On the Formalization of Foundations of Tarski's System of Geometry

Pierre Boutry

University of Strasbourg - ICube - CNRS

Computations and Proofs - Specfun - March 2016

Motivations

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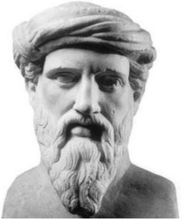
- Metamathematics;
- Education.



Pythagoras
(580 B.C. - 495 B.C.)

Motivations

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- The **missing** concept in *Euclid's Elements*

Euclid
(325 B.C. - 265 B.C.)

Motivations

- The **missing** concept in *Euclid's Elements*: the betweenness.

Moritz Pasch
(1843 - 1930)

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Archimedes
(287 B.C. - 212 B.C.)

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Posidonius
(135 B.C. - 51 B.C.)

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Ptolemy
(90 - 168)

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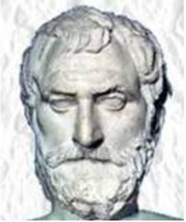

Proclus
(412 - 485)

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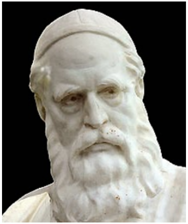

Omar Khayyam
(1048 - 1131)

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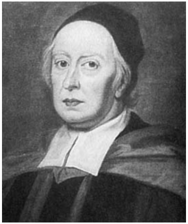

John Wallis
(1616 - 1703)

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Jean-Henri Lambert
(1728 - 1777)

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Adrien-Marie Legendre
(1752 - 1833)

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- We can still make **mistakes**.

It soon became clear that the only real long-term solution to the problems that I encountered is to start using computers in the verification of mathematical reasoning.

Vladimir Voevodsky
(1966 -)

(Vladimir Voevodsky, talk in March 2014 at the Institute for Advanced Studies at Princeton)

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Il n'en est pas moins certain que le théorème sur la somme des trois angles du triangle doit être regardé comme l'une de ces vérités fondamentales qu'il est impossible de contester, et qui sont un exemple toujours subsistant de la certitude mathématique qu'on recherche sans cesse et qu'on n'obtient que bien difficilement dans les autres branches des connaissances humaines.

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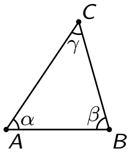

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Formalizations of foundations of geometry

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Euclide.

Les éléments.
Presses Universitaires de France, 1998.
Traduit par Bernard Vitrac.

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Foundations of Geometry (Grundlagen der Geometrie).

Open Court, La Salle, Illinois, 1960.
Second English edition, translated from the tenth German edition by Leo Unger. Original publication date, 1899.

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Alfred Tarski
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Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski.
Metamathematische Methoden in der Geometrie.
Springer-Verlag, Berlin, 1983.

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George David Birkhoff
(1884 - 1944)

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George David Birkhoff.
A set of postulates for plane geometry (based on scale and protractors).
Annals of Mathematics, 33, 1932.

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Felix Klein
(1849 - 1925)

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- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
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- Erlangen program: a geometry is defined as a space of objects and a group of transformations acting on it.

Felix C. Klein.

A comparative review of recent researches in geometry, 1872.

Outline

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The axioms

Alfred Tarski
(1901 - 1983)

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- A single primitive type: point.

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- 11 axioms.

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- 11 axioms.
- A parameter controls the dimension.

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The axioms

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
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- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoritical properties.

Alfred Tarski
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Axioms about congruence

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Axiom (Pseudo-transitivity for congruence)

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

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Axiom (Identity for congruence)

$$AB \equiv CC \Rightarrow A = B$$

Axiom about betweenness

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Axiom (Identity for betweenness)

$$A\text{—}B\text{—}A \Rightarrow A = B$$

Five-Segment Axiom

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Axiom (Five-Segment)

$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$
$$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$$
$$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$

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Axiom of Segment Construction

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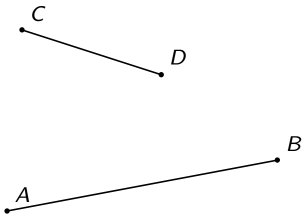
Axiom (Segment Construction)

$$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$$



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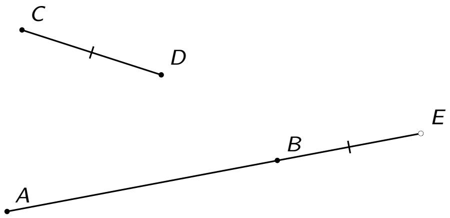

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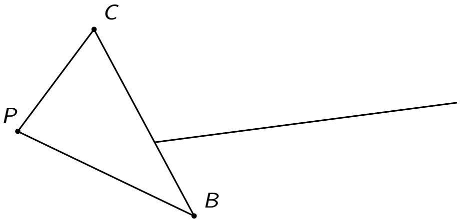

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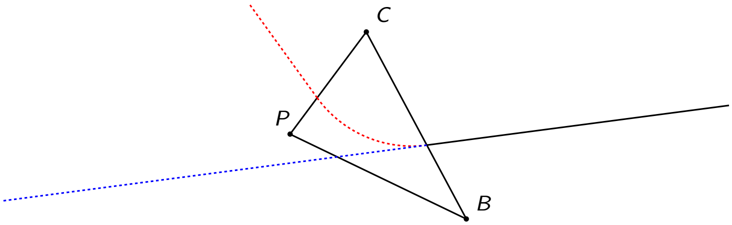

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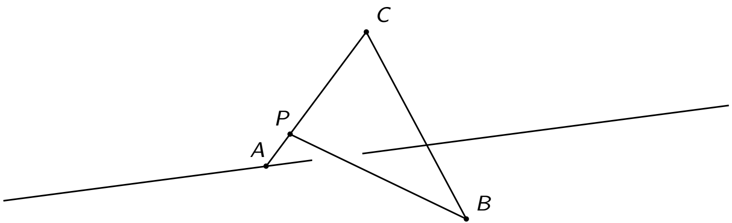

