Computational Complexity of the Fisher Information

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Motivation

- **Epidemiology**

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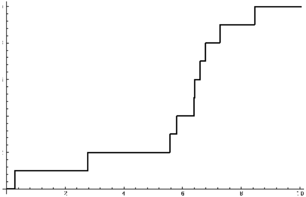
- A **Growing** Population



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- It is **Markovian**, that is

$$\Pr(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n, \ldots, , X_{t_1} = x_1) = \Pr(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n) ,$$

for all possible values of n and $t_1, \ldots, t_{n+1}$.

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  for all possible values of n and $t_1, \ldots, t_{n+1}$.
- The **transition probability** is equal to

  $$\Pr(X_{s+t} = j | X_s = i) \;=\; \binom{j-1}{i-1} e^{-\lambda t i}(1 - e^{-\lambda t})^{j-i}\,.$$

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- It can be shown that

$$\mathcal{FI}_{(X_{t_1},\ldots,X_{t_n})}(\lambda) \quad = \quad E_{\mathcal{L}}\left[\left(\frac{d}{d\lambda}\ln(\mathcal{L}(X_{t_1},\ldots,X_{t_n};\lambda))\right)^2\right].$$

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- Hence, $(t_1^*, \ldots, t_n^*) \in \operatorname{argmax}\{\mathcal{FI}_{(X_{t_1},\ldots,X_{t_n})}(\lambda)\}$.

Fisher Information and Optimal Observation Times

Proposition (Becker and Kersting, 1983)

The **Fisher information** *for a SBP with the parameter* λ, *the initial value of* x_0 *and the observation times of* $(t_1, \ldots, t_n)$ *is as follows:*

$$\mathcal{FI}_{(x_{t_1},\cdots,x_{t_n})}(\lambda) = x_0 \sum_{i=1}^{n} \frac{(t_i - t_{i-1})^2}{e^{-\lambda t_{i-1}} - e^{-\lambda t_i}} .$$

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Optimal Observation Times (Becker and Kersting, 1983)

$$\mathbf{t_i^* \approx \frac{3}{\lambda} \log\left(1 + \frac{i}{n}(e^{\frac{\lambda\tau}{3}} - 1)\right) \quad for\ i = 1, \ldots, n}$$

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- We call the stochastic process $\{Y_t : t \in \mathrm{R}_0^+\}$ the **partially-observable simple birth process (POSBP)** with parameters (λ, p).
- $\text{POSBP}(\lambda, 1) \equiv \text{SBP}(\lambda)$.

Markovian or non-Markovian?

Theorem (Bean, Elliott, Eshragh and Ross; 2014)

The POSBP $\{Y_t : t \in \mathrm{R}_0^+\}$ *with parameters* (λ, p) *is* **not Markovian**.

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The POSBP $\{Y_t : t \in \mathbb{R}_0^+\}$ with parameters (λ, p) is **not Markovian**.

- However,

$$\Pr(Y_{t_1} = y_{t_1}, \ldots, Y_{t_n} = y_{t_n} | X_{t_1} = x_{t_1}, \ldots, X_{t_n} = x_{t_n})$$

$$= \prod_{i=1}^{n} \Pr(Y_{t_i} = y_{t_i} | X_{t_i} = x_{t_i}) .$$

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where $q := 1 - p$ and $\upsilon_{i-1,i} := e^{-\lambda(t_i - t_{i-1})}$.

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&= \sum_{y_{t_1},\ldots,y_{t_n}} \frac{\left(\frac{d\mathcal{L}(y_{t_1},\ldots,y_{t_n};\lambda)}{d\lambda}\right)^2}{\mathcal{L}(y_{t_1},\ldots,y_{t_n};\lambda)}.
\end{aligned}$$

Theoretical Result

Proposition (Bean, Eshragh and Ross; 2014)

For a POSBP with n observations and time horizon τ, the FI is an **increasing** *function of t_n. Hence, the* **optimal observation time** *for the last observation, that is t_n^*, is equal to $\boldsymbol{\tau}$.*

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Proposition (Bean, Eshragh and Ross; 2014)

If $t_1^*, \ldots, t_n^*$ are optimal observation times for a POSBP with parameters (λ, p) and time-horizon τ, then $\frac{\mathbf{t_1^*}}{\boldsymbol{\tau}}, \ldots, \frac{\mathbf{t_n^*}}{\boldsymbol{\tau}}$ are **optimal observation times** *for a POSBP with parameters $(\boldsymbol{\lambda\tau}, p)$ and time-horizon $\mathbf{1}$.*

Truncated Summation

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$$\mathcal{FI}_{(Y_{t_1},\ldots,Y_{t_n})}(\lambda) \quad = \quad \sum_{y_{t_1},\ldots,y_{t_n}} \frac{\left(\frac{d\mathcal{L}(y_{t_1},\ldots,y_{t_n};\,\lambda)}{d\lambda}\right)^2}{\mathcal{L}(y_{t_1},\ldots,y_{t_n};\,\lambda)}.$$

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where $\upsilon_{i-1,i} := e^{-\lambda(t_i-t_{i-1})}$.

