Guessing Linear Recurrence Relations of Sequence Tuples and P-recursive Sequences with Linear Algebra

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C-relations and P-relations.

The binomial sequence

$$\boldsymbol{b} = \begin{pmatrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

satisfies the relation

$$b_{i+1,j+1} = b_{i,j+1} + b_{i,j}.$$

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$$b_{i+1,j+1} = b_{i,j+1} + b_{i,j} \rightsquigarrow$$ constant coefficients $\to$ C-relation,

$$\begin{aligned} (i-j+1)\, b_{i+1,j} &= (i+1)\, b_{i,j} \\ (j+1)\, b_{i,j+1} &= (i-j)\, b_{i,j}. \end{aligned}$$

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Question.

How to compute or guess efficiently these C-relations and P-relations?

Dynamical Dictionary of Mathematical Functions (DDMF).

Generates dynamically and automatically a web page on function $\sum_{i_1,\dots,i_n\geq 0} u_{i_1,\dots,i_n}\, x^{i_1}\cdots x^{i_n}$ thanks to the P-relations satisfied by $(u_{i_1,\dots,i_n})_{(i_1,\dots,i_n)\in\mathbb{N}^n}$.

[Benoit, Chyzak, Darrasse, Gerhold, Mezzarobba, Salvy, 2010]

The Special Function e^x

rendering link

[-] 1. Differential Equation

The function e^x satisfies the differential equation

$$\frac{d}{dx}y(x) - y(x) = 0$$

with initial value $y(0) = 1$.

[+] 2. Plot

[-] 3. Numerical Evaluation

$$e^i \approx 0.54030231 + 0.84147098\, i.$$

(Below, `path` may be either a point z or a broken-line path $[z_1, z_2, \dots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form `x + y*i`.)

path = i precision = 8 Valider

[-] 4. Derivative in Terms of Lower-Order Derivatives

$$\frac{d^6}{dx^6}e^x = e^x.$$

order = 6 Valider

[-] 5. Taylor Expansion at 0

- Taylor coefficients:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

See the recurrence relations for the coefficients of the Taylor expansion.

Figure. Web page of $\exp(x) = \sum_{i\geq 0} u_i\, x^i$ from $\begin{cases} y(0) = 1 \\ \frac{d}{dx} y(x) = y(x), \end{cases}$ i.e. $\begin{cases} u_0 = 1 \\ (i+1)\, u_{i+1} = u_i. \end{cases}$

Planar and 3D-space walks.

Walk: The sequence $(u_{n,i_1,\ldots,i_d})_{(n,i_1,\ldots,i_d)\in\mathbb{N}^{d+1}}$ counts the number of ways to end in $(i_1,\ldots,i_d)$ starting from 0 with n steps in $\mathfrak{S}\subseteq\{-1,0,1\}^d$ while remaining in $\mathbb{N}^d$.

(planar walks) [Bousquet-Mélou, Mishna, 2010] [Bousquet-Mélou, Petkovšek, 2010] [Bostan, Raschel, Salvy, 2014]

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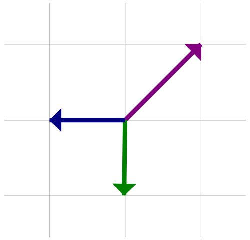

Figure. Kreweras's walk $\boldsymbol{K}$: $\mathfrak{S}=\{(-1,0),(0,-1),(1,1)\}$.

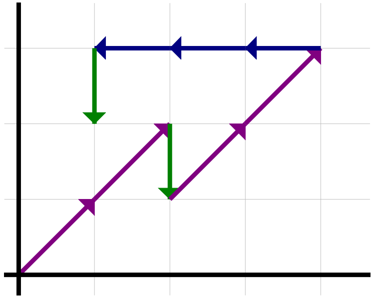

Figure. One of the $K_{9,1,2}=368$ ways to end in $(1,2)$ in 9 steps.

- At least one C-relation thanks to $\mathfrak{S}$: $K_{n+1,i,j}=K_{n,i+1,j}+K_{n,i,j+1}+K_{n,i-1,j-1}$. Any non trivial P-relations?

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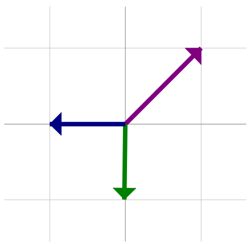

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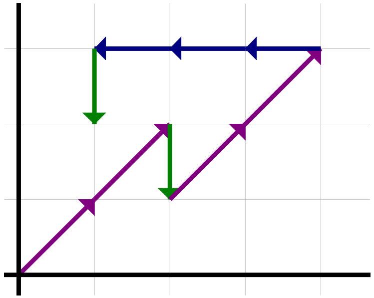

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- At least one C-relation thanks to $\mathfrak{S}$: $K_{n+1,i,j}=K_{n,i+1,j}+K_{n,i,j+1}+K_{n,i-1,j-1}$. Any non trivial P-relations? Yes: $i^2\,K_{n+3,i,j}=2\,(i+1)^2\,K_{n+2,i+1,j}-(i+2)^2\,K_{n+1,i+2,j}+4\,i^2\,K_{n+2,i,j+1}$ $-\,4\,(i+2)^2\,K_{n,i+2,j+1}-4\,i^2\,K_{n+1,i,j+2}-4\,(i+1)^2\,K_{n,i+1,j+2}+2\,i\,j\,K_{n+3,i,j}-6\,(i+1)\,j\,K_{n+2,i+1,j}-\cdots$

$\rightarrow$ Need for efficient computations!

- Planar walks classified but some 3D-space walks are still open problems!

Guessing the C-relations of a sequence.

→ The Berlekamp – Massey algorithm in dimension 1.

[Berlekamp, 1968] [Massey, 1969]

→ The Berlekamp – Massey – Sakata algorithm in dimension n.

[Sakata, 1988, 1990, 2009]

→ The Scalar-FGLM algorithm in dimension n. [B., Boyer, Faugère, 2015]

Guessing the P-relations of a sequence.

→ The Beckermann – Labahn algorithm in dimension 1.

[Beckermann, Labahn, 1994]

Proving parameterized sums or integrals.

→ The creative telescoping method.

[Zeilberger, 1990] and see [Chyzak, 2014] for a nice survey:

[Bostan, Chen, Chyzak, Kauers, Koutschan, Salvy, Zeilberger,...]

Past results.

- Computation of $\mathbb{C}$-relations (constant coefficients) for a sequence.
  - Use of linear algebra techniques.
  - Algebraic complexity.
  - Number of sequence queries.

