Transcendence of solutions of Mahler equations

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Abstract

- Generating functions of automatic sequences are solutions of Mahler equations

$$\phi_p^n y + a_{n-1}\phi_p^{n-1} y + \cdots + a_0 y = 0,$$

where $p \geq 2$, $\phi_p y(z) := y(z^p)$ $a_i \in \mathbb{C}(z)$, $0 \neq a_0$.
- Many authors are interested about the differential-algebraic properties of such generating functions.
- In this talk we use parametrized differential Galois theory to study this question in a systematic way.

Case $n = 1$

Proposition (D., Hardouin, Roques)

Let $f \neq 0$ such that $\phi_p(f) = a_0 f$. The following statements are equivalent:

1. *f is hyperalgebraic over $\mathbb{C}(z)$*[1]*;*
2. *there exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $u \in \mathbb{C}(z)^\times$ such that $a_0 = cz^m \frac{\phi_p(u)}{u}$.*

[1] We say that f is hyperalgebraic over $\mathbb{C}(z)$ if there is an algebraic relation over $\mathbb{C}(z)$ between f and its derivatives.

Case $n = 2$

Theorem (D., Hardouin, Roques)

Let $f(z) \in \mathbb{C}((z))$ be a nonzero solution of

$$\phi_p^2 y + a_1 \phi_p y + a_0 y = 0. \tag{1}$$

Assume that (1) can not be reduced into an order one equation[2]. Then, f is hypertranscendental over $\mathbb{C}(z)$.

[2] More formally, we assume that the difference Galois group contains $\mathrm{SL}_2(\mathbb{C})$.

The Baum-Sweet sequence

Example

The generating function of the Baum-Sweet sequence satisfies

$$\phi_2^2 y + z\phi_2 y - y = 0.$$

It is hypertranscendental.

The Rudin-Shapiro sequence

Example

The generating function of the Rudin-Shapiro sequence satisfies

$$\phi_2^2 y + \frac{1}{2z}\phi_2 y - \frac{1}{2z} y = 0.$$

It is hypertranscendental.

Difference Galois theory

Parametrized difference Galois theory

Hypertranscendence of solutions of Mahler equations

Consider the field

$$\mathbf{K} := \bigcup_{j\geq 1} \mathbb{C}\left(z^{1/j}\right),$$

we equip with the automorphism ϕ_p. Let

$$\phi_p Y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} Y = AY, \qquad (2)$$

which is equivalent to

$$\phi_p^2 y + a\phi_p y + by = 0,$$

with $a, b \in \mathbb{C}(z)$, $b \neq 0$.

Picard-Vessiot extension

A Picard-Vessiot ring for (2) over $\mathbf{K}$ is a difference ring extension $R|\mathbf{K}$ such that

1) there exists $U \in \mathrm{GL}_2(R)$ such that $\phi_p(U) = AU$;
2) R is generated, as a $\mathbf{K}$-algebra, by the entries of U and $\det(U)^{-1}$;
3) the only ϕ_p-ideals of R are $\{0\}$ and R.

Difference Galois group

Let $R|\mathbf{K}$ be a Picard-Vessiot ring for (2). The difference Galois group $\mathrm{Gal}(R/\mathbf{K})$ of R over $\mathbf{K}$ is the group of $\mathbf{K}$-automorphisms of R commuting with ϕ_p:

$$\mathrm{Gal}(R/\mathbf{K}) := \{\sigma \in \mathrm{Aut}(R/\mathbf{K}) \mid \phi_p \circ \sigma = \sigma \circ \phi_p\}.$$

The image

$$\begin{array}{ccc} \mathrm{Gal}(R/\mathbf{K}) & \rightarrow & \mathrm{GL}_2(\mathbb{C}) \\ \sigma & \mapsto & U^{-1}\sigma(U) \end{array}$$

is an algebraic subgroup of $\mathrm{GL}_2(\mathbb{C})$.