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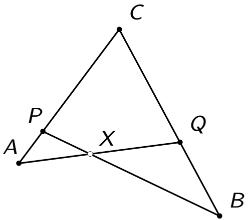

2-Dimensional Axiom

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Axiom (Lower 2-Dimensional)

$$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$$

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$$\exists ABC, \neg A\text{—}B\text{—}C \land \neg B\text{—}C\text{—}A \land \neg C\text{—}A\text{—}B$$

Axiom (Upper 2-Dimensional)

$$AP \equiv AQ \land BP \equiv BQ \land CP \equiv CQ \land P \neq Q \Rightarrow$$
$$A\text{—}B\text{—}C \lor B\text{—}C\text{—}A \lor C\text{—}A\text{—}B$$

Euclid's axiom

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Axiom (Euclid)

$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
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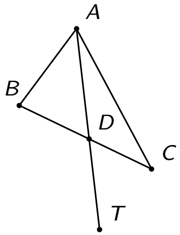

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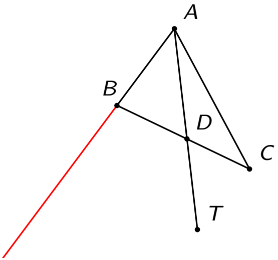

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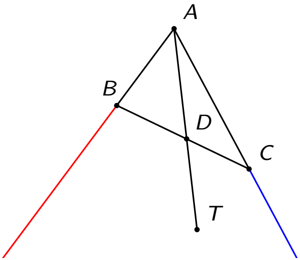

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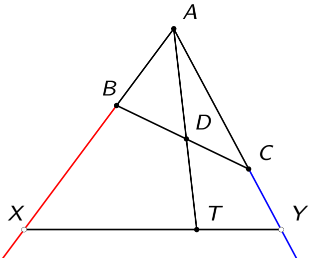

The axioms (summary)

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Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
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Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q$ $\Rightarrow A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X—B—Y)$

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Overview of the formalization

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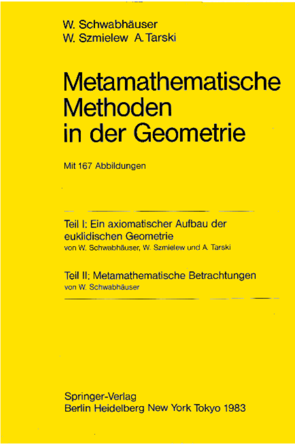

W. Schwabhäuser
W. Szmielew A. Tarski

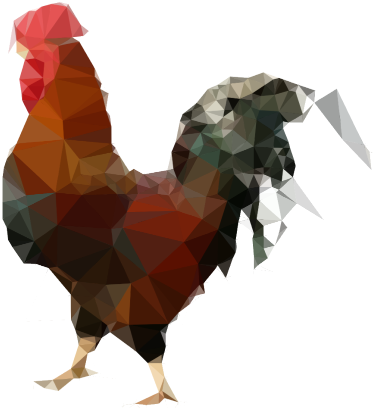
Metamathematische Methoden in der Geometrie

Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983



geocoq.github.io/GeoCoq/

Overview of the formalization

Chapter	Neutral $\geq$ 2D	$=$ 2D	Euclid	Continuity
Ch 2: Properties about betweenness	✓			
Ch 3: Properties about congruence	✓			
Ch 4: Properties about bet. et cong.	✓			
Ch 5: Order relation on points	✓			
Ch 6: Collinearity	✓			
Ch 7: Midpoint	✓			
Ch 8: Orthogonality	✓			
Ch 9: Planes	✓			
Ch 10: Reflection	✓	✓		
Ch 11: Angles	✓	✓		
Ch 12: Parallelism	✓	✓	✓	
Ch 13: Pappus and Desargues	✓	✓	✓	
Ch 14: Ordered field	✓	✓	✓	
Ch 15: Pythagorean ordered field	✓	✓	✓	
Ch 16: Coordinates	✓	✓	✓	

Outline

1. Introduction
2. Tarski's system of geometry
3. Parallel postulates
   - A syntaxic proof of the independence
   - Decidability of the predicates of the development
   - Equivalent statements
4. Arithmetization of geometry
5. Perspectives

Outline

Euclid's axiom

Axiom (Euclid)

$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
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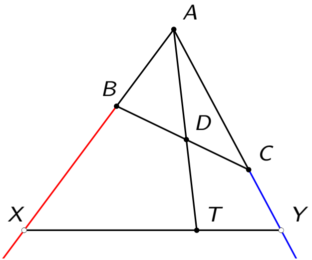

Types of independence proofs

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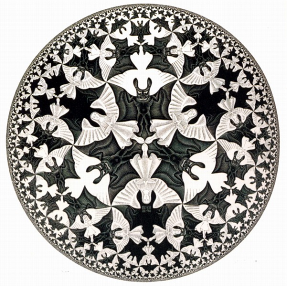
- Semantic proofs: prove the consistency of non-Euclidean geometry.

Types of independence proofs

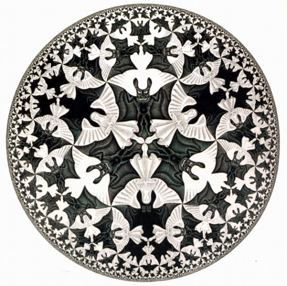
- Semantic proofs: prove the consistency of non-Euclidean geometry.



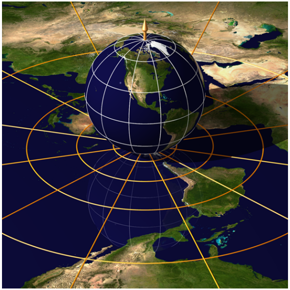
Hyperbolic geometry

Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.



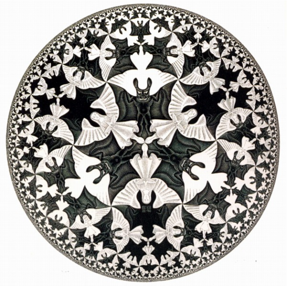
Hyperbolic geometry



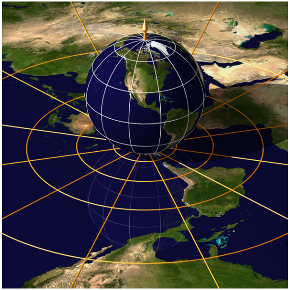
Elliptic geometry

Types of independence proofs

- Semantic proofs: prove the consistency of non-Euclidean geometry.