[B., Boyer, Faugère, 2015]

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New results.

- Computation of C-relations for a sequence tuple.
- Computation of P-relations (polynomial coefficients) for a sequence.
    - Use of linear algebra techniques.
    - Use of Gröbner basis computations.
    - Algebraic complexity.
    - Number of sequence queries.

Notation – Link between C-relations and polynomials.

For a sequence $\boldsymbol{u}$, $[\boldsymbol{x}^{\boldsymbol{i}}]_{\boldsymbol{u}} = u_{\boldsymbol{i}}$.

$$\rightarrow \quad \begin{aligned}[(x\,y - y - 1)\,x^i\,y^j]_{\boldsymbol{b}} &= [x^{i+1}\,y^{j+1} - x^i\,y^{j+1} - x^i\,y^j]_{\boldsymbol{b}} \\ &= b_{i+1,j+1} - b_{i,j+1} - b_{i,j}.\end{aligned}$$

Definition.

The set $I = \{P \in \mathbb{K}[\boldsymbol{x}],\ \forall \boldsymbol{i} \in \mathbb{N}^n,\ [P\boldsymbol{x}^{\boldsymbol{i}}]_{\boldsymbol{u}} = 0\}$ is the ideal of C-relations of $\boldsymbol{u}$.

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Example.

Sequences	$\boldsymbol{u} = (2^i\,(i + 3^j))_{(i,j)\in\mathbb{N}^2}$	$\boldsymbol{b} = \left(\binom{i}{j}\right)_{(i,j)\in\mathbb{N}^2}$
Initial terms	$\begin{cases} u_{0,0} = 1 \\ u_{1,0} = 4 \\ u_{0,1} = 3 \end{cases}$	$\begin{cases} b_{i,0} = 1, & \forall i \in \mathbb{N} \\ b_{0,j} = 0, & \forall j \in \mathbb{N}^* \end{cases}$
C-relations	$\begin{cases} u_{i+2,j} = 4\,u_{i+1,j} - 4\,u_{i,j} \\ u_{i+1,j+1} = u_{i+1,j} + 2\,u_{i,j+1} - 2\,u_{i,j} \\ u_{i,j+2} = 4\,u_{i,j+1} - 3\,u_{i,j} \end{cases}$	$b_{i+1,j+1} = b_{i,j+1} + b_{i,j}$
Ideal of C-relations	$\langle x^2 - 4\,x + 4,\ x\,y - x - 2\,y + 2,\ y^2 - 4\,y + 3\rangle$	$\langle x\,y - y - 1\rangle$

Main idea.

Find relations $[\sum_{\boldsymbol{s}\in T}\alpha_{\boldsymbol{s}}\,\boldsymbol{x}^{\boldsymbol{s}+\boldsymbol{v}}]_{\boldsymbol{u}} = 0$ valid for all $\boldsymbol{v}\in\mathcal{U}$.

Definition.

For two ordered sets of terms T and U in $\boldsymbol{x}$, the multi-Hankel matrix $H_{U,T}$ is

$$\begin{array}{c} \\ \vdots \\ v\in U \\ \vdots \end{array} \begin{array}{c} \cdots \quad s\in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & [s\,v]_{\boldsymbol{u}} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix}. \end{array}$$

The set of the first linearly independent columns is the staircase of $H_{U,T}$.

Example.

$$H_{\{1,x,x^2,x^3\},\{1,x,x^2,x^3\}} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{matrix} 1 & x & x^2 & x^3 \end{matrix} \\ \begin{pmatrix} u_{0,0} & u_{1,0} & u_{2,0} & u_{3,0} \\ u_{1,0} & u_{2,0} & u_{3,0} & u_{4,0} \\ u_{2,0} & u_{3,0} & u_{4,0} & u_{5,0} \\ u_{3,0} & u_{4,0} & u_{5,0} & u_{6,0} \end{pmatrix} \end{array}$$

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Example.

$$H_{\{1,x,y,xy\},\{1,x,y,xy\}} = \begin{array}{c c} & \begin{array}{cccc} 1 & x & y & x\,y \end{array} \\ \begin{array}{c} 1 \\ x \\ y \\ x\,y \end{array} & \begin{pmatrix} u_{0,0} & u_{1,0} & u_{0,1} & u_{1,1} \\ u_{1,0} & u_{2,0} & u_{1,1} & u_{2,1} \\ u_{0,1} & u_{1,1} & u_{0,2} & u_{1,2} \\ u_{1,1} & u_{2,1} & u_{1,2} & u_{2,2} \end{pmatrix} \end{array}$$

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The set of the first linearly independent columns is the staircase of $H_{U,T}$.

Example.

For $\boldsymbol{u}=(2^i\,(i+3^j))_{(i,j)\in\mathbb{N}^2}$ and $T=U=\{1,x,y,x^2,x\,y,y^2\}$, $H_{T,T}$ has rank 3 and the staircase is $\{1,x,y\}$.

$$H_{T,T}=\begin{array}{c} \\ 1 \\ x \\ y \\ x^2 \\ x\,y \\ y^2 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & x & y & x^2 & x\,y & y^2 \end{array} \\ \left(\begin{array}{cccccc} 1 & 4 & 3 & 12 & 8 & 9 \\ 4 & 12 & 8 & 32 & 20 & 20 \\ 3 & 8 & 9 & 20 & 20 & 27 \\ 12 & 32 & 20 & 80 & 48 & 44 \\ 8 & 20 & 20 & 48 & 44 & 56 \\ 9 & 20 & 27 & 44 & 56 & 81 \end{array}\right). \end{array}$$

Informal version of the Scalar-FGLM algorithm.

Input.

- A sequence $\boldsymbol{u} = (u_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^n}$ over $\mathbb{K}$;
- The ordered set T of all monomials in $\boldsymbol{x}$ of degree at most d wrt. $\prec$.

Output.

→ A reduced d-truncated Gröbner basis of the ideal of C-relations of $\boldsymbol{u}$.

1. Compute S the staircase of $H_{T,T} := \begin{array}{c} \\ \vdots \\ v \in T \\ \vdots \end{array} \begin{array}{c} \cdots \quad s \in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & [s\, v]_{\boldsymbol{u}} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}$.
2. $L := T \setminus S$.
3. $\mathcal{G} := \varnothing$.
4. **While** $L \neq \varnothing$ **do**
   a. $\tau := \min_{\prec}(L)$.
   b. Find $\boldsymbol{\alpha} = (\alpha_s)_{s \in S}$ s.t. $H_{S,S}\, \boldsymbol{\alpha} + H_{S,\{\tau\}} = 0$.
   c. $\mathcal{G} := \mathcal{G} \cup \{\tau + \sum_{s \in S} \alpha_s\, s\}$ and remove multiples of τ from L.
5. **Return** $\mathcal{G}$.