Proposition

The algebraic dimension of $R|\mathbf{K}$ equals to the dimension of $\mathrm{Gal}(R/\mathbf{K}) \subset \mathrm{GL}_2(\mathbb{C})$.

Theorem (Roques)

One of the three following cases occurs.

1. $\mathrm{Gal}(R/\mathbf{K})$ *is conjugated to a group on upper triangular matrices. This happens if and only if there exists a solution* $u \in \mathbf{K}$ *of the Riccati equation* $(\phi_p(u) + a)u = -b$.
2. *The first case does not occur and* $\mathrm{Gal}(R/\mathbf{K})$ *is conjugated to a subgroup of* $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & \gamma \\ \varepsilon & 0 \end{pmatrix} \middle| \gamma, \varepsilon \in \mathbb{C}^\times \right\}$. *This happens if and only if the first case does not occur and there exists a solution* $u \in \mathbf{K}$ *of the Riccati equation*
$$\left(\phi_p^2(u) + \left(\phi_p^2\left(\frac{b}{a}\right) - \phi_p(a) + \frac{\phi_p(b)}{a}\right)\right) u = -\frac{\phi_p(b)b}{a^2}.$$
3. $\mathrm{Gal}(R/\mathbf{K})$ *contains* $\mathrm{SL}_2(\mathbb{C})$.

Differentially closed field

Definition

Let $\mathbf{C}$ *be a field equipped with a derivation* δ*.*
We say that $(\mathbf{C}, \delta)$ *is differentially closed if, for every (finite) set of* δ*-polynomials* $\mathcal{F}$ *in coefficients in* $\mathbf{C}$*, if the system of differential equations* $\mathcal{F} = 0$ *has a solution with entries in some* δ*-field extension* $\mathbf{L}$*, then it has a solution with entries in* $\mathbf{C}$*.*

Any δ-field $\mathbf{C}$ has a differential closure $\widetilde{\mathbf{C}}$.

Consider the derivation

$$\delta := z\log(z)\partial_z, \text{ such that } \delta \circ \phi_p = \phi_p \circ \delta.$$

Let $(\widetilde{\mathbb{C}}, \delta)$ be a differential closure of $(\mathbb{C}, \delta)$. Let

$$\mathbf{L} := \mathrm{Frac}\left(\widetilde{\mathbb{C}} \otimes_{\mathbb{C}} \mathbf{K}(\log)\right).$$

Parametrized Picard-Vessiot extension

A parametrized Picard-Vessiot ring for (2) over $\mathbf{L}$ is a differential-difference ring extension $S|\mathbf{L}$ such that

1) there exists $U \in \mathrm{GL}_2(S)$ such that $\phi_p(U) = AU$;
2) S is generated, as a δ-$\mathbf{L}$-algebra, by the entries of U, and $\det(U)^{-1}$;
3) the only (δ, ϕ_p)-ideals of S are $\{0\}$ and S.

Parametrized difference Galois group

Let $S|\mathbf{L}$ be a parametrized Picard-Vessiot ring for (2). The parametrized difference Galois group $\mathrm{PGal}(S/\mathbf{L})$ of S over $\mathbf{L}$ is the group of $\mathbf{L}$-automorphisms of S commuting with ϕ_p and δ:

$$\mathrm{PGal}(S/\mathbf{L}) := \{\sigma \in \mathrm{Aut}(S/\mathbf{L}) \mid \phi_p \circ \sigma = \sigma \circ \phi_p, \delta \circ \sigma = \sigma \circ \delta\}.$$

Linear differential algebraic group

Definition

We say that a subgroup G of $\mathrm{GL}_2(\widetilde{\mathbb{C}})$ is a differential algebraic group if there exist $P_1, \dots, P_k$, δ-polynomials in 4 variables and in coefficients in $\widetilde{\mathbb{C}}$ such that for $A = (a_{i,j}) \in \mathrm{GL}_2(\widetilde{\mathbb{C}})$,

$$A \in G \Longleftrightarrow P_1(a_{i,j}) = \dots = P_k(a_{i,j}) = 0.$$

The image

$$\begin{array}{lcl} \mathrm{PGal}(S/\mathbf{L}) & \rightarrow & \mathrm{GL}_2(\widetilde{\mathbb{C}}) \\ \sigma & \mapsto & U^{-1}\sigma(U) \end{array}$$

is a differential algebraic subgroup of $\mathrm{GL}_2(\widetilde{\mathbb{C}})$.