- By exploiting **Chebyshev's inequality**, we have

$$\begin{aligned}\Pr\left(E[Z]-12\sqrt{Var(Z)} \le Z \le E[Z]+12\sqrt{Var(Z)}\right) &\ge 1-\frac{1}{12^2} \\ &= 99.3\%.\end{aligned}$$

Conditional Expectations

- Motivating from Chebyshev's inequality:

$$0 \;\leq\; y_{t_i} \;\leq\; E[Y_{t_i}] + 12\sqrt{Var(Y_{t_i})}$$

$$\max\{1, y_{t_1}, \ldots, y_{t_n}\} \;\leq\; x_{t_n} \;\leq\; E[X_{t_n}|Y_{t_n} = y_{t_n}] + 12\sqrt{Var(X_{t_n}|Y_{t_n} = y_{t_n})}$$

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Lemma (Eshragh, Bean and Ross; 2014)

If $\{X_t\}$ is a **SBP** *with parameter λ and $\{Y_t\}$ is the corresponding* **POSBP** *with parameters (λ, p), then we have*

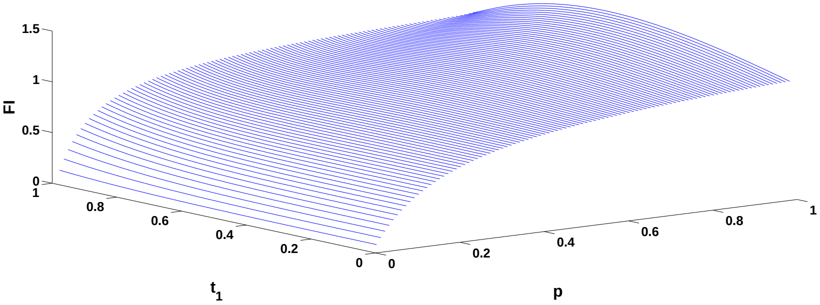
$$E[Y_t] = pe^{\lambda t}, \quad Var(Y_t) = p(pe^{2\lambda t} + (1-2p)e^{\lambda t})$$

$$E[X_t | Y_t = y_t] = \frac{y_t e^{\lambda t} + (1-p)(e^{\lambda t} - 1)}{pe^{\lambda t} + 1 - p}$$

$$Var(X_t | Y_t = y_t) = \frac{(y_t + 1)(1-p)e^{\lambda t}(e^{\lambda t} - 1)}{(pe^{\lambda t} + 1 - p)^2}.$$

Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Fisher Information vs. t_1 and p



Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

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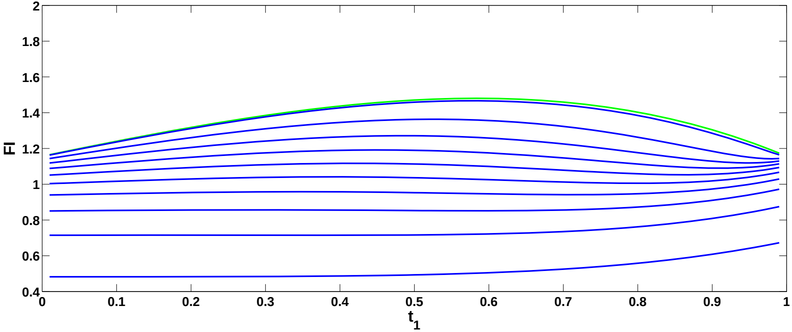
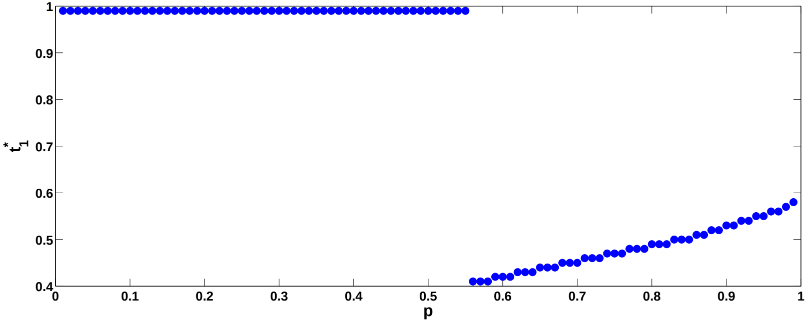


Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Optimal observation time t_1^* vs. p