Motivation.

- The sequence $\boldsymbol{u} = (u_i)_{i\in\mathbb{N}} = \left(\left\lfloor \frac{i}{2} \right\rfloor! + (-1)^i\right)_{i\in\mathbb{N}}$ satisfies, for all $i \in \mathbb{N}$,

$$\begin{aligned} u_{2i+3} - u_{2i+2} - u_{2i+1} + u_{2i} &= 0 \\ u_{i+1}^{(1)} - u_{i+1}^{(0)} - u_i^{(1)} + u_i^{(0)} &= 0, \end{aligned}$$

  with $\boldsymbol{u}^{(0)} = (u_{2i})_{i\in\mathbb{N}}$ and $\boldsymbol{u}^{(1)} = (u_{2i+1})_{i\in\mathbb{N}}$.

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  with $\boldsymbol{u}^{(0)} = (u_{2i})_{i \in \mathbb{N}}$ and $\boldsymbol{u}^{(1)} = (u_{2i+1})_{i \in \mathbb{N}}$.

  However, for all $i \in \mathbb{N}$,

$$u_{2i+4} - u_{2i+3} - u_{2i+2} + u_{2i+1} \neq 0.$$

  Hence no global C-relation satisfied by $\boldsymbol{u}$.

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- The P-relation $(i - j + 1)\, b_{i+1,j} = (i+1)\, b_{i,j}$ can be rewritten as

$$\begin{array}{rcl} (i+1)\, b_{i+1,j} - j\, b_{i+1,j} & = & i\, b_{i,j} + b_{i,j} \\ b_{i+1,j}^{(1)} - b_{i+1,j}^{(2)} & = & b_{i,j}^{(1)} + b_{i,j}^{(0)}, \end{array}$$

  with $\boldsymbol{b}^{(0)} = (b_{i,j})_{(i,j) \in \mathbb{N}^2}, \boldsymbol{b}^{(1)} = (i\, b_{i,j})_{(i,j) \in \mathbb{N}^2}$ and $\boldsymbol{b}^{(2)} = (j\, b_{i,j})_{(i,j) \in \mathbb{N}^2}$.

First idea for computing P-relations.

Find C-relations between the sequences $(u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}$, $(i_1\ u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}$, …, $(i_n\ u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}$, $(i_1^2\, u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}, (i_1\, i_2\, u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}, \ldots (i_n^2\, u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}, \ldots$

Problem.

How can we find C-relations between multiple sequences?

Notation – Link between C-relations and polynomials vectors.

For sequences $\boldsymbol{u}^{(0)}, \ldots, \boldsymbol{u}^{(m)}$, $[e_0\, P_0 + \cdots + e_m\, P_m] := [P_0]_{\boldsymbol{u}^{(0)}} + \cdots + [P_m]_{\boldsymbol{u}^{(m)}}$.

$$\to \quad \begin{aligned}[][(e_1\,(x-1) - e_0\,(x-1))\,x^i]_{\boldsymbol{u}} &= [x^{i+1} - x^i]_{\boldsymbol{u}^{(1)}} - [x^{i+1} - x^i]_{\boldsymbol{u}^{(0)}}\\ &= u^{(1)}_{i+1} - u^{(1)}_i - u^{(0)}_{i+1} + u^{(0)}_{i,j},\end{aligned}$$

$$\to \quad \begin{aligned}[][(e_1\,(x-1) - e_2\,x - e_0)\,x^i\,y^j]_{\boldsymbol{b}} &= [x^{i+1}\,y^j - x^i\,y^j]_{\boldsymbol{b}^{(1)}} - [x^{i+1}\,y^j]_{\boldsymbol{b}^{(2)}} - [x^i\,y^j]_{\boldsymbol{b}^{(0)}}\\ &= b^{(1)}_{i+1,j} - b^{(1)}_{i,j} - b^{(2)}_{i+1,j} - b^{(0)}_{i,j}.\end{aligned}$$

Main idea.

Find relations $[\sum_{\ell=1}^{m}\sum_{\boldsymbol{s}\in\mathcal{S}}\alpha_{\ell,\boldsymbol{s}}\,\boldsymbol{x}^{\boldsymbol{s}+\boldsymbol{v}}\,e_\ell]=0$ valid for all $\boldsymbol{v}\in\mathcal{U}$.

Example.

For $\boldsymbol{u}^{(1)}=(3^i)_{(i,j)\in\mathbb{N}^2}$, $\boldsymbol{u}^{(2)}=(3^i+2^j)_{(i,j)\in\mathbb{N}^2}$, $T=\{e_1,e_2,x\,e_1,x\,e_2,y\,e_1,y\,e_2\}$ and $U=\{1,x,y,x^2,x\,y,y^2\}$, the matrix $H_{U,T}$ has rank 2,

$$H_{U,T}=\begin{array}{c}\\ 1\\ x\\ y\\ x^2\\ x\,y\\ y^2\end{array}\begin{array}{c}\begin{array}{cccccc} e_1 & e_2 & x\,e_1 & x\,e_2 & y\,e_1 & y\,e_2\end{array}\\ \begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 3\\ 3 & 4 & 9 & 10 & 3 & 5\\ 1 & 3 & 3 & 5 & 1 & 5\\ 9 & 10 & 27 & 28 & 9 & 11\\ 3 & 5 & 9 & 11 & 3 & 7\\ 1 & 5 & 3 & 7 & 1 & 9\end{pmatrix}\end{array}.$$

$\to$ Relations

$$\begin{aligned}
[e_1\,(x-3)\,x^i\,y^j] &= u^{(1)}_{i+1,j}-3\,u^{(1)}_{i,j} &&= 0,\\
[e_2\,(x-1)\,x^i\,y^j-e_2\cdot 2\,x^i\,y^j] &= u^{(2)}_{i+1,j}-u^{(2)}_{i,j}-2\,u^{(1)}_{i,j} &&= 0,\\
[e_1\,(y-1)\,x^i\,y^j] &= u^{(1)}_{i,j+1}-u^{(1)}_{i,j} &&= 0,\\
[e_2\,(y-2)\,x^i\,y^j+e_1\,x^i\,y^j] &= u^{(2)}_{i,j+1}-2\,u^{(2)}_{i,j}+u^{(1)}_{i,j} &&= 0.
\end{aligned}$$

The MultiScalar-FGLM algorithm.

