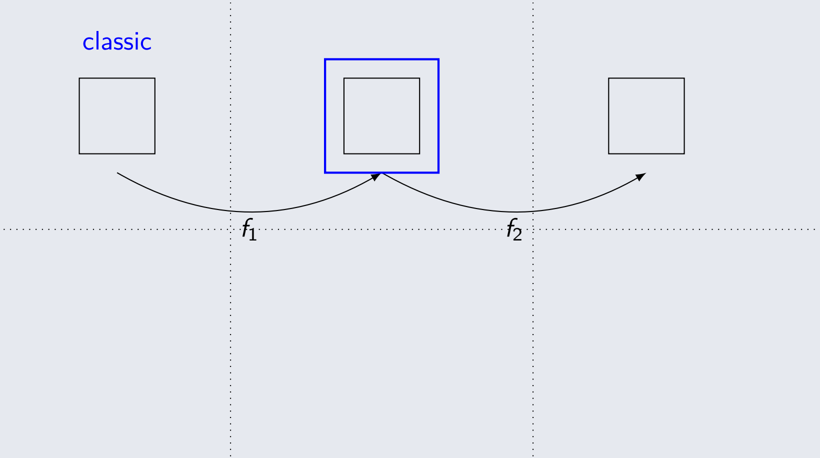

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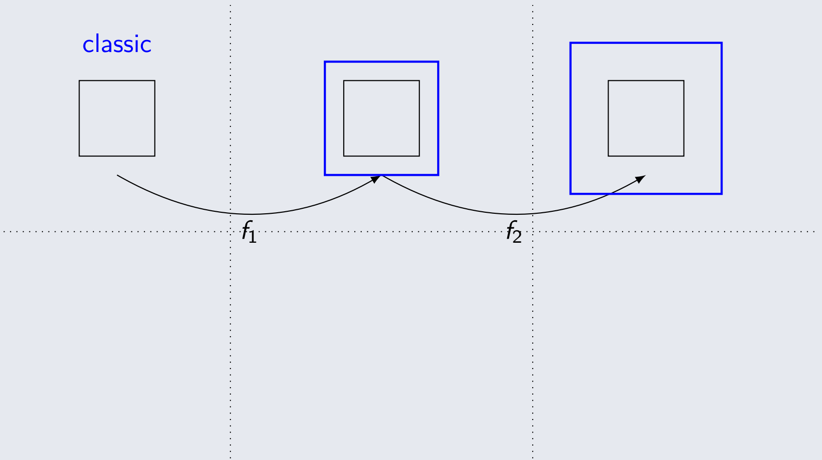

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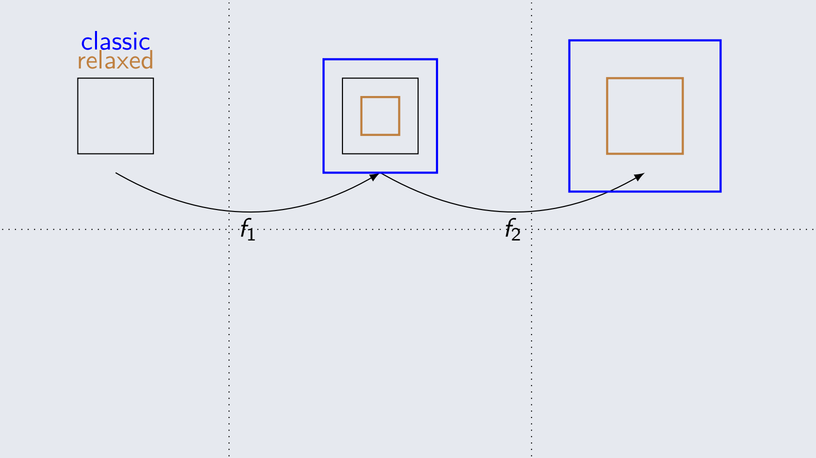

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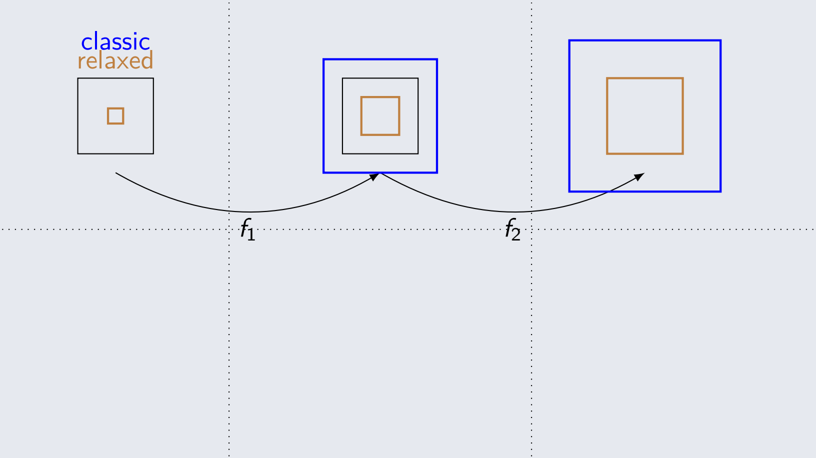

