On Generalized Hypergeometric Solutions of First-Order Linear Differential Systems

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In the literature...

Maple sheet

In these slides...

The literature offers algorithms to detect and construct whether a second order differential equation has Bessel, Kummer, Gauss,... type solutions.

What about equations of orders 3, 4, . . . ?

What about systems of equations?

Generalized hypergeometric series

$$ {}_pF_q(\mathbf{a};\mathbf{b};x) = {}_pF_q\left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!}, $$

- p and q are two natural integers; $p \leq q+1$;
- $a_1, a_2, \dots, a_p$ (resp. $b_1, b_2, \dots, b_q$) are called numerator parameters (resp. denominator parameters);
- $(a)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol;
- With $\theta = x\frac{d}{dx}$, the series ${}_pF_q(\mathbf{a};\mathbf{b};x)$ satisfies:

$$ (\theta(\theta+b_1-1)(\theta+b_2-1)\dots(\theta+b_q-1)-x(\theta+a_1)(\theta+a_2)\dots(\theta+a_p))y = 0. $$

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Matricial representation of the differential equation $(C(x), \frac{d}{dx})$

$$[H_{q,g}]\ \frac{d}{dx}Y = H_{q,g}(x)Y = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & & & 1 \\ -\frac{\alpha_0}{x^q(1-x)^g} & -\frac{\alpha_1 x+\beta_1}{x^q(1-x)^g} & & & -\frac{\alpha_q x+\beta_q}{x(1-x)^g} \end{bmatrix} Y$$

- $Y = (y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^qy}{dx^q})^T$
- the α_k's and the β_k's are parameters expressible explicitly as sums of products of the a_j's and b_j's.
- $$g = \begin{cases} 0 & \text{if } p \le q\text{: } 0 \text{ (resp. } \infty\text{) is a regular (resp. irregular) singular point} \\ 1 & \text{if } p = q+1\text{: } 0, 1, \infty \text{ are regular singular points} \end{cases}$$

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Transformations

$$[S] \qquad \frac{d}{dx}Y = S(x)Y, \quad S(x) \in \mathcal{M}_{q+1}(C(x))$$

$$\downarrow$$

$$[\tilde{S}] \qquad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad \tilde{S}(x) \in \mathcal{M}_{q+1}(C(x))$$

Gauge transformation: $T(x) \in GL_{q+1}(C(x))$

$Y = T(x)Z, \quad \tilde{S}(x) = T^{-1}(x)(S(x)T(x) - \frac{d}{dx}T(x))$

Exponential transformation $\rightarrow_E$: $b(x) \in C(x)$

$Y = Z\exp(-\int \frac{b(x)}{x}dx), \quad \tilde{S}(x) = S(x) + \frac{b(x)}{x}$

Change of variable $\rightarrow_C$

$Z(x) = Y(f(x)), \quad \tilde{S}(x) = (f'(x))^{-1}S(f(x))$ Maple sheet: Examples

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Reformulating our quest

Equivalence problem

Input system

$$[A]$$

Algorithm to detect and construct

$$[H_{q,g}] \rightarrow_C [M_1] \rightarrow_E [M_2] \rightarrow_G [M_3] \rightarrow_E [M_4] \rightarrow_C \ldots [A]$$

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Where to get information about $[H_{q,g}]$ and the transformations?

- Dimension of the system (assumed irreducible- Barkatou'2007, van Hoeij'1996, van der hoeven'2007): q, p?
- Singularity Structure: g?
- Order, number, kind of transformations: $\rightarrow_G$, $\rightarrow_E$, $\rightarrow_C$?
- Invariants under transformations?
- ...?

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Order of transformations

Input system

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Proposition

Consider two systems $[H_{q,g}]$ and $[A]$ with $H_{q,g}(x), A(x) \in \mathcal{M}_{q+1}(C(x))$. If $[H_{q,g}] \rightarrow [A]$, then there exists an algebraic function f and a system $[M]$ with $M(x) \in \mathcal{M}_{q+1}(C(x))$ such that

$$[H_{q,g}] \xrightarrow{f}_{C} [M] \rightarrow_{EG} [A].$$

f: pullback function

Order of transformations

Input system

$$[A]$$

Consequence

$$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$$

(Barkatou-Pfluegel'1998)

But!!

To construct $[M]$, we need to construct $[H_{q,g}]$ and $\rightarrow C$ using information from $[A]$. So are there certain invariants under the transformations involved to give us some insight?