→ An algorithm for computing C-relations of a sequence tuple.

Useless computations.

For $\boldsymbol{b}^{(0)} = \left(\binom{i}{j}\right)_{(i,j)\in\mathbb{N}^2}$, $\boldsymbol{b}^{(1)} = \left(i\binom{i}{j}\right)_{(i,j)\in\mathbb{N}^2}$, $\boldsymbol{b}^{(2)} = \left(j\binom{i}{j}\right)_{(i,j)\in\mathbb{N}^2}$, up to degree 2, we find the 3 relations

$$\begin{aligned}
[e_1\,(x-1)\,x^i\,y^j - e_2\,x^{i+1}\,y^j - e_0\,x^i\,y^j] &= b^{(1)}_{i+1,j} - b^{(1)}_{i,j} - b^{(2)}_{i+1,j} - b^{(0)}_{i,j} &= 0,\\
[e_2\,(y+1)\,x^i\,y^j - e_1\,x^i\,y^j] &= b^{(2)}_{i,j+1} + b^{(2)}_{i,j} - b^{(1)}_{i,j} &= 0,\\
[e_0\,(x\,y-y-1)\,x^i\,y^j] &= b^{(0)}_{i+1,j+1} - b^{(0)}_{i,j+1} - b^{(0)}_{i,j} &= 0.
\end{aligned}$$

and an extra, useless, relation:

$$\begin{aligned}
[e_1\,(x\,y-y-1)\,x^i\,y^j - e_0\,(y+1)\,x^i\,y^j] &= b^{(1)}_{i+1,j+1} - b^{(1)}_{i,j+1} - b^{(1)}_{i,j} - b^{(0)}_{i,j+1} - b^{(0)}_{i,j}\\
&= 0.
\end{aligned}$$

Problem.

→ Computing the P-relations is not a simple extension of computing the C-relations.

Definition. **[Lipschitz, 1989, Definition 3.2]**

A sequence $\boldsymbol{u} = (u_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^n}$ is P-recursive if $\exists k \in \mathbb{N}$ s.t.

- $\forall j \in \{1, \ldots, n\}, \forall \boldsymbol{\ell} \in \{0, \ldots, k\}^n, \exists p_{\boldsymbol{\ell}}^{(j)} \in \mathbb{K}[t]$ not all zero s.t.

$$\forall \ell_1 \leq i_1, \ldots, \ell_n \leq i_n, \quad \sum_{\boldsymbol{\ell} \in \{0,\ldots,k\}^n} p_{\boldsymbol{\ell}}^{(j)}(i_j)\, u_{i_1-\ell_1,\ldots,i_n-\ell_n} = 0;$$

- all the subsequences with fixed indices set between 0 and $k-1$ are P-recursives.

Example.

Sequences	$\boldsymbol{b} = \left(\binom{i}{j}\right)_{(i,j)\in\mathbb{N}^2}$	$\boldsymbol{K} = (K_{n,i,j})_{(n,i,j)\in\mathbb{N}^3}$, the Kreweras's walk, with $\mathfrak{S} = \{(-1,0), (0,-1), (1,1)\}$
Initial terms	$b_{0,0} = 1$	$\begin{cases} K_{0,0,0} = 1 \\ K_{0,i,j} = 0, \ \forall (i,j) \in \mathbb{N}^2 \setminus \{(0,0)\} \\ K_{n,i,j} = 0, \text{ if } n < 0 \text{ or } i < 0 \text{ or } j < 0. \end{cases}$
P-relations	$\begin{cases} (i-j+1)\, b_{i+1,j} = (i+1)\, b_{i,j} \\ (j+1)\, b_{i,j+1} = (i-j)\, b_{i,j} \\ b_{i+1,j+1} = b_{i,j+1} + b_{i,j} \end{cases}$	$K_{n+1,i,j} = \sum_{\sigma\in\mathfrak{S}} K_{n,i-\sigma_1,j-\sigma_2}$

Definition.

$\mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x} \rangle$: ring of quasi-commutative polynomials in $t_1, \ldots, t_n, x_1, \ldots, x_n$, where

- $\forall k,\ t_k\, x_k - x_k\, t_k = x_k$;
- $\forall k \neq \ell$, t_k, t_ℓ, x_k, x_ℓ commute!

Notation – Link between P-relations and polynomials.

For a sequence $\boldsymbol{u}$, $[\boldsymbol{t}^{\boldsymbol{j}}\, \boldsymbol{x}^{\boldsymbol{i}}]_{\boldsymbol{u}} = \boldsymbol{i}^{\boldsymbol{j}}\, u_{\boldsymbol{i}}$.

→ $[(t_1 + 1)\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} = (i_1 + 1)\, b_{i_1, i_2}$;

→ $[(t_1 - t_2)\, x_1^{i_1+1}\, x_2^{i_2}]_{\boldsymbol{b}} = (i_1 + 1 - i_2)\, b_{i_1+1, i_2}$;

→ $$\begin{aligned} [((t_1 - t_2)\, x_1 - (t_1 + 1))\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} &= [(t_1 - t_2)\, x_1^{i_1+1}\, x_2^{i_2} - (t_1 + 1)\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} \\ &= (i_1 + 1 - i_2)\, b_{i_1+1, i_2} - (i_1 + 1)\, b_{i_1, i_2} \end{aligned};$$

→ $$\begin{aligned} [((t_1 - t_2)\, x_1 - (t_1 + 1))\, t_1\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} &= [((t_1^2 - t_1\, t_2 - t_1 + t_2)\, x_1 - (t_1^2 + t_1))\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} \\ &= ((i_1 + 1)^2 - (i_1 + 1)\, i_2 - (i_1 + 1) + i_2)\, b_{i_1+1, i_2} \\ &\quad - (i_1^2 + i_1)\, b_{i_1, i_2}. \end{aligned}$$

Definition.

$\mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x}\rangle$: ring of quasi-commutative polynomials in $t_1, \dots, t_n, x_1, \dots, x_n$, where

- $\forall k,\ t_k\, x_k - x_k\, t_k = x_k$;
- $\forall k \neq \ell$, t_k, t_ℓ, x_k, x_ℓ commute!

Notation – Link between P-relations and polynomials.