Hyperbolic geometry



Elliptic geometry

- Syntaxic proofs: prove there does not exist a derivation of the axiom from the others.

Syntaxic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X\text{—}B\text{—}Y)$

Syntaxic proof

Identity for betweenness	$A—B—A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$

Syntaxic proof

Identity for betweenness	$A–B–A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A–B–E \wedge BE \equiv CD$
Pasch	$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A–B–C \vee B–C–A \vee C–A–B$
Euclid	$A–D–T \wedge B–D–C \wedge A \neq D \Rightarrow$ $\exists XY, A–B–X \wedge A–C–Y \wedge X–T–Y$

Syntaxic proof

Identity for betweenness	A—B—$A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A$—B—$E \wedge BE \equiv CD$
Pasch	A—P—$C \wedge B$—Q—$C \Rightarrow \exists X, P$—$X$—$B \wedge Q$—$X$—$A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ A—B—$C \wedge A'$—B'—$C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A$—B—$C \wedge \neg B$—C—$A \wedge \neg C$—A—B
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ A—B—$C \vee B$—C—$A \vee C$—A—B
Euclid	A—D—$T \wedge B$—D—$C \wedge A \neq D \Rightarrow$ $\exists XY, A$—B—$X \wedge A$—C—$Y \wedge X$—T—Y

Syntaxic proof

Identity for betweenness	$A{-}B{-}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
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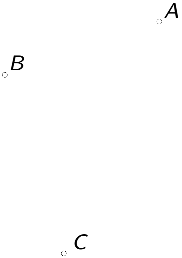
Pasch	$A{-}P{-}C \wedge B{-}Q{-}C \Rightarrow \exists X, P{-}X{-}B \wedge Q{-}X{-}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A{-}B{-}C \wedge A'{-}B'{-}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A{-}B{-}C \vee B{-}C{-}A \vee C{-}A{-}B$
Euclid	$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$ $\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$

Syntaxic proof

A—B—$A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A$—B—$E \wedge BE \equiv CD$
A—P—$C \wedge B$—Q—$C \Rightarrow \exists X, P$—$X$—$B \wedge Q$—$X$—$A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ A—B—$C \wedge A'$—B'—$C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A$—B—$C \wedge \neg B$—C—$A \wedge \neg C$—A—B
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ A—B—$C \vee B$—C—$A \vee C$—A—B
A—D—$T \wedge B$—D—$C \wedge A \neq D \Rightarrow$ $\exists XY, A$—B—$X \wedge A$—C—$Y \wedge X$—T—Y

Syntaxic proof



$A\text{—}B\text{—}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$

$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$

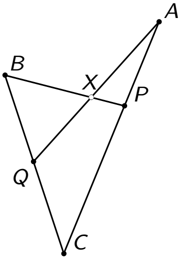
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$

$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntaxic proof



$A\text{—}B\text{—}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$

$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$

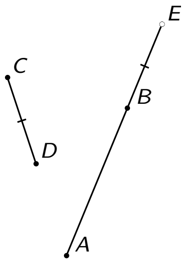
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$

$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntaxic proof



$A–B–A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A–B–E \wedge BE \equiv CD$
$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A–B–C \vee B–C–A \vee C–A–B$
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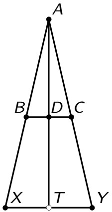
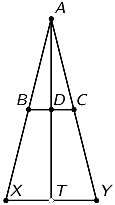
Syntaxic proof



$A—B—A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A—B—E \wedge BE \equiv CD$
$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$

Syntaxic proof



$A\text{—}B\text{—}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$

$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$

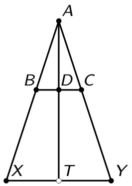
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$

$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntaxic proof



$A–B–A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A–B–E \wedge BE \equiv CD$

$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$

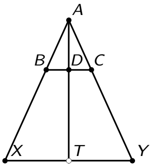
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A–B–C \vee B–C–A \vee C–A–B$

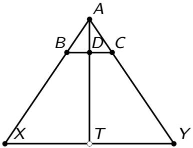
$A–D–T \wedge B–D–C \wedge A \neq D \Rightarrow$
$\exists XY, A–B–X \wedge A–C–Y \wedge X–T–Y$

Syntaxic proof



$A–B–A \Rightarrow A = B$
$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
$AB \equiv BA$
$AB \equiv CC \Rightarrow A = B$
$\exists E, A–B–E \wedge BE \equiv CD$
$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$
$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A–B–C \vee B–C–A \vee C–A–B$
$A–D–T \wedge B–D–C \wedge A \neq D \Rightarrow$ $\exists XY, A–B–X \wedge A–C–Y \wedge X–T–Y$

Syntaxic proof



$A–B–A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A–B–E \wedge BE \equiv CD$

$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$

$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A–B–C \vee B–C–A \vee C–A–B$

$A–D–T \wedge B–D–C \wedge A \neq D \Rightarrow$
$\exists XY, A–B–X \wedge A–C–Y \wedge X–T–Y$

Syntaxic proof

Identity for betweenness	$A—B—A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$

Syntaxic proof

Identity for betweenness	$A–B–A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
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Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A–B–E \wedge BE \equiv CD$
Pasch	$A–P–C \wedge B–Q–C \Rightarrow \exists X, P–X–B \wedge Q–X–A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A–B–C \wedge A'–B'–C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A–B–C \wedge \neg B–C–A \wedge \neg C–A–B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A–B–C \vee B–C–A \vee C–A–B$
Euclid	$A–D–T \wedge B–D–C \wedge A \neq D \Rightarrow$ $\exists XY, A–B–X \wedge A–C–Y \wedge X–T–Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A–X–Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X–B–Y)$

Outline

One remark

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Axiom (Playfair)

In a plane, there is at most one line parallel to another given line and passing by a given point.