Order of transformations

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In the literature...

- $q \geq 1$:
  - $\rightarrow_{EG}$ in a more general context: van Hoeij'1996, Barkatou-Pfluegel'1998
  - $f = x$, finding all hypergeometric solutions: Petkovsek-Salvy'1993
  - $q = 2$, f has a special form: Cheb Terrab-Roche'2008
- $q = 1$:
  - $\xrightarrow{f}_C$: Bronstein-Lafaille'2002
  - $\xrightarrow{f}_C \rightarrow_{EG}$, 2007-: W. Koepf., M. Van Hoeij, Q. Yuan, V. J. Kunwar, E. Imamoglu, R. Debeerst, ...

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A closer look: Formal Solutions

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- A Fundamental Matrix of Formal Solutions (FMFS) is given by:

$$(Y =) \ \text{Sol} = \Phi(t) \ \exp^{\int \frac{U(t)+\Lambda}{t} \, dt}$$

  - Φ lies in $\mathcal{M}_{q+1}(\bar{C}((t^{1/r})))$; $r \in \mathbb{N}^*$;
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Classification of singularities

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Invariants?

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A closer look: Formal Solutions

Equivalence transformations: Two lemmas

$$[S] \quad \frac{d}{dx} Y = S(x) Y, \quad Y = \Phi(t) \ \exp^{\frac{\int U(t)+\Lambda}{t}} \ dt$$

$$\downarrow_{EG}$$

$$[\tilde{S}] \quad \frac{d}{dx} Z = \tilde{S}(x) Z, \quad Z = \tilde{\Phi}(t) \ \exp^{\frac{\int \tilde{U}(t)+\tilde{\Lambda}}{t}} \ dt$$

Gauge	Exp-product
$T(x) \in GL_{q+1}(C(x))$	$b(x) \in C(x)$
$Y = TZ$	$Y = Z \exp(-\int \frac{b(x)}{x} dx)$
$\tilde{r} = r$	$\tilde{r} = r$
$\tilde{U}(t) = U(t)$	$\tilde{U}(t) = U(t) + b(t) I_{q+1}$
$\tilde{\Lambda} = \Lambda \mod \frac{1}{r}\mathbb{Z}$	$\tilde{\Lambda} = \Lambda + b I_{q+1}$ if $b \in C$
singularity structure can be tracked	singularity structure can be tracked

Maple sheet: Examples on effect of gauge and exp-product

Recall that...

Input system

$$[A]$$

What about $\xrightarrow{f}_C$?

$$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$$

Singularity tracking under $\xrightarrow{f}_C$?

$[H_{q,g}] \xrightarrow{f}_C [M]$: $Sol(x) \xrightarrow{f}_C Sol(f)$

- If x_0 is a zero of $f(x)$ then it is a regular singularity of $[M]$
- If x_0 is a pole of $f(x)$ then it is an irregular singularity of $[M]$

Maple sheet: The inverse is not necessarily true, possible scenarios

Recall that...

Input system

$$[A]$$

What about $\xrightarrow{f}_C$?

$$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$$

Singularity tracking under $\xrightarrow{f}_C$?

$[H_{q,g}] \xrightarrow{f}_C [M]$: $Sol(x) \xrightarrow{f}_C Sol(f)$

- If x_0 is a zero of $f(x)$ then it is a regular singularity of $[M]$
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Maple sheet: The inverse is not necessarily true, possible scenarios

$[H_{q,g}] \to [A]$ for $g \in \{0,1\}$?

Step 0. Fix the order:

$$[H_{q,g}] \xrightarrow{f\text{: pullback function}}_C [M] \to_{EG} [A];$$

Step 1. Recover the poles and zeros of f from the singularity structure of $[A]$:

- Step 1.1. Filter the information: Distinguish between removable singularities and zeroes and poles of f, determine g from the structure of singularities;
- Step 1.2. Recover the lost information about the zeroes and/or poles of f, and determine p;

Step 2. Compute candidates for the coefficients of $[H_{q,g}]$, once a candidate for f is computed;

Step 3. Compute $[M]$ for each candidate f and set of candidates for the coefficients of $[H_{q,g}]$. Then test whether $[M] \to_{EG} [A]$.

Maple file: Example on our approach

Removable singularities

Definition

Definition

A pole x_0 of $S(x)$ is said to be a removable singularity of $[S]\ \frac{d}{dx}Y = S(x)Y$ if there exists a system $[\tilde{S}]$ such that:

- $[S] \rightarrow_{EG} [\tilde{S}]$;
- x_0 is an ordinary (non-singular) point of $[\tilde{S}]$.