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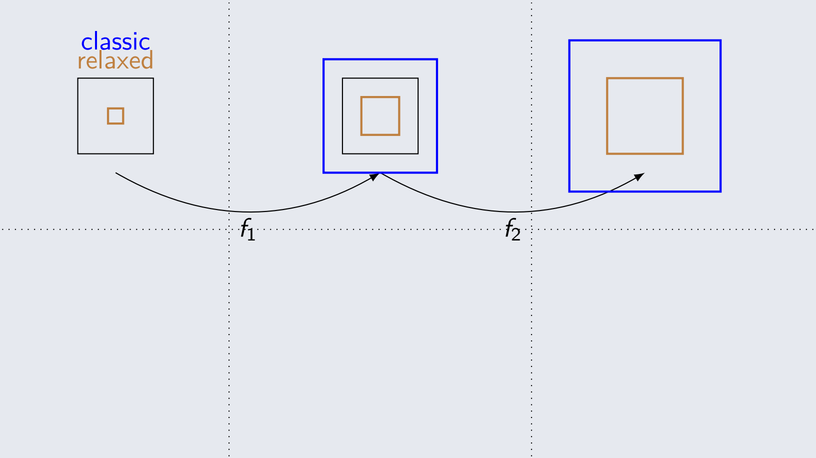

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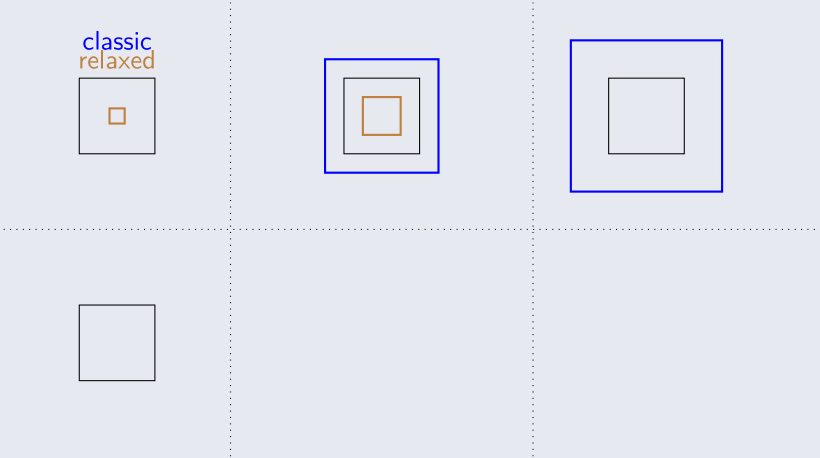

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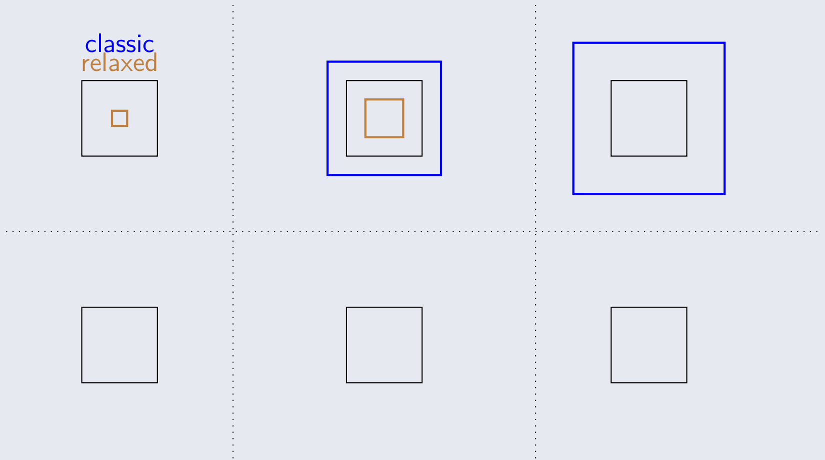

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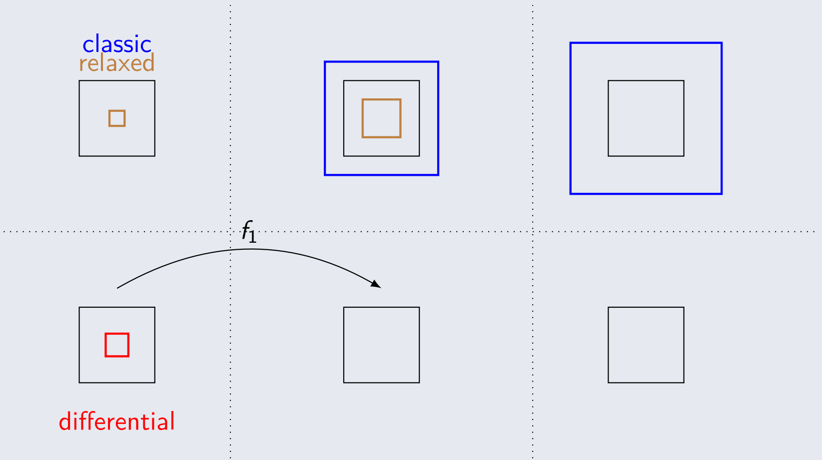