For a sequence $\boldsymbol{u}$, $[\boldsymbol{t}^{\boldsymbol{j}}\, \boldsymbol{x}^{\boldsymbol{i}}]_{\boldsymbol{u}} = \boldsymbol{i}^{\boldsymbol{j}}\, u_{\boldsymbol{i}}$.

$\rightarrow\quad [((t_1 - t_2)\, x_1 - (t_1 + 1))\, x_1^{i_1}\, x_2^{i_2}]_{\boldsymbol{b}} = (i_1 + 1 - i_2)\, b_{i_1+1, i_2} - (i_1 + 1)\, b_{i_1, i_2}$.

Theorem. **[Lipschitz, 1989, Theorem 3.8 (vii)]**

If a sequence $\boldsymbol{u} = (u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^n}$ is P-recursive, then $\forall x_k, \exists d, \boldsymbol{e}$, s.t. $\boldsymbol{t}^{\boldsymbol{e}}\, x_k^d \in \mathrm{LT}(I)$, where $I = \{P \in \mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x}\rangle,\ \forall \boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^n,\ [P\boldsymbol{t}^{\boldsymbol{j}}\, \boldsymbol{x}^{\boldsymbol{i}}]_{\boldsymbol{u}} = 0\}$ is the right ideal of P-relations of $\boldsymbol{u}$.

Example.

The sequence $\boldsymbol{b} = \left(\binom{i_1}{i_2}\right)_{\boldsymbol{i}\in\mathbb{N}^2}$ is P-recursive. Since
$$\left\{\begin{aligned} (i_1 + 1 - i_2)\binom{i_1+1}{i_2} - (i_1+1)\binom{i_1}{i_2} &= 0,\\ (i_2+1)\binom{i_1}{i_2+1} + (i_2 - i_1)\binom{i_1}{i_2} &= 0,\\ \binom{i_1+1}{i_2+1} - \binom{i_1}{i_2+1} - \binom{i_1}{i_2} &= 0,\end{aligned}\right.$$
its ideal of P-relations is $\langle (t_1 - t_2)\, x_1 - (t_1 + 1), t_2\, x_2 + (t_2 - t_1), x_1\, x_2 - x_2 - 1\rangle$.

Definition.

For two ordered sets of terms T in $\boldsymbol{t}, \boldsymbol{x}$ and U in $\boldsymbol{x}$, the multi-Hankel matrix $H_{U,T}$ is

$$\begin{array}{c} \\ \vdots \\ v \in U \\ \vdots \end{array} \begin{array}{c} \cdots \quad s \in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & [s\,v]_{\boldsymbol{u}} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Example.

For $T = \{1, t_1, t_2, t_1^2, t_1\,t_2, t_2^2\}, U = \{1, x_1, x_2, x_1^2, x_1\,x_2, x_2^2\}$,

$$H_{U,T} = \begin{array}{c|cccccc} & 1 & t_1 & t_2 & t_1^2 & t_1\,t_2 & t_2^2 \\ \hline 1 & u_{0,0} & 0 \cdot u_{0,0} & 0 \cdot u_{0,0} & 0^2 \cdot u_{0,0} & 0 \cdot 0 \cdot u_{0,0} & 0^2 \cdot u_{0,0} \\ x_1 & u_{1,0} & 1 \cdot u_{1,0} & 0 \cdot u_{1,0} & 1^2 \cdot u_{1,0} & 1 \cdot 0 \cdot u_{1,0} & 0^2 \cdot u_{1,0} \\ x_2 & u_{0,1} & 0 \cdot u_{0,1} & 1 \cdot u_{0,1} & 0^2 \cdot u_{0,1} & 0 \cdot 1 \cdot u_{0,1} & 1^2 \cdot u_{0,1} \\ x_1^2 & u_{2,0} & 2 \cdot u_{2,0} & 0 \cdot u_{2,0} & 2^2 \cdot u_{2,0} & 2 \cdot 0 \cdot u_{2,0} & 0^2 \cdot u_{2,0} \\ x_1\,x_2 & u_{1,1} & 1 \cdot u_{1,1} & 1 \cdot u_{0,1} & 1^2 \cdot u_{1,1} & 1 \cdot 1 \cdot u_{1,1} & 1^2 \cdot u_{0,1} \\ x_2^2 & u_{0,2} & 0 \cdot u_{0,2} & 2 \cdot u_{0,2} & 0^2 \cdot u_{0,2} & 0 \cdot 2 \cdot u_{0,2} & 2^2 \cdot u_{0,2} \end{array}.$$

Definition.

For two ordered sets of terms T in $\boldsymbol{t}, \boldsymbol{x}$ and U in $\boldsymbol{x}$, the multi-Hankel matrix $H_{U,T}$ is

$$\begin{array}{c} \\ \vdots \\ v \in U \\ \vdots \end{array} \begin{array}{c} \cdots \quad s \in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & ⋰ \\ \cdots & [s\,v]_{\boldsymbol{u}} & \cdots \\ ⋰ & \vdots & \ddots \end{pmatrix} \end{array}.$$

Example.

For $T = \{1, t_1, x_1, t_1^2, t_1\,x_1, x_1^2\}, U = \{1, x_1, x_2, x_1^2, x_1\,x_2, x_2^2\}$,

$$H_{U,T} = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1\,x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & t_1 & x_1 & t_1^2 & t_1\,x_1 & x_1^2 \end{array} \\ \begin{pmatrix} u_{0,0} & 0 \cdot u_{0,0} & u_{1,0} & 0^2 \cdot u_{0,0} & 1 \cdot u_{1,0} & u_{2,0} \\ u_{1,0} & 1 \cdot u_{1,0} & u_{2,0} & 1^2 \cdot u_{1,0} & 2 \cdot u_{2,0} & u_{3,0} \\ u_{0,1} & 0 \cdot u_{0,1} & u_{1,1} & 0^2 \cdot u_{0,1} & 1 \cdot u_{1,1} & u_{2,1} \\ u_{2,0} & 2 \cdot u_{2,0} & u_{3,0} & 2^2 \cdot u_{2,0} & 3 \cdot u_{3,0} & u_{4,0} \\ u_{1,1} & 1 \cdot u_{1,1} & u_{2,1} & 1^2 \cdot u_{1,1} & 2 \cdot u_{2,1} & u_{3,1} \\ u_{0,2} & 0 \cdot u_{0,2} & u_{1,2} & 0^2 \cdot u_{0,2} & 1 \cdot u_{1,2} & u_{2,2} \end{pmatrix} \end{array}.$$

Definition.