Shanin-Craster'2002, Chen-Kauers-Singer'2014, Bostan-Chyzak-Van Hoeij-Pech,...

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Removable singularities

Detection for possible removal: a singular point

$$[S] \quad \frac{d}{dx} Y = S(x) Y, \quad Y = \Phi(t)\ \exp^{\frac{\int U(t)+\Lambda}{t}\ dt}$$

$$\downarrow_{EG}$$

$$[\tilde{S}] \quad \frac{d}{dx} Z = \tilde{S}(x) Z, \quad Z = \tilde{\Phi}(t)\ \exp^{\frac{\int \tilde{U}(t)+\tilde{\Lambda}}{t}\ dt}$$

Gauge	Exp-product
$T(x) \in GL_{q+1}(C(x))$	$b(x) \in C(x)$
$Y = TZ$	$Y = Z \exp(-\int \frac{b(x)}{x} dx)$
$\tilde{r} = r$	$\tilde{r} = r$
$\tilde{U}(t) = U(t)$	$\tilde{U}(t) = U(t) + b(t) I_{q+1}$
$\tilde{\Lambda} = \Lambda \mod \frac{1}{r}\mathbb{Z}$	$\tilde{\Lambda} = \Lambda + b I_{q+1}$ if $b \in C$
singularity structure can be tracked	singularity structure can be tracked

Removable singularities

Detection for possible removal: Irregular singular point

$$[S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{U(t)+\Lambda}{t}\ dt}$$

$$\downarrow_E$$

$$[\tilde{S}] \qquad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\ \exp^{\int \frac{\tilde{U}(t)+\tilde{\Lambda}}{t}\ dt}$$

Gauge	Exp-product
	$b(x) \in C(x)$
	$Y = Z\exp(-\int \frac{b(x)}{x}dx)$
	$\tilde{r} = r$
	$\tilde{U}(t) = U(t) + b(t)I_{q+1}$
	$\tilde{\Lambda} = \Lambda + bI_{q+1}$ if $b \in C$

Removable singularities

Detection for possible removal: Irregular singular point

$$[S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\frac{\int U(t)+\Lambda}{t}\ dt}$$

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	$Y = Z \exp(-\int \frac{b(x)}{x} dx)$
	$\tilde{U}(t) = U(t) + b(t)I_{q+1}$

Removable singularities

Removal of a (partially) removable irregular singular point

Input system: [S] $\frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{U(t)+\Lambda}{t}\,dt}$

If

$$U(t) = \tilde{U}(t) - b(t)I_{q+1}$$

for some $b(t) \in C(x)$ then

Input system: [S] $\frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{U(t)+\Lambda}{t}\,dt}$

$$\downarrow_E \ Y = Z\exp\left(-\int \frac{b(x)}{x}dx\right) \qquad \uparrow_E \ Y = Z\exp\left(+\int \frac{b(x)}{x}dx\right)$$

Output system: $[\tilde{S}]$ $\frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\ \exp^{\int \frac{\tilde{U}(t)+\tilde{\Lambda}}{t}\,dt}$

- If $\tilde{U}(t) \neq 0$ then x_0 is *partially removable*.
- If $\tilde{U}(t) = 0$ then x_0 is *totally removable*, and we continue..

Removable singularities

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- If $\tilde{U}(t) \neq 0$ then x_0 is *partially removable*.
- If $\tilde{U}(t) = 0$ then x_0 is *totally removable*, and we continue..

Removable singularities

Detection for possible removal: Regular singular point

$$[S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\frac{\int \Lambda}{t}\ dt}$$

$$\downarrow_E$$

$$[\tilde{S}] \quad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\ \exp^{\frac{\int \tilde{\Lambda}}{t}\ dt}$$

Gauge	Exp-product
	$b \in C$
	$Y = Z \exp(-\int \frac{b}{x} dx)$
	$\tilde{\Lambda} = \Lambda + bI_{q+1}$ if $b \in C$

Removable singularities

Removal of a (partially) removable regular singular point

$$\text{Input system: } [S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{\Lambda}{t}\, dt}$$

If

$$\Lambda = \tilde{\Lambda} - bI_{q+1}$$

for some $b \in C(x)$ then

$$\text{Input system: } [S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{\Lambda}{t}\, dt}$$

$$\downarrow_E \; Y = Z\exp(-\int \frac{b}{x}dx) \qquad \uparrow_E \; Y = Z\exp(+\int \frac{b}{x}dx)$$

$$\text{Output system: } [\tilde{S}] \quad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\ \exp^{\int \frac{\tilde{\Lambda}}{t}\, dt}$$

x_0 is not an ordinary point yet:

We now have to investigate Λ with a gauge transformation!