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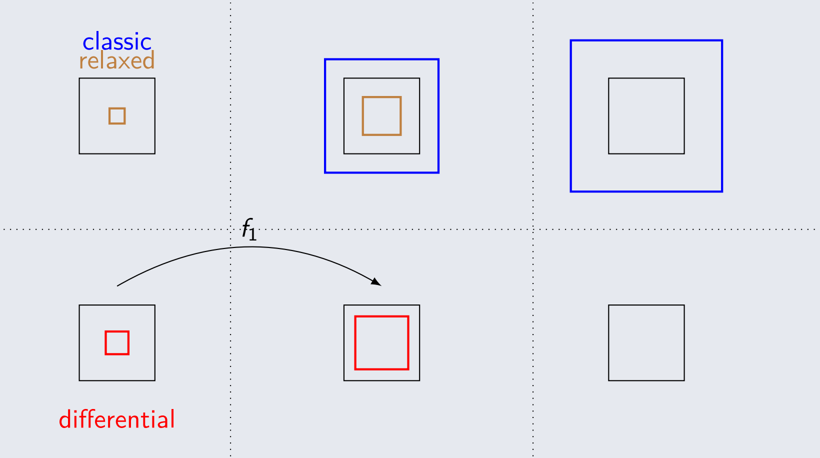

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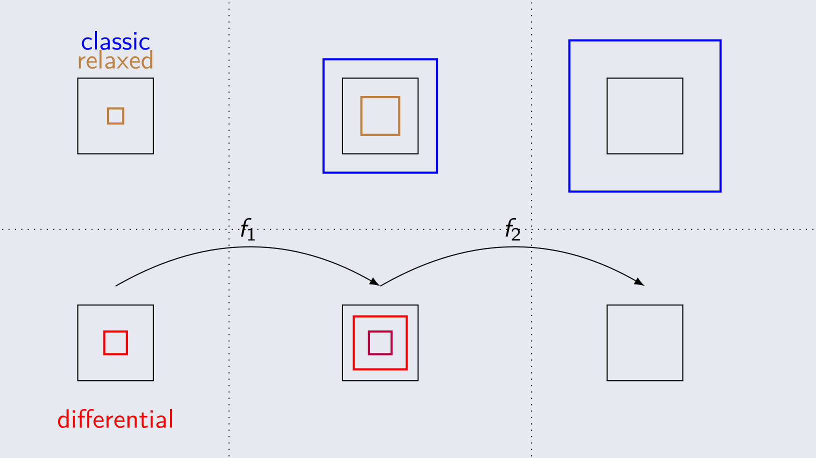

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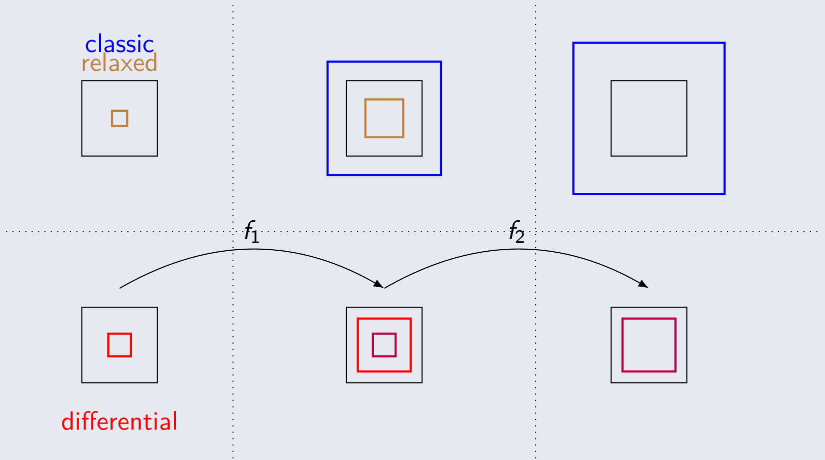

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Some calculus

Differential of the euclidean division

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Then,

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Therefore, δQ and δR are determined by the division of $\delta A - Q\delta B$ by B.

About matrices

Differential of the LU factorization

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Therefore, $\begin{aligned} \delta L &= L \times \left(L^{-1} \times \delta M \times U^{-1}\right)_{\mathrm{Low}} \\ \delta U &= \left(L^{-1} \times \delta M \times U^{-1}\right)_{\mathrm{Up}} \times U \end{aligned}$

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then δr is the remainder of the division of $f - (\delta q_1 \times f_1 + \dots \delta q_s \times f_s)$ by $(f_1, \dots, f_s)$.

Back to GB

Differential of reduced GB

Let $(f_1, \dots, f_s)$ satisfying **Structure**.

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$(\delta g_1, \dots, \delta g_t)$ is the remainder of the divisions of $(\delta f_1, \dots, \delta f_s) \times A$ by $(g_1, \dots, g_t)$.

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- New framework : differentials and lattices.

References

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