The Chain Rule

- The likelihood function

$$\mathcal{L}(y_{t_1}, y_{t_2}|\lambda) = \Pr(Y_{t_2} = y_{t_2} | Y_{t_1} = y_{t_1}, \lambda) \Pr(Y_{t_1} = y_{t_1}|\lambda) \,.$$

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- Accordingly,

$$\log\left(\mathcal{L}(y_{t_1}, y_{t_2}|\lambda)\right) = \log\left(\Pr(Y_{t_2} = y_{t_2}|Y_{t_1} = y_{t_1}, \lambda)\right) + \log\left(\Pr(Y_{t_1} = y_{t_1}|\lambda)\right).$$

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- The Fisher Information:

$$\mathcal{FI}_{(Y_{t_1}, Y_{t_2})}(\lambda) = \mathcal{FI}_{(Y_{t_2}|Y_{t_1})}(\lambda) + \mathcal{FI}_{(Y_{t_1})}(\lambda).$$

Two-Parameter Geometric Distribution

Definition

A discrete random variable V has the "**Two-Parameter Geometric**" distribution with parameters $\alpha \in [0, 1)$ and $\beta \in (0, 1)$, denoted by **TPG**(α, β), if its **probability mass function** (**p.m.f.**) is

$$P_V(v) = \begin{cases} \alpha & \text{for } v = 0 \\ (1-\alpha)\beta(1-\beta)^{v-1} & \text{for } v = 1, 2, \dots . \end{cases}$$

Three-Parameter Negative Binomial Distribution

Definition

Suppose $V_1, \ldots, V_r$ are **i.i.d.** random variables with common $\mathrm{TPG}(\alpha, \beta)$ distribution. If $\mathbf{W} := \sum_{\mathbf{i}=1}^{\mathbf{r}} \mathbf{V_i}$, then W has "**Three-Parameter Negative Binomial**" distribution with parameters $\mathbf{r}$, α and β, denoted by $\mathbf{TPNB}(\mathbf{r}, \alpha, \beta)$.

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Proposition (Bean, Eshragh and Ross; 2014)

If W follows the $TPNB(r, \alpha, \beta)$ distribution, then its **p.m.f.** *is*

$$P_W(w) = \begin{cases} \alpha^r & \text{for } w = 0 \\ \displaystyle\sum_{\xi=1}^{\min\{r,w\}} \binom{w-1}{\xi-1} \beta^{\xi}(1-\beta)^{w-\xi} \binom{r}{\xi} (1-\alpha)^{\xi} \alpha^{r-\xi} & \text{for } w \geq 1 \end{cases}$$

The Distribution of Y_t

Theorem (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 \geq 1$. *For any real value* $t > 0$, *the random variable* Y_t *follows the* $\mathbf{TPNB(x_0, (1-p)\beta_t, \beta_t)}$ *distribution where*

$$\beta_t := \frac{e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} .$$

The Fisher Information for a Single Observation

Proposition (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p)*. The Fisher Information of a single observation* Y_{t_1} *for parameter* λ *is equal to*

$$\mathcal{FI}_{\mathbf{Y}_1}(\lambda) = \frac{pt_1^2\left(p+(1-p)(1-e^{-\lambda t_1})e^{-\lambda t_1}\right)}{(1-e^{-\lambda t_1})(p+(1-p)e^{-\lambda t_1})^2}.$$

The Distribution of $(Y2|Y1 = y_{t_1})$

Theorem (Bean, Eshragh and Ross; 2014)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p). *Then*

$$\mathbf{W} \stackrel{\mathbf{d}}{=} (\mathbf{Y_{t_2}}|\mathbf{Y_{t_1}} = \mathbf{y_{t_1}}) + \mathbf{V}$$

where $(Y_{t_2}|Y_{t_1} = y_{t_1})$ *and* V *are mutually independent and*

$$W \sim TPNB(y_{t_1} + 1, (1-p)\beta^{\circ}, \beta^{\circ})$$

and

$$V \sim TPG((1-p)\beta_{t2-t1}, \beta_{t2-t1}).$$

Bounds for the General Form of the Fisher Information

Theorem

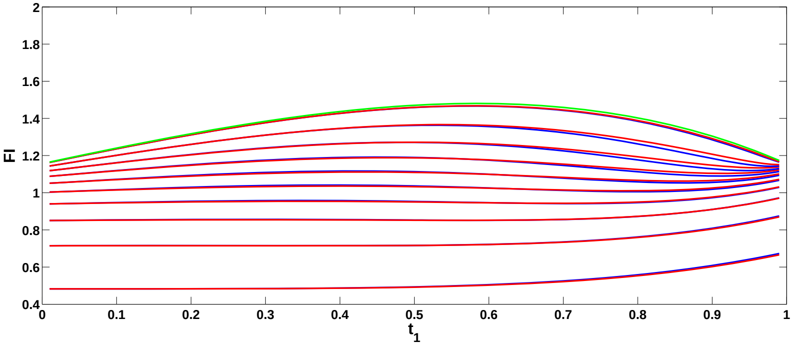
If $Z_1, \ldots, Z_n$ are independent random variables from distributions with common unknown parameter γ and $\mathbf{g} : \mathbb{R}^{\mathbf{n}} \rightarrow \mathbb{R}$ is a real-value function, then