Removable singularities

Removal of a (partially) removable regular singular point

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Removable singularities

Removal of a removable regular singular point

Input system: [S] $\frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ \exp^{\int \frac{\Lambda}{t}\ dt}$

Rewriting:

Input system: [S] $\frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\ x^{\Lambda}$

Removable singularities

Removal of a removable regular singular point

$$\text{Input system:} \quad [S] \quad \frac{d}{dx} Y = S(x) Y, \quad Y = \Phi(t) \, \exp^{\int \frac{\Lambda}{t} \, dt}$$

Rewriting:

$$\text{Input system:} \quad [S] \quad \frac{d}{dx} Y = S(x) Y, \quad Y = \Phi(t) \, x^{\Lambda}$$

Removable singularities

Removal of a removable regular singular point

$$[S] \quad \frac{d}{dx}Y = S(x)Y, \quad Y = \Phi(t)\, x^{\Lambda}$$

$$\downarrow_G$$

$$[\tilde{S}] \quad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\, x^{\tilde{\Lambda}}$$

Gauge	Exp-product
$T(x) \in GL_{q+1}(C(x))$	
$Y = TZ$	
$\tilde{r} = r = 1$	
$\tilde{\Lambda} = \Lambda \bmod \mathbb{Z}$	

Λ can be altered:

Only by integers!!

Removable singularities

Removal of a removable regular singular point

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$$[\tilde{S}] \quad \frac{d}{dx}Z = \tilde{S}(x)Z, \quad Z = \tilde{\Phi}(t)\, x^{\tilde{\Lambda}}$$

$\Lambda = D + N$:

- If $N = 0$ and D has integer entries, singularity can be removed by a Gauge transformation;
- Otherwise the singularity is nonremovable;
- Special case of $N = 0$ and D has non-negative integer entries: Apparent singularity (Barkatou-Maddah'2015).

Available information

Input system $[A]$

Recovering $\xrightarrow{f}_C$?

$$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$$

Singularities of $[A]$ are classified into S_{reg}, S_{irr}, and S_{remov}.

- The information from the generalized exponents is filtered;
- one-to-one correspondence between S_{irr} and poles of f;
- the points of S_{reg} are zeroes of f;

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- But are they the only zeroes?!
- How to recover f from the filtered information?!

Maple sheet: recall possible scenarios

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Maple sheet: recall possible scenarios

Consequences on generalized exponents $[H_{q,1}] \xrightarrow{f}_C [M]$

$p = q + 1$, num-param:$\{a_1, \ldots, a_p\}$, denom-param: $\{b_1, \ldots, b_q\}$, $b_0 = 1$, $\Phi(t)\ \exp^{\frac{\int \Lambda}{t}\ dt}$

$[H_{q,1}]$: Generalized exponents at zero

$$\Lambda = \mathrm{diag}(0, 1 - b_1, \ldots, 1 - b_q)$$

$[H_{q,1}]$: Generalized exponents at ∞

$$\Lambda = \mathrm{diag}(a_1, a_2, \ldots, a_p)$$

$[H_{q,1}]$: Generalized exponents at 1

$$\Lambda = \mathrm{diag}(0, 1, \ldots, q - 1, \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k)$$

Consequences on generalized exponents $[H_{q,1}] \xrightarrow{f}_C [M]$

$p = q + 1$, num-param:$\{a_1, \dots, a_p\}$, denom-param: $\{b_1, \dots, b_q\}$, $b_0 = 1$, $\Phi(t)\ \exp^{\int \frac{\Lambda}{t}\, dt}$

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Consequences on generalized exponents $[H_{q,1}] \xrightarrow{f}_C [M]$

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Consequences on generalized exponents $[H_{q,1}] \xrightarrow{f}_C [M]$

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Consequences on generalized exponents $[H_{q,1}] \xrightarrow{f}_C [M]$

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$[M]$: Generalized exponents at a zero of f of multiplicity m

$$\Lambda = \mathrm{diag}(0, m(1 - b_1), \ldots, m(1 - b_q))$$

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$[M]$: Generalized exponents at a zero of $1 - f$ of multiplicity m