For two ordered sets of terms T in $\boldsymbol{t}, \boldsymbol{x}$ and U in $\boldsymbol{x}$, the multi-Hankel matrix $H_{U,T}$ is

$$\begin{array}{c} \\ \vdots \\ v \in U \\ \vdots \end{array} \begin{array}{c} \cdots \quad s \in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & [s\,v]_{\boldsymbol{u}} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Example.

For $T' = \{1, x_1, x_2\}, T = T' \cup t_1 T' \cup t_2 T', U = \{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_1^3, x_1^2 x_2, x_1 x_2^2\}$,

$$H_{U,T} = \begin{array}{c|ccc|ccc|ccc} & 1 & x_1 & x_2 & t_1 & t_1 x_1 & t_1 x_2 & t_2 & t_2 x_1 & t_2 x_2 \\ \hline 1 & u_{0,0} & u_{1,0} & u_{0,1} & 0 \cdot u_{0,0} & 1 \cdot u_{1,0} & 0 \cdot u_{0,1} & 0 \cdot u_{0,0} & 0 \cdot u_{1,0} & 1 \cdot u_{0,1} \\ x_1 & u_{1,0} & u_{2,0} & u_{1,1} & 1 \cdot u_{1,0} & 2 \cdot u_{2,0} & 1 \cdot u_{1,1} & 0 \cdot u_{1,0} & 0 \cdot u_{2,0} & 1 \cdot u_{1,1} \\ x_2 & u_{0,1} & u_{1,1} & u_{0,2} & 0 \cdot u_{0,1} & 1 \cdot u_{1,1} & 0 \cdot u_{0,2} & 1 \cdot u_{0,1} & 1 \cdot u_{1,1} & 2 \cdot u_{0,2} \\ x_1^2 & u_{2,0} & u_{3,0} & u_{2,1} & 2 \cdot u_{2,0} & 3 \cdot u_{3,0} & 2 \cdot u_{2,1} & 0 \cdot u_{2,0} & 0 \cdot u_{3,0} & 1 \cdot u_{2,1} \\ x_1 x_2 & u_{1,1} & u_{2,1} & u_{1,2} & 1 \cdot u_{1,1} & 2 \cdot u_{2,1} & 1 \cdot u_{1,2} & 1 \cdot u_{1,1} & 1 \cdot u_{2,1} & 2 \cdot u_{1,2} \\ x_2^2 & u_{0,2} & u_{1,2} & u_{0,3} & 0 \cdot u_{0,2} & 1 \cdot u_{1,2} & 0 \cdot u_{0,3} & 2 \cdot u_{0,2} & 2 \cdot u_{1,2} & 3 \cdot u_{0,3} \\ x_1^3 & u_{3,0} & u_{4,0} & u_{3,1} & 3 \cdot u_{3,0} & 4 \cdot u_{4,0} & 3 \cdot u_{3,1} & 0 \cdot u_{3,0} & 0 \cdot u_{4,0} & 1 \cdot u_{3,1} \\ x_1^2 x_2 & u_{2,1} & u_{3,1} & u_{2,2} & 2 \cdot u_{2,1} & 3 \cdot u_{3,1} & 2 \cdot u_{2,2} & 1 \cdot u_{2,1} & 1 \cdot u_{3,1} & 2 \cdot u_{2,2} \\ x_1 x_2^2 & u_{1,2} & u_{2,2} & u_{1,3} & 1 \cdot u_{1,2} & 2 \cdot u_{2,2} & 1 \cdot u_{1,3} & 2 \cdot u_{1,2} & 2 \cdot u_{2,2} & 3 \cdot u_{1,3} \end{array}.$$

Useless columns.

For $\boldsymbol{u} = (i!)_{i\in\mathbb{N}}$, $T = \{1, t, x, t\,x, x^2\}$ and $U = \{1, x, x^2, x^3, x^4\}$,

$$H_{U,T} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & t & x & t\,x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 & 6 \\ 2 & 4 & 6 & 18 & 24 \\ 6 & 18 & 24 & 96 & 120 \\ 24 & 96 & 120 & 600 & 720 \end{pmatrix} \end{array}.$$

$H_{U,T}$ has rank 3 and its column rank profile is $\{1, t, t\,x\}$.

→ From $u_0 = 1$, $[x\,x^i] = [(t+1)\,x^i] \Longleftrightarrow u_{i+1} = (i+1)\,u_i$ allows us to compute any term.

Useless columns.

For $\boldsymbol{u} = (i!)_{i \in \mathbb{N}}$, $T = \{1, t, x, t\,x, x^2\}$ and $U = \{1, x, x^2, x^3, x^4\}$,

$$H_{U,T} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & t & x & t\,x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 & 6 \\ 2 & 4 & 6 & 18 & 24 \\ 6 & 18 & 24 & 96 & 120 \\ 24 & 96 & 120 & 600 & 720 \end{pmatrix} \end{array}.$$

$H_{U,T}$ has rank 3 and its column rank profile is $\{1, t, t\,x\}$.

→ From $u_0 = 1$, $[x\,x^i] = [(t+1)\,x^i] \iff u_{i+1} = (i+1)\,u_i$ allows us to compute any term.

→ But $(i+1)\,u_{i+1} = (i^2 + 2\,i + 1)\,u_i \iff [t\,x\,x^i] = [(t^2 + 2\,t + 1)\,x^i]$.

→ Column $t\,x$ seems useless!

Useless columns.

For $\boldsymbol{u} = (i!)_{i \in \mathbb{N}}$, $T = \{1, t, x, t\,x, x^2\}$ and $U = \{1, x, x^2, x^3, x^4\}$,

$$H_{U \cup \{x^5\}, T \cup \{t^2\}} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & t & x & t^2 & t\,x & x^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 1 & 4 & 6 \\ 2 & 4 & 6 & 8 & 18 & 24 \\ 6 & 18 & 24 & 54 & 96 & 120 \\ 24 & 96 & 120 & 384 & 600 & 720 \\ 120 & 600 & 720 & 3000 & 4320 & 5040 \end{pmatrix} \end{array}.$$

$H_{U,T}$ has rank 3 and its column rank profile is $\{1, t, t^2\}$.

Remark.

Relations form a right ideal of $\mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x}\rangle$ ⇝ take this structure into account.