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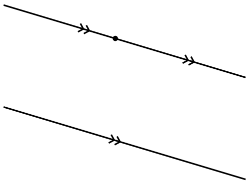
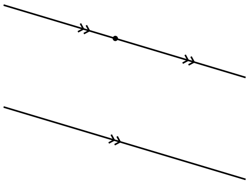
In a plane, there is at most one line parallel to another given line and passing by a given point.

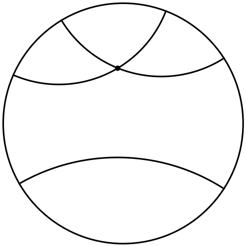


One remark

Axiom (Playfair)

In a plane, there is at most one line parallel to another given line and passing by a given point.





Intuitionistic Logic

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Axiom (Excluded middle (not admitted))

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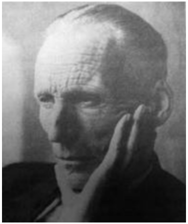

L. E. J. Brouwer
(1881 - 1966)

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A particular instance of the excluded middle

$$\forall ABCD, (\exists I, \text{Col } A\, B\, I \wedge \text{Col } C\, D\, I) \vee \neg(\exists I, \text{Col } A\, B\, I \wedge \text{Col } C\, D\, I)$$

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The most frequent instance of the excluded middle

$$\forall AB : Point, A = B \vee A \neq B$$

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- $\forall AB : Point, A = B \vee A \neq B$;
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- $\forall ABCD, AB \equiv CD \vee \neg AB \equiv CD$.

Results

Results

Identity for betweenness	$A{-}B{-}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A{-}B{-}E \wedge BE \equiv CD$
Pasch	$A{-}P{-}C \wedge B{-}Q{-}C \Rightarrow \exists X, P{-}X{-}B \wedge Q{-}X{-}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A{-}B{-}C \wedge A'{-}B'{-}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A{-}B{-}C \vee B{-}C{-}A \vee C{-}A{-}B$
Euclid	$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$ $\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$

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Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Decidability of equality	$A = B \vee A \neq B$

Results

We proved the decidability of:

- Bet $\forall ABC, A - B - C \lor \neg A - B - C$;
- Cong $\forall ABCD, AB \equiv CD \lor \neg AB \equiv CD$;
- Col $\forall ABC, \text{Col } A\ B\ C \lor \neg \text{ Col } A\ B\ C$;
- Out $\forall ABC, A \nleftrightarrow B \leftrightarrow C \lor \neg A \nleftrightarrow B \leftrightarrow C$;
- Per $\forall ABC, \llcorner A\ B\ C \lor \neg \llcorner A\ B\ C$;
- Perp_at $\forall ABCDP, AB \underset{P}{\perp} CD \lor \neg AB \underset{P}{\perp} CD$;
- TS $\forall ABCD, A \overset{D}{\underset{C}{\nparallel}} B \lor \neg A \overset{D}{\underset{C}{\nparallel}} B$;
- OS $\forall ABCD, A \underset{CD}{\nparallel} B \lor \neg A \underset{CD}{\nparallel} B$;
- CongA $\forall ABCDEF, A\ B\ C \mathrel{\widehat{=}} D\ E\ F \lor \neg A\ B\ C \mathrel{\widehat{=}} D\ E\ F$;
- Reflect ...

Results

Chapter	Neutral $\geq$ 2D	$=$ 2D	Euclid	Decidability of equality	Excluded middle
Ch 2: Properties about betweenness	✓			✓	
Ch 3: Properties about congruence	✓			✓	
Ch 4: Properties about bet. et cong.	✓			✓	
Ch 5: Order relation on points	✓			✓	
Ch 6: Collinearity	✓			✓	
Ch 7: Midpoint	✓			✓	
Ch 8: Orthogonality	✓			✓	
Ch 9: Planes	✓			✓	
Ch 10: Reflection	✓	✓		✓	
Ch 11: Angles	✓	✓		✓	
Ch 12: Parallelism	✓	✓	✓	✓	
Ch 13: Pappus and Desargues	✓	✓	✓	✓	
Ch 14: Ordered field	✓	✓	✓	✓	
Ch 15: Pythagorean ordered field	✓	✓	✓	✓	
Ch 16: Coordinates	✓	✓	✓	✓	

Outline

Equivalent statements

Equivalent statements

1. Postulate of existence of a right Saccheri quadrilateral
2. Postulate of existence of a right Lambert quadrilateral
3. Lambert's postulate
4. Posidonius' postulate
5. Existential Thales' postulate
6. Thales' converse postulate
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8. Postulate of existence of similar triangles
9. Triangle postulate
10. Postulate of existence of a triangle whose angles sum to 2 rights
11. Saccheri's hypothesis of right angle
12. Postulate of parallelism of perpendicular transversals
13. Proclus' second postulate
14. Alternative Playfair's postulate
15. Alternate interior angles postulate
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18. Postulate of transitivity of parallelism
19. Playfair's postulate
20. Perpendicular transversal postulate
21. Strong parallel postulate
22. Triangle circumscription principle
23. Tarski's version of the parallel postulate
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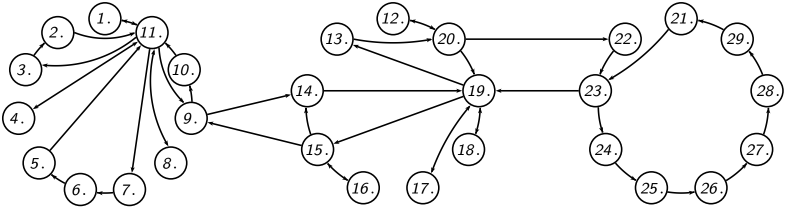

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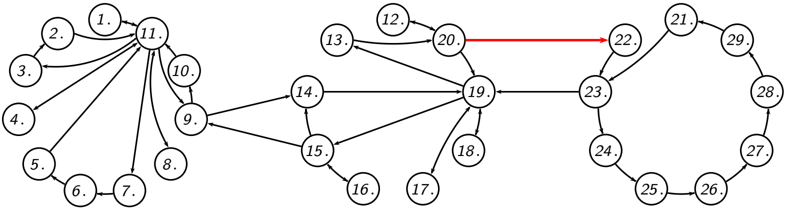