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Consequences on generalized exponents $[H_{q,1}] \rightarrow [A]$

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[A]: Generalized exponents at a zero of f of multiplicity m

$$\Lambda = \text{diag}(0, m(1 - b_1), \ldots, m(1 - b_q)) \quad \text{mod } \mathbb{Z}$$

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Consequences on generalized exponents $[H_{q,1}] \to [A]$

$p = q+1$, num-param:$\{a_1, \ldots, a_p\}$, denom-param: $\{b_1, \ldots, b_q\}$, $b_0 = 1$, $\Phi(t)$ $\exp^{\frac{\int \Lambda}{t}} dt$

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[A]: Generalized exponents at a zero of f of multiplicity m

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$$\Lambda = \mathrm{diag}(0, m(1), \ldots, m(q-1), m(\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k)) \mod \mathbb{Z}$$

Consequences on generalized exponents $[H_{q,1}] \to [A]$

$p = q + 1$, num-param:$\{a_1, \ldots, a_p\}$, denom-param: $\{b_1, \ldots, b_q\}$, $b_0 = 1$, $\Phi(t) \exp^{\frac{\int \Lambda}{t}} dt$

$[A]$: Generalized exponents at a zero of f of multiplicity m

$$\Lambda = \operatorname{diag}(0, m(1-b_1), \ldots, m(1-b_q)) \mod \mathbb{Z}$$

$[A]$: Generalized exponents at a pole of f of multiplicity m

$$\Lambda = \operatorname{diag}(ma_1, ma_2, \ldots, ma_p) \mod \mathbb{Z}$$

$[A]$: Generalized exponents at a zero of $1-f$ of multiplicity m

$$\Lambda = \operatorname{diag}(0, m(1), \ldots, m(q-1), m(\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k)) \mod \mathbb{Z}$$

Consequences on generalized exponents $[H_{q,0}] \xrightarrow{f}_C [M]$

$p \leq q$, num-param:$\{a_1, \ldots, a_p\}$, denom-param: $\{b_1, \ldots, b_q\}$, $b_0 = 1$, $\Phi(t)\ \exp^{\int \frac{U(t)+\Lambda}{t}\, dt}$

$[H_{q,0}]$: Generalized exponents at zero

$$U(t) = 0, \quad \Lambda = \mathrm{diag}(0, 1 - b_1, \ldots, 1 - b_q)$$

$[H_{q,0}]$: Generalized exponents at ∞: $t = 1/x = (-T)^{q-p+1}$

$$U(t) = \mathrm{diag}(0, \ldots, 0, \frac{1}{T}, \ldots \frac{1}{T})$$

$$\Lambda = \mathrm{diag}(a_1, \ldots, a_p, \alpha, \ldots, \alpha), \quad \alpha = \frac{q - p - 2(\sum_{k=1}^{p} a_k + \sum_{k=1}^{q} b_k)}{q - p + 1}.$$

Consequences on generalized exponents $[H_{q,0}] \xrightarrow{f}_C [M]$

$p \leq q$, num-param:$\{a_1, \ldots, a_p\}$, denom-param: $\{b_1, \ldots, b_q\}$, $b_0 = 1$, $\Phi(t)\ \exp^{\int \frac{U(t)+\Lambda}{t}\, dt}$

$[M]$: Generalized exponents at a zero x_0 of f of multiplicity m

$$U(t) = 0, \quad \Lambda = \mathrm{diag}(0, m(1-b_1), \ldots, m(1-b_q))$$

$[M]$: Generalized exponents at a pole x_0 of f of multiplicity m

$f = t^{-m} \sum_{i=0}^{\infty} f_i t^i$, $f_i \in \bar{C}$, $f_0 \neq 0$, $s = \max_{k \in \mathbb{N}}\{k < \frac{m}{q-p+1}\}$,

$$U(t) = \mathrm{diag}(0, \ldots, 0, \beta, \ldots \beta), \quad \beta = \sum_{i=0}^{s} (i(q-p+1)-m) g_i T^{i(q-p+1)-m}$$

where $t = T^{q-p+1}$, $g = T^{-m} \sum_{i=0}^{\infty} g_i T^i$, and $f = (-g)^{(q-p+1)}$.