⇝ If relation
$$\begin{aligned}[\boldsymbol{t}^{\boldsymbol{e}}\,\boldsymbol{x}^{\boldsymbol{d}+\boldsymbol{v}}] &= [\textstyle\sum_{\boldsymbol{r}\in\mathcal{R},\boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}}\,\boldsymbol{t}^{\boldsymbol{r}}\,\boldsymbol{x}^{\boldsymbol{s}+\boldsymbol{v}}] \\ (\boldsymbol{d}+\boldsymbol{v})^{\boldsymbol{e}}\,u_{\boldsymbol{d}+\boldsymbol{v}} &= \textstyle\sum_{\boldsymbol{r}\in\mathcal{R},\boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}}\,(\boldsymbol{s}+\boldsymbol{v})^{\boldsymbol{r}}\,u_{\boldsymbol{s}+\boldsymbol{v}}\end{aligned}$$
is valid for all $\boldsymbol{v}\in\mathcal{U}$,

then

$$[t_k\,\boldsymbol{t}^{\boldsymbol{e}}\,\boldsymbol{x}^{\boldsymbol{d}+\boldsymbol{v}}] = (d_k+v_k)\,(\boldsymbol{d}+\boldsymbol{v})^{\boldsymbol{e}}\,u_{\boldsymbol{d}+\boldsymbol{v}} = \sum_{\boldsymbol{r}\in\mathcal{R},\boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}}\,(d_k+v_k)\,(\boldsymbol{s}+\boldsymbol{v})^{\boldsymbol{r}}\,u_{\boldsymbol{s}+\boldsymbol{v}} = \Big[\sum \cdots\Big]$$

is also valid for all $\boldsymbol{v}\in\mathcal{U}$.

Example.

$\boldsymbol{u} = (i!)_{i\in\mathbb{N}}$ ⇝ $I = \langle x-(t+1)\rangle \subseteq \mathbb{K}\langle t, x\rangle$. Hence $(x-(t+1))\,t \in I$ but

$$\begin{aligned}(x-(t+1))\,t &= x\,t-(t+1)\,t \\ &= t\,x - x - (t+1)\,t \\ &= t\,x-(t+1)-(t+1)\,t-(x-(t+1)) \\ &= t\,x-(t+1)^2-(x-(t+1)).\end{aligned}$$

Therefore, $(t\,x-(t+1)^2)\in I$.

Remark.

Relations form a right ideal of $\mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x} \rangle \rightsquigarrow$ take this structure into account.

$\rightsquigarrow$ If relation
$$\begin{aligned} [\boldsymbol{t}^{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{d}+\boldsymbol{v}}] &= [\textstyle\sum_{\boldsymbol{r}\in\mathcal{R}, \boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}} \boldsymbol{t}^{\boldsymbol{r}} \boldsymbol{x}^{\boldsymbol{s}+\boldsymbol{v}}] \\ (\boldsymbol{d}+\boldsymbol{v})^{\boldsymbol{e}} u_{\boldsymbol{d}+\boldsymbol{v}} &= \textstyle\sum_{\boldsymbol{r}\in\mathcal{R}, \boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}} (\boldsymbol{s}+\boldsymbol{v})^{\boldsymbol{r}} u_{\boldsymbol{s}+\boldsymbol{v}} \end{aligned}$$
is valid for all $\boldsymbol{v} \in \mathcal{U}$, then

$$[t_k \boldsymbol{t}^{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{d}+\boldsymbol{v}}] = (d_k + v_k)(\boldsymbol{d}+\boldsymbol{v})^{\boldsymbol{e}} u_{\boldsymbol{d}+\boldsymbol{v}} = \sum_{\boldsymbol{r}\in\mathcal{R}, \boldsymbol{s}\in\mathcal{S}} \alpha_{\boldsymbol{r},\boldsymbol{s}} (d_k+v_k)(\boldsymbol{s}+\boldsymbol{v})^{\boldsymbol{r}} u_{\boldsymbol{s}+\boldsymbol{v}} = [\textstyle\sum \cdots]$$

is also valid for all $\boldsymbol{v} \in \mathcal{U}$.

Proposition.

Let T (resp. U) be an ordered set of terms in $\boldsymbol{t}, \boldsymbol{x}$ (resp. $\boldsymbol{x}$). If column $\boldsymbol{t}^{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{d}}$ is discarded in matrix $H_{U,T}$, then so is any column $\boldsymbol{t}^{\boldsymbol{e}+\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{d}}$.

Example.

For $\boldsymbol{u} = (i!)_{i\in\mathbb{N}}$, $T = \{1, t, x, t\,x\}$ and $U = \{1, x, x^2, x^3\}$,
$$H_{U,T} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & t & x & t\,x \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 2 & 4 & 6 & 18 \\ 6 & 18 & 24 & 96 \end{pmatrix} \end{array}.$$

Column x is discarded, hence so is column $t\,x$ and the staircase is $S = \{1, t\}$.

Informal version of the PRECURSIVE-FGLM algorithm.

Input.

- A sequence $\boldsymbol{u} = (u_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^n}$ over $\mathbb{K}$;
- The ordered set T of all monomials in $\boldsymbol{t}, \boldsymbol{x}$ of bidegree at most $(d_{\boldsymbol{t}}, d_{\boldsymbol{x}})$ wrt. $\prec$.
- The ordered set U of all monomials in $\boldsymbol{x}$ wrt. $\prec$ such that $\#U = \#T$.

Output.

→ A reduced $(d_{\boldsymbol{t}}, d_{\boldsymbol{x}})$-truncated Gröbner basis of the ideal of P-relations of $\boldsymbol{u}$.

1. Compute S the staircase of $H_{U,T} := \begin{array}{c} \\ \vdots \\ v \in U \\ \vdots \end{array} \begin{array}{c} \cdots \quad s \in T \quad \cdots \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & [s\,v]_{\boldsymbol{u}} & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}$.
2. Compute V the row rank profile of $H_{U,S}$.
3. $L := T \setminus S$.
4. **While** $L \neq \varnothing$ **do**
   a. $\tau := \min_{\prec}(L)$.
   b. Find $\boldsymbol{\alpha} = (\alpha_s)_{s \in S}$ s.t. $H_{V,S}\,\boldsymbol{\alpha} + H_{V,\{\tau\}} = 0$.
   c. $\mathcal{G} := \mathcal{G} \cup \{\tau + \sum_{s \in S} \alpha_s\, s\}$ and remove multiples of τ from L.
5. **Return** $\mathcal{G}$.

Remark.