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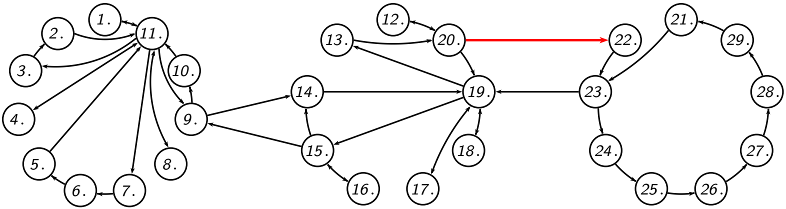

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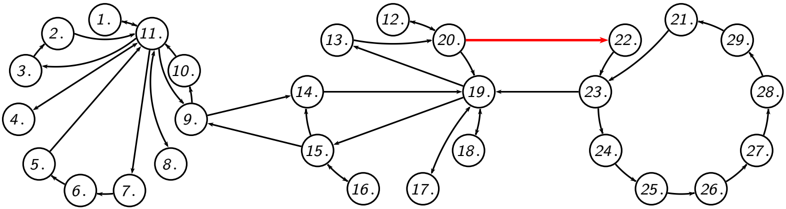

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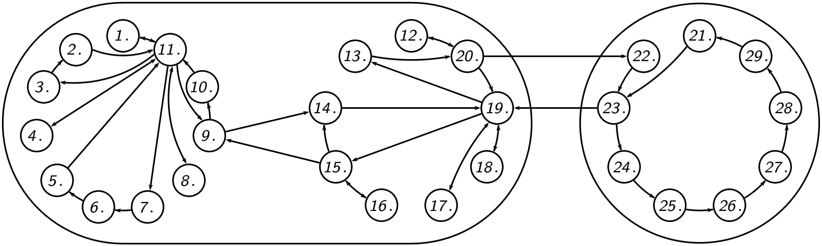

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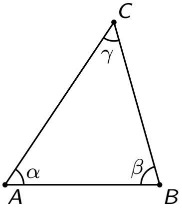

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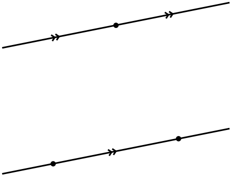

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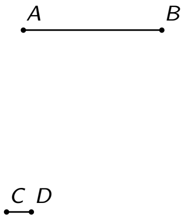

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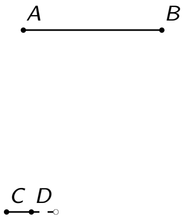

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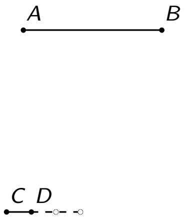

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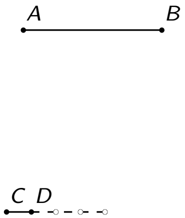

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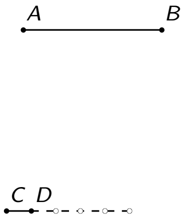

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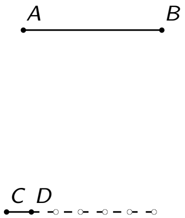

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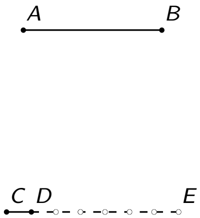

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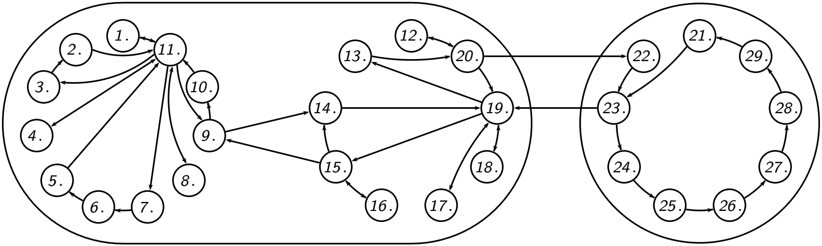

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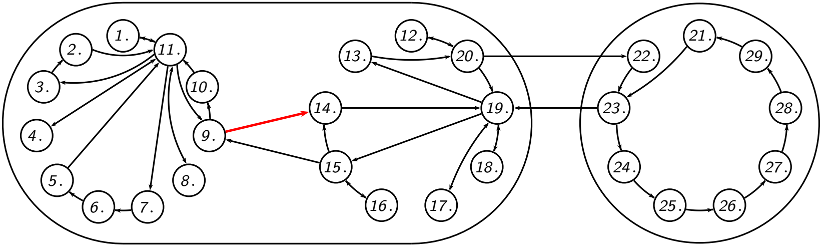

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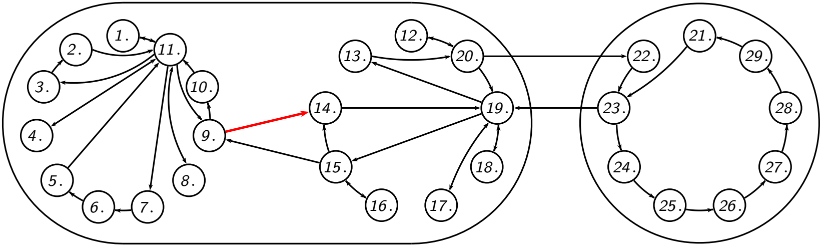

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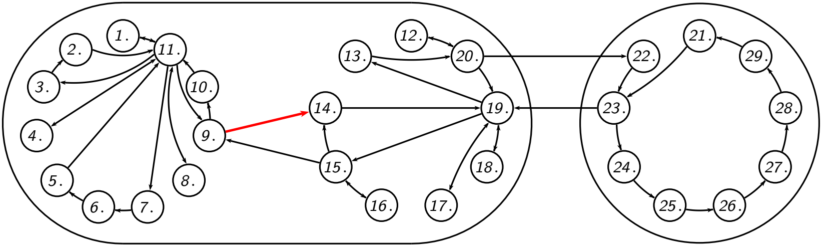