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Recovering f

$$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$$

For $[A]$, we compute S_{reg}, S_{irr}, and S_{remov}, and the generalized exponents:

- If $x_0 \in S_{reg}$ then x_0 is a zero of f (however, the multiplicity had been shifted by an integer);
- If x_0 is a zero of f then $x_0 \in S_{remov}$ or $x_0 \in S_{reg}$;
- $x_0 \in S_{irr}$ iff x_0 is a pole of f. The generalized exponents of $[A]$ and $[M]$ are equal modulo $\frac{1}{q-p+1}\mathbb{Z}$ (p can be calculated from the rmaification index).
- If $S_{irr} = \emptyset$ then $g = 1$.
- If $S_{irr} \neq \emptyset$ then $g = 0$.

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- If $S_{irr} = \emptyset$ then $g = 1$.
- If $S_{irr} \neq \emptyset$ then $g = 0$.
- If $q = p$ then we can recover the polar part of $f = (-g)^{q-p+1}$ from the generalized exponents at a point in S_{irr}!

$[H_{q,0}]$: Recovering f

$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$

For $[A]$, we compute S_{reg}, S_{irr}, and S_{remov}, and the generalized exp:

- $f \in C(x)$
- Let $f = \frac{A}{B}$ where $A, B \in C[x]$, B is monic, and $gcd(A, B) = 1$.
- We compute B from S_{irr}:

$$B = \prod_{x_0 \in S_{irr}} (x - x_0)^{m_{x_0}}, \quad x_0 \neq \infty.$$

- We compute a bound d_A for the degree of $A = \sum_{i=0}^{d_A} a_i x^i$. Set

$$d_A = \begin{cases} deg(B) + m_\infty & \text{if} \infty \in S_{irr} \\ deg(B) & \text{otherwise} \end{cases}$$

:

  - If $\infty \in S_{reg}$ then $deg(A) < d_A$.
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  - Otherwise $deg(A) \leq d_A$.
- Either insure that there is one-to-one correspondence btw poles of f and S_{reg} or find $d_A + 1$ equations to compute A!!

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$[H_{q,0}]$: Logarithmic case

Example: Maple file

$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$

- x_0 is a logarithmic singularity of $[A]$ if and only if x_0 is a logarithmic singularity of $[M]$ if and only if zero is a logarithmic singularity of $[H_{q,0}]$.
- If x_0 is a logarithmic singularity of $[A]$ then there exists one-to-one correspondence between the singularities of $[A]$ and zeros of f

Method:

- $f = c\frac{A}{B}, \quad c \in C$
- Find B, d_A and S_{reg}
- For each possible degree configuration, find c: We always have enough number of equations from S_{irr}!

$[H_{q,0}]$: Logarithmic case

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$[H_{q,0}]$: Irrational case

Example: Maple file

$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$

- If at least one of the generalized exponents at x_0 lies in $C \backslash \mathbb{Q}$ then there exists at least one $k \in \{1, ..., q\}$ such that $b_k \in C \backslash \mathbb{Q}$.
- There exists one-to-one correspondence between the singularities of $[A]$ and zeros of f

Method:

- $f = c\frac{A}{B}, \quad c \in C$
- Find B, d_A, and S_{reg}
- We can also compute the multiplicities of the zeroes of f!
- Find c: We always have enough number of equations from S_{irr}!

$[H_{q,0}]$: Irrational case

Example: Maple file

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$[H_{q,0}]$: Rational non-logarithmic case

$[H_{q,g}] \xrightarrow{f}_C [M] \rightarrow_{EG} [A]$

- The zeros of f are regular singularities of $[M]$.
- If the generalized exponents of the zeros of f in $[M]$ are integers, they might be removed by a gauge transformation.
- So, we do not have a one-to-one correspondence between the singularities of $[A]$ and zeros of f. Example: Maple file

Method:

- $f = c\frac{A_1 A_2^d}{B}, \quad c \in C$
- Find B, d_A, and S_{reg}
- Find candidates for $(d, A_1, deg(A_2))$
- For each candidate, find c: We always have enough number of equations from S_{irr}!

$[H_{q,0}]$: Rational non-logarithmic case

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Summary and further investigations

- We give an algorithm which detects and removes removable singularities of an input first-order system
- We apply this algorithm to detect whether an input differential equation of an arbitrary order or a first-order system has generalized hypergeometric solutions ($p \leq q+1$)
- We give a decision algorithm in the case $p \leq q$
- We give a method to "filter" the information in the case $p = q+1$
- MAPLE package GenHypSols

Further investigations:

- Recovering f in the case $p = q+1$

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Thank you