$$\mathcal{FI}_{\mathbf{g}(\mathbf{Z}_1,\ldots,\mathbf{Z}_n)}(\gamma) \leq \sum_{\mathbf{i}=1}^{\mathbf{n}} \mathcal{FI}_{\mathbf{Z}_\mathbf{i}}(\gamma) .$$

Furthermore, equality occurs if and only if g is a sufficient estimator for γ.

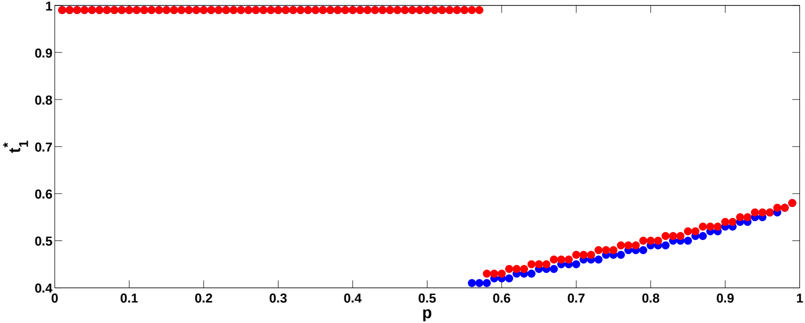
Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- The Fisher Information (blue) and its Approximation (red) vs. t_1



Results for $\lambda = 2$, $n = 2$ and $t_2^* = \tau = 1$

- Optimal observation time t_1^* vs. p



Bounds for the Fisher Information

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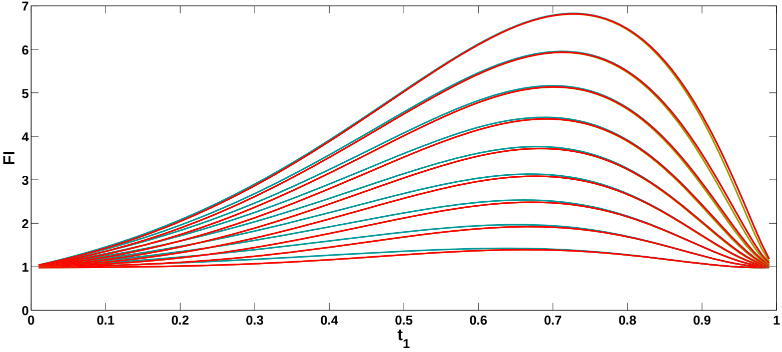
The approximation function for the Fisher Information ***lies within*** *the lower and upper bounds found for the Fisher Information.*

Theorem (Bean, Eshragh and Ross; 2014)

The lower and upper bounds for the Fisher Information ***approach together*** *as λ tends to infinity.*

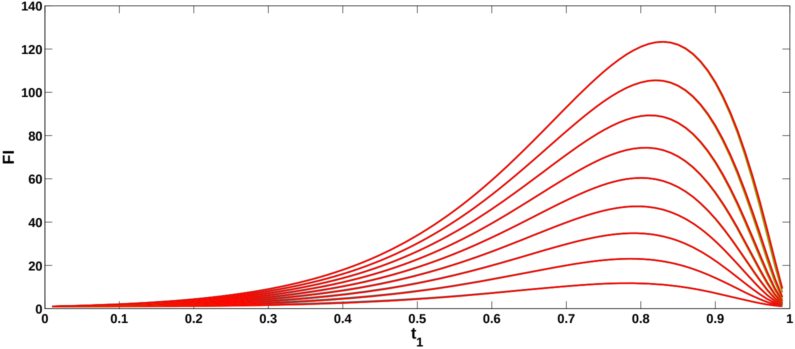
Results for $\lambda = 6$, $n = 2$ and $t_2^* = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Results for $\lambda = 10$, $n = 2$ and $t_2^* = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Further Developments

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- Developing analogous approximation for **higher values** of n .
- Investigating the quality of the **approximation**

$$\mathcal{FI}^{\mathbf{x}_0}(\lambda) \approx \mathbf{x}_0 \mathcal{FI}^1(\lambda)$$

  for $x_0 > 1$.
- Finding the Fisher Information to estimate parameter **p** along with λ, both together.

End

Thank you $\cdots$ Questions?