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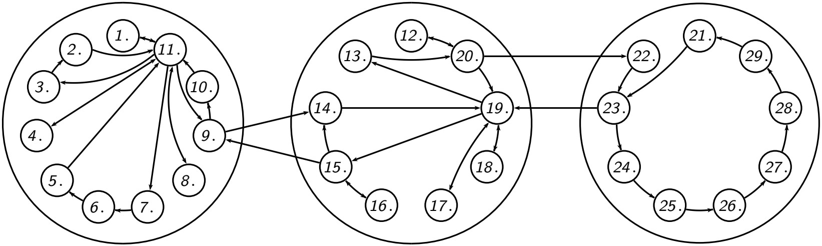

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Outline

Several ways to define the foundations of geometry

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- Synthetic approach: geometric objects and axioms about them.
  - Euclid
  - Hilbert
  - Tarski
- Analytic approach: a field $\mathbb{F}$ is assumed and the space is defined as $\mathbb{F}^n$.
- Mixed analytic/synthetic approach: existence of a field and geometric axioms.
  - Birkhoff
- Erlangen program: a geometry is defined as a space of objects and a group of transformations acting on it.

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René Descartes
(1596 - 1650)

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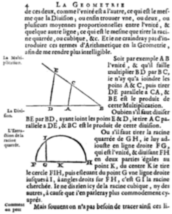

4 LA GEOMETRIE

de ces deux, comme l'vnité est a l'autre, ce qui est le mesme que la Diuision, ou enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne, ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, afin de me rendre plus intelligible.

La Multiplication.

Soit par exemple AB l'vnité, & qu'il faille multiplier BD par BC, ie n'ay qu'a ioindre les poins A & C, puis tirer DE parallele a CA, & BE est le produit de cette Multiplication.

La Diuision.

Oubien s'il faut diuiser BE par BD, ayant ioint les poins E & D, ie tire AC parallele a DE, & BC est le produit de cette diuision.

L'Extraction de la racine quarrée.

Ou s'il faut tirer la racine quarrée de GH, ie luy adiouste en ligne droite FG, qui est l'vnité, & diuisant FH en deux parties égales au point K, du centre K ie tire le cercle FIH, puis esleuant du point G vne ligne droite iusques à I, à angles droits sur FH, c'est GI la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cause que i'en parleray plus commodement cy-aprés.

Comment on peut

Mais souuent on n'a pas besoin de tracer ainsi ces li-

A page from *La Géométrie* of Descartes

Arithmetization of geometry

- These approches seem very different.
- In 1637, Descartes proved that the analytic approach can be derived from the synthetic approach.
- This is called arithmetization and coordination of geometry.

Arithmetization of geometry

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As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality, and thenceforth marched on at a rapid pace toward perfection.

(Joseph-Louis Lagrange, Leçons élémentaires sur les mathématiques; quoted by Morris Kline, Mathematical Thought from Ancient to modern Times, p. 322)

Joseph-Louis Lagrange
(1736 - 1813)

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René Descartes
(1596 - 1650)

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Alfred Tarski
(1901 - 1983)

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Outline

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Arithmetic operations

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These properties are summarized as:

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Definition Ar2 O E E' A B C :=
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Addition (a first approach)

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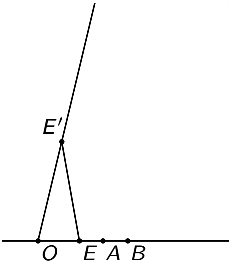
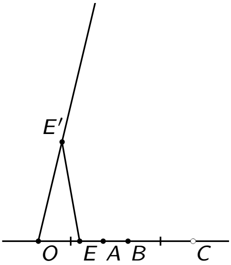


Addition (a first approach)

- Let us prolong $\overline{OB}$ but the length of $\overline{OA}$.



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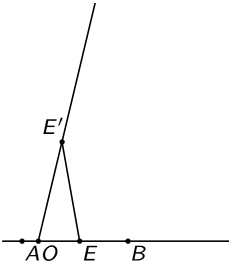

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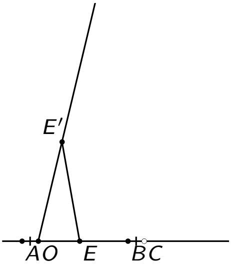

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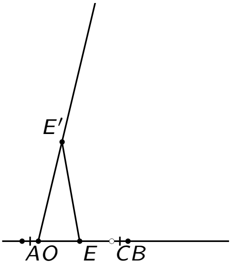

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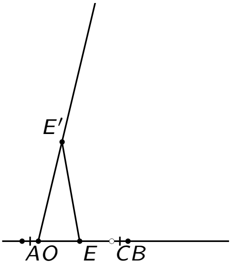
- Therefore, we need to be able to handle the **negative** points.



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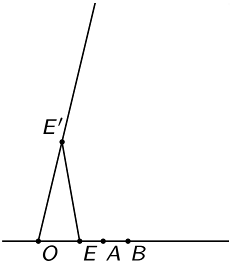

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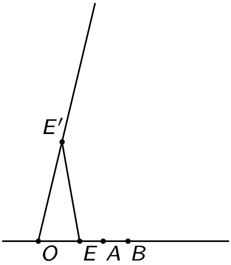

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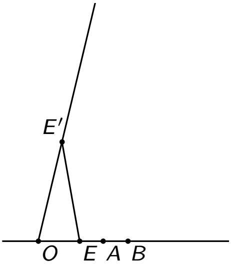

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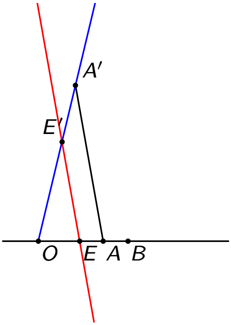

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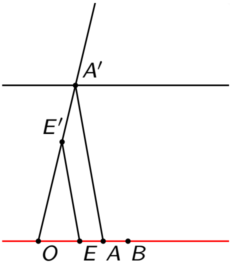