When a relation $[P_1]=0$ is true, we do not need relations $[Q]=0$ s.t. $P_1 \mid Q$.

→ When relations $[P_1]=\cdots=[P_r]=0$ are true, we should not need relations $[Q]=0$ with $Q \in \langle P_1,\ldots,P_r\rangle \subseteq \mathbb{K}\langle \boldsymbol{t},\boldsymbol{x}\rangle$.

Idea.

Computation of a (truncated) Gröbner basis of $J=\langle P_1,\ldots,P_r\rangle \subseteq \mathbb{K}\langle \boldsymbol{t},\boldsymbol{x}\rangle$:

→ new polynomials $P_{r+1},\ldots,P_s \in J$, s.t. $\mathrm{LT}(P_{r+1}),\ldots,\mathrm{LT}(P_s) \notin \langle \mathrm{LT}(P_1),\ldots,\mathrm{LT}(P_r)\rangle$

→ guessing new relations without new queries to the sequence!

Implementation.

- In Maple using the F4 algorithm in Ore algebras.
- In C, integrated in the FGb library. [Faugère, 2010]

Remark.

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Example.

$\boldsymbol{G}=(G_{n,i,j})_{(n,i,j)\in\mathbb{N}^3}$, the Gessel walk, with $\mathfrak{S}=\{(-1,0),(-1,-1),(0,1),(1,1)\}$.

→ If $\boldsymbol{G}$ is P-recursive, then so is $\boldsymbol{G}'=(G_{n,0,j})_{(n,i)\in\mathbb{N}^2}$.

→ Computation of relations of $\boldsymbol{G}'$ in bidegree $(5,5)$ with P-recursive-FGLM:

→ $[t_0\, t_2^3\, x_0^2\, x_2^3 - \cdots] = [t_0^4\, x_0^4\, x_2 - \cdots] = [t_0^4\, t_2\, x_0^4\, x_2 - \cdots] = [t_0^3\, t_2^2\, x_0^4\, x_2 - \cdots] = [t_0^3\, t_2^2\, x_0^2\, x_2^3 - \cdots] = 0$.

→ Gröbner basis computation in $\mathbb{K}\langle t_0, t_2, x_0, x_2 \rangle$:

→ $[t_0^5\, t_2^2\, x_0^4 - \cdots] = [t_0^5\, x_0^6 - \cdots] = [t_0^3\, t_2^4\, x_2^4 - \cdots] = [t_0^3\, t_2^3\, x_2^5 - \cdots] = 0$.

→ Zero-dimensional ideal of relations in x_0, x_2:

→ We can suspect that $\boldsymbol{G}'$ is P-recursive.

Remark.

When a relation $[P_1]=0$ is true, we do not need relations $[Q]=0$ s.t. $P_1 \mid Q$.

→ When relations $[P_1]=\cdots=[P_r]=0$ are true, we should not need relations $[Q]=0$ with $Q \in \langle P_1, \ldots, P_r\rangle \subseteq \mathbb{K}\langle \boldsymbol{t}, \boldsymbol{x}\rangle$.

Example.

$\boldsymbol{W} = (W_{n,i,j,k})_{(n,i,j,k)\in\mathbb{N}^4}$, a 3D-space walk, with $\mathfrak{S} = \{(-1,-1,-1), (-1,-1,1), (-1,1,0), (1,0,0)\}$.

[Bostan, Bousquet-Mélou, Kauers, Melczer, 2014, Section 4.3]

→ If $\boldsymbol{W}$ is P-recursive, then so is $\boldsymbol{W}' = (W_{n,i,j,0})_{(n,i,j)\in\mathbb{N}^3}$.

$(d_{\boldsymbol{t}}, d_{\boldsymbol{x}})$	Matrix size	Timing	Gröbner bases	Timing	Relations	Staircase
$(2,2)$	270×100	19s	N/A		$[t_0\, t_1\, x_0\, x_1 - \cdots]=0$	Not closed
$(2,4)$	1448×350	756s	max $(d_{\boldsymbol{t}}, d_{\boldsymbol{x}})$: $(4,3), (3,4), (2,5), (1,7)$	1s	$[t_0\, t_1\, t_2\, x_1^2 - \cdots]=0$	Not closed
$(3,4)$	2959×600	4310s	truncated to total degree 9	1s	$[t_0^3\, t_1\, t_2\, x_0^4 - \cdots]=0$ $[t_0\, t_1^2\, t_2\, x_2^4 - \cdots]=0$	Closed!

→ We guess relations in bidegree $(5,4)$ for free!

→ We can suspect that $\boldsymbol{W}'$ is P-recursive.

Proposition.

- Adaptive variant of $\textsc{Precursive-FGLM}$:
  - Increase S and V step by step such that $H_{V,S}$ is always full rank.
- Denote $S_{\boldsymbol{x}} = S \cap \mathbb{K}[\boldsymbol{x}]$. Let the Dth dilatation of $S_{\boldsymbol{x}}$ contain V, then with this adaptive algorithm, we can compute the relations with at most

$$\#(S_{\boldsymbol{x}}\, V) \leq D^n\, \#(2\, S_{\boldsymbol{x}})$$

queries.
$\rightsquigarrow$ If $S_{\boldsymbol{x}}$ is included in a parallelotope, $\exists c,\ \#(2\, S_{\boldsymbol{x}}) \leq c\, S_{\boldsymbol{x}}$. [Rusza, 1994]

Proposition.

If $\delta, d_1, \ldots, d_n$ are the maximal degrees of polynomials in $\boldsymbol{t}, x_1, \ldots, x_n$, D is as above, and μ is the number of polynomials in the Gröbner basis, then the number of operations in the base field to compute the Gröbner basis is no more than

$$O(\mu\, (D^n\, n^\delta\, d_2 \cdots d_n)^{\omega - 1}\, \mathsf{M}(d_1 \cdots d_n) \log(d_1 \cdots d_n)),$$

using fast quasi-Hankel algorithms. [Bostan, Jeannerod, Schost, 2007]

Conclusion.

- Algorithm for guessing C-relations of sequence tuples.
- Algorithm for guessing P-relations of sequences.
- Mixed approach:
  - Non trivial computations on $3\mathrm{D}$-space walks.
- Estimation on the number of sequence queries.
- Estimation on the complexity.

Perspectives.

- Improving the complexity estimates using the structure.
- Unclassified $3\mathrm{D}$-space walks:
  - Can we exhibit a 0-dim. ideal of P-relations for their first terms?