Proposition (Hardouin-Singer)

The differential dimension of $S|\mathbf{L}$ equals to the dimension of $\mathrm{PGal}(S/\mathbf{L}) \subset \mathrm{GL}_2\left(\widetilde{\mathbb{C}}\right)$.

Case $n = 1$

Proposition (D., Hardouin, Roques)

Let $f \neq 0$ such that $\phi_p(f) = af$ with $a \neq 0$. We have the following alternative:

1. *f is hypertranscendental over $\mathbb{C}(z)$. In this case $\mathrm{PGal}(S/\mathbf{L}) = \widetilde{\mathbb{C}}^{\times}$;*
2. *f is hyperalgebraic over $\mathbb{C}(z)$. In this case $\mathrm{PGal}(S/\mathbf{L})$ is conjugated to a subgroup of $\mathbb{C}^{\times}$.*

Furthermore, the last case occurs if and only if there exist $c \in \mathbb{C}^{\times}$, $m \in \mathbb{Z}$ and $u \in \mathbb{C}(z)^{\times}$ such that $a = cz^m \frac{\phi_p(u)}{u}$.

From now, we consider

$$\phi_p Y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} Y = AY, \tag{2}$$

and assume that $\mathrm{Gal}(R/\mathbf{K})$ contains $\mathrm{SL}_2(\mathbb{C})$. This implies that $\mathrm{PGal}(S/\mathbf{L})$ contains $\mathrm{SL}_2(\mathbb{C})$.
Let $U \in \mathrm{GL}_2(S)$ be a fundamental solution. $\det(U)$ is solution of

$$\phi_p \det(U) = \det(A) \det(U) = b \det(U).$$

$\det(U)$ is hypertranscendental

Assume that $\det(U)$ is hypertranscendental over $\mathbb{C}(z)$. We have the following alternative:

1. $\mathrm{PGal}(S/\mathbf{L})$ is conjugated to $\widetilde{\mathbb{C}}^{\times}\mathrm{SL}_2(\mathbb{C})$;
2. $\mathrm{PGal}(S/\mathbf{L})$ is equal to a $\mathrm{GL}_2(\widetilde{\mathbb{C}})$.

Moreover, the first case holds if and only if there exists $B \in \mathbf{K}^{2\times 2}$ such that

$$p\phi_p(B) = ABA^{-1} + z\partial_z(A)A^{-1} - \frac{1}{2}z\partial_z(b)b^{-1}I_2.$$

$\det(U)$ is hypertranscendental

Theorem (D., Hardouin, Roques)

Assume that $\det(U)$ *is hypertranscendental over* $\mathbb{C}(z)$. *Assume that* $\phi_p^2 y + a\phi_p y + by = 0$ *admits a nonzero solution* $f \in \mathbb{C}((z))$. *Then,* f *is hypertranscendental over* $\mathbb{C}(z)$.

$\det(U)$ is hyperalgebraic

Theorem (D., Hardouin, Roques)

Assume that $\det(U)$ *is hyperalgebraic over* $\mathbb{C}(z)$*. Then, the parametrized difference Galois group* $\mathrm{PGal}(S/\mathbf{L})$ *is a subgroup of* $\mathbb{C}^{\times}\mathrm{SL}_2(\widetilde{\mathbb{C}})$ *containing* $\mathrm{SL}_2(\widetilde{\mathbb{C}})$*.*
Furthermore, if $\phi_p^2 y + a\phi_p y + by = 0$ *admits a nonzero solution* $f \in \mathbb{C}((z))$*, then* f *is hypertranscendental over* $\mathbb{C}(z)$*.*