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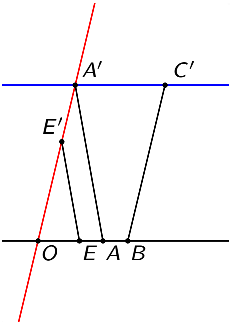

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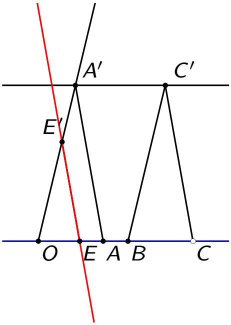

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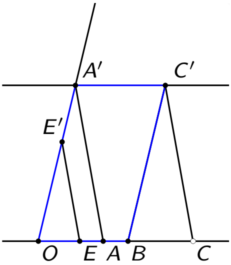

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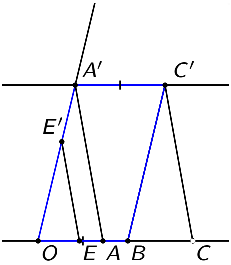

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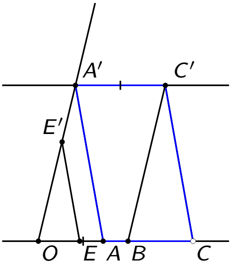

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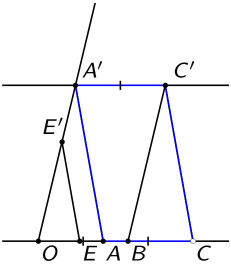

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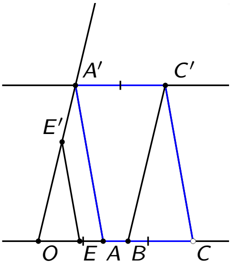
Properties of parallelograms to prove properties about Sum.



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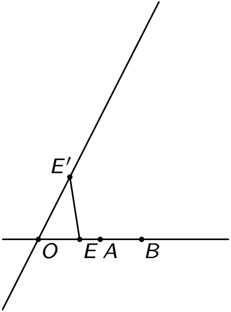

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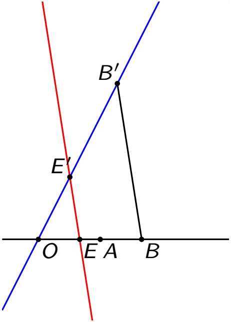

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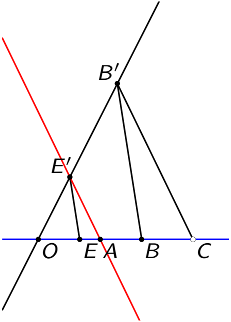

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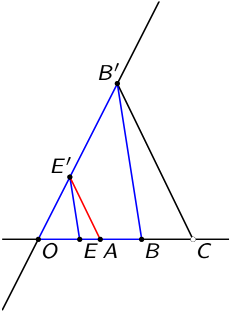

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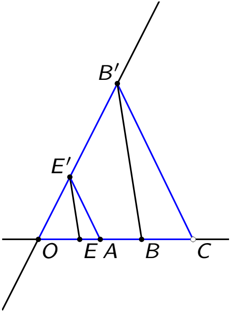

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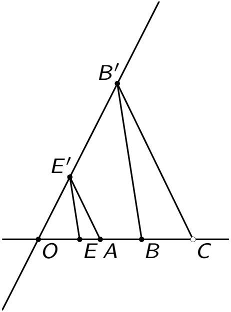

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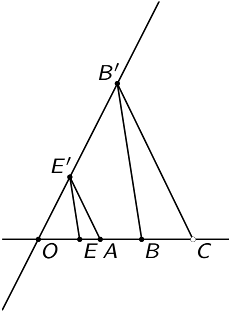

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However, this axiom turns the intuitionistic logic of Coq into an almost classical logic.

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Definition F : Type := {P: Tpoint | Col O E P}.
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Finally, we can prove we have a field:

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Lemma fieldF :
  (field_theory OF OneF AddF MulF SubF OppF DivF InvF EqF).
```

Outline

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Lemma characterization_of_congruence_F : forall A B C D,
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  (Ax - Bx) * (Ax - Bx) + (Ay - By) * (Ay - By) -
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Characterization of the predicates of the theory

We formalized the characterizations of the predicates of the theory.

For example, the characterization of the congruence predicate is:

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- Wu's approach: prove manually the characterizations **then** use these characterizations to obtain theorems automatically.

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- Our approach: prove manually only the first three characterizations and obtain **automatically** the others.

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But an equivalent statement:

```
Lemma characterization_of_parallelism_F_aux :
  forall A B C D,
  Par A B C D <->
  A <> B /\ C <> D /\
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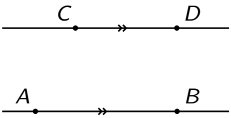

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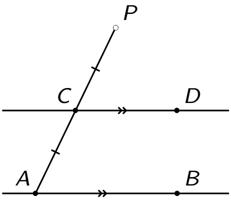

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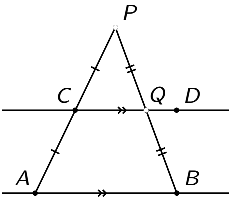

Automated proofs of characterizations

Geometric predicate	Algebraic Characterization
$AB \equiv CD$	$(x_A - x_B)^2 + (y_A - y_B)^2 - (x_C - x_D)^2 + (y_C - y_D)^2 = 0$
A—B—C	$\exists t, 0 \leq t \leq 1 \wedge t(x_C - x_A) = x_B - x_A \wedge t(y_C - y_A) = y_B - y_A$
Col $A\ B\ C$	$(x_A - x_B)(y_B - y_C) - (y_A - y_B)(x_B - x_C) = 0$
A-I-B	$2x_I - (x_A + x_B) = 0 \wedge 2y_I - (y_A + y_B) = 0$
$\triangle A\ B\ C$	$(x_A - x_B)(x_B - x_C) + (y_A - y_B)(y_B - y_C) = 0$
$AB \parallel CD$	$(x_A - x_B)(x_C - x_D) + (y_A - y_B)(y_C - y_C) = 0 \wedge (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$
$AB \perp CD$	$(x_A - x_B)(y_C - y_D) - (y_A - y_B)(x_C - x_D) = 0 \wedge (x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge (x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$

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$A - I - B$	$2x_I - (x_A + x_B) = 0 \wedge 2y_I - (y_A + y_B) = 0$
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Col $A\ B\ C$	$(x_A - x_B)(y_B - y_C) - (y_A - y_B)(x_B - x_C) = 0$
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We first proved the characterization of the midpoint predicate manually and then automatically and the script of the proof by computation was eight times shorter than our original one.

An example of proof by computation

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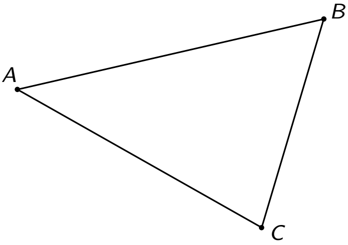
Our example is the nine-point circle theorem which states that the following nine points are concyclic:



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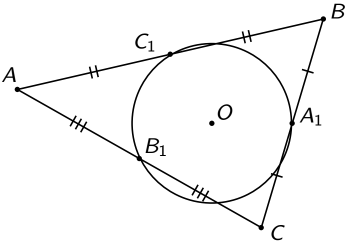
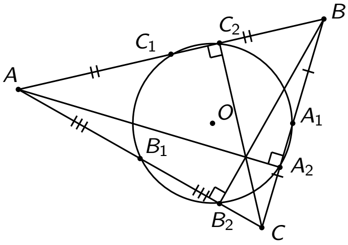


An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

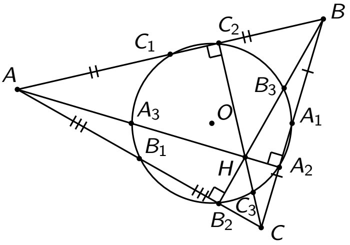
- The midpoints of each side of the triangle;
- The feet of each altitude;



An example of proof by computation

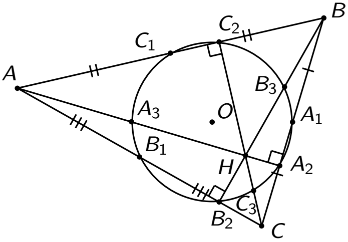
Our example is the nine-point circle theorem which states that the following nine points are concyclic:

- The midpoints of each side of the triangle;
- The feet of each altitude;
- The midpoints of the line-segments from each vertex of the triangle to the orthocenter.



An example of proof by computation

```
Lemma nine_point_circle:
forall A B C A1 B1 C1 A2 B2 C2 A3 B3 C3 H O,
  ~ Col A B C ->
  Col A B C2 -> Col B C A2 -> Col A C B2 ->
  Perp A B C C2 -> Perp B C A A2 -> Perp A C B B2 ->
  Perp A B C2 H -> Perp B C A2 H -> Perp A C B2 H ->
  Midpoint A3 A H -> Midpoint B3 B H -> Midpoint C3 C H ->
  Midpoint C1 A B -> Midpoint A1 B C -> Midpoint B1 C A ->
  Cong O A1 O B1 -> Cong O A1 O C1 ->
  Cong O A2 O A1 /\ Cong O B2 O A1 /\ Cong O C2 O A1 /\
  Cong O A3 O A1 /\ Cong O B3 O A1 /\ Cong O C3 O A1.
```



An example of proof by computation

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Lemma nine_point_circle:
forall A B C A1 B1 C1 A2 B2 C2 A3 B3 C3 H O,
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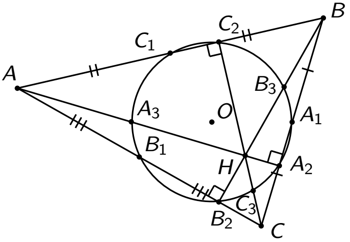
  Col A B C2 -> Col B C A2 -> Col A C B2 ->
  Perp A B C C2 -> Perp B C A A2 -> Perp A C B B2 ->
  Perp A B C2 H -> Perp B C A2 H -> Perp A C B2 H ->
  Midpoint A3 A H -> Midpoint B3 B H -> Midpoint C3 C H ->
  Midpoint C1 A B -> Midpoint A1 B C -> Midpoint B1 C A ->
  Cong O A1 O B1 -> Cong O A1 O C1 ->
  Cong O A2 O A1 /\ Cong O B2 O A1 /\ Cong O C2 O A1 /\
  Cong O A3 O A1 /\ Cong O B3 O A1 /\ Cong O C3 O A1.
```



- We did not prove a theorem about polynomials but a geometric statement.

An example of proof by computation

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Lemma nine_point_circle:
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  Perp A B C2 H -> Perp B C A2 H -> Perp A C B2 H ->
  Midpoint A3 A H -> Midpoint B3 B H -> Midpoint C3 C H ->
  Midpoint C1 A B -> Midpoint A1 B C -> Midpoint B1 C A ->
  Cong O A1 O B1 -> Cong O A1 O C1 ->
  Cong O A2 O A1 /\ Cong O B2 O A1 /\ Cong O C2 O A1 /\
  Cong O A3 O A1 /\ Cong O B3 O A1 /\ Cong O C3 O A1.
```

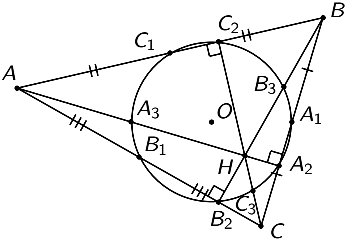

- We did not prove a theorem about polynomials but a geometric statement.
- The nine-point circle theorem is true in *any* model of Tarski's Euclidean geometry axioms (without continuity) and not only in a specific one.

Perspectives

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- Verify the **constructive** version of the arithmetization of geometry introduced by Beeson.
- Formalize the arithmetization of **hyperbolic** geometry.
- Extend our formalization of geometry to **higher dimension** geometry.

Thank you!