Algorithmic proof for the transcendence of D-finite power series

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Overview

Context and goal

Algebraic and transcendental power series

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

- A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$.
- A power series that is not algebraic is called *transcendental*.

▷ Task: Given a power series, either in explicit or in implicit form, determine whether it is algebraic or transcendental.

Stanley's question

Given a linear differential equation with polynomial coefficients, together with suitable initial conditions, satisfied by a power series y, give an algorithm suitable for computer implementations for deciding whether y is algebraic.

[Stanley, 1980]

Motivations

- Number theory: first step towards proving the transcendence of a complex number is to prove that a power series is transcendental
- Combinatorics: nature of generating series may reveal strong underlying structures
- Computer science: are algebraic power series (intrinsically) easier to manipulate?

Quiz: is it algebraic, or transcendental?

- $\displaystyle\sum_n n^{2017} t^n, \quad \sum_n \frac{1}{n} t^n, \quad \sum_n \frac{1}{n^2+1} t^n, \quad \sum_n F_n t^n, \quad \sum_n \frac{1}{F_n} t^n$

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- $\displaystyle\sum_n \frac{1}{n!} t^n, \quad \sum_n \frac{(2n)!}{4^n (n!)^2 (2n+1)} t^n, \quad \sum_n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) t^n$

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- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n, \quad \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^n$

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Classical transcendence criteria

Two simple (though not very useful) criteria

Establishing transcendence of values at an algebraic point constitutes in principle the most straightforward transcendence criterion for functions, although it is almost invariably the most difficult to apply.

[Flajolet, 1987]

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- There exists a $z \in \overline{\mathbb{Q}}$ such that $f(z) \notin \overline{\mathbb{Q}}$
- There exists a prime number p such that $f_p = f \bmod p$ is well-defined in $\mathbb{F}_p[[t]]$ and f_p is not algebraic over $\mathbb{F}_p(t)$

then the power series f is transcendental

Main properties of algebraic series

- Algebraic properties
  - Algebraic series are D-finite [Abel, 1827]
  - They are diagonals of bivariate rational functions [Furstenberg, 1967]
  - Their resolvents admit a basis of algebraic solutions [Singer, 1979]
- Arithmetic properties
  - Algebraic series are globally bounded [Eisenstein, 1852]
  - Their resolvents have zero p-curvature for $p \gg 0$ [Katz, 1972]
  - The coefficient sequence of an algebraic series of degree > 2 is not p-Lucas [Allouche, Gouyou-Beauchamps, Skordev, 1998]
- Analytic properties
  - The coefficient sequence of an algebraic series has bounded gaps
  - It has "nice" asymptotics [Puiseux, 1850; Flajolet, 1987]
  - Resolvents of algebraic series are Fuchsian [Fuchs, 1865]

Transcendence criteria

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- f is not globally bounded $\qquad \sum_n \frac{1}{n} t^n$
- $(a_n)_n$ has "ugly" asymptotics $\qquad \sum_n p_n t^n$ (p_n = n-th prime)
- f is not D-finite $\qquad \prod_n \frac{1}{1-t^n}$
- f is D-finite, but $L_f^{\min}$ has non-zero p-curvatures $\qquad \sum_n \sum_k \binom{n}{k}^3 t^n$
- f is D-finite, but $L_f^{\min}$ is not Fuchsian $\qquad \sum_n \frac{1}{n!^2} t^n$
- f is D-finite, but $L_f^{\min}$ has a log singularity $\qquad \sum_n \sum_k \binom{n}{k}^2 \binom{n+k}{k} t^n$
- $(a_n)_n$ is p-Lucas and $f^2 \notin \mathbb{Q}(t)$ $\qquad \mathrm{Diag}\, \frac{1}{(1-x-y)(1-z-t)-xyzt}$
- $(a_n)_n$ has too large gaps $\qquad \sum_n t^{n^2}$

then the power series f is transcendental

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is D-finite, and if one of the following holds

- f is not globally bounded $\qquad \displaystyle\sum_n \frac{1}{n} t^n$
- $(a_n)_n$ has "ugly" asymptotics $\qquad a_0 = 0, a_1 = 1,\ a_{n+2} = \dfrac{7n+11}{2n+1} a_{n+1} - a_n$
- $L_f^{\min}$ has non-zero p-curvatures $\qquad \displaystyle\sum_n \sum_k \binom{n}{k}^3 t^n$
- $L_f^{\min}$ is not Fuchsian $\qquad \displaystyle\sum_n \frac{1}{n!^2} t^n$
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then the power series f is transcendental

Transcendence criteria for D-finite and globally bounded series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is D-finite and globally bounded†
and if one of the following holds

- $(a_n)_n$ has "ugly" asymptotics $\qquad \sum_n \frac{(3n)!}{n!^3} t^n$
- $L_f^{\min}$ has non-zero p-curvatures $\qquad \sum_n \sum_k \binom{n}{k}^3 t^n$
- $L_f^{\min}$ has a log singularity $\qquad \sum_n \sum_k \binom{n}{k}^2 \binom{n+k}{k} t^n$
- $(a_n)_n$ is p-Lucas and $f^2 \notin \mathbb{Q}(t)$ $\qquad \operatorname{Diag} \frac{1}{(1-x-y)(1-z-t)-xyzt}$

then the power series f is transcendental

† Conjecturally, f is then the diagonal of a rational function [Christol, 1990]

Christol's example

Christol's conj. (1990): Is any D-finite glob. & bounded series a diagonal?

Concrete open problem: Is $f(t) = {}_3F_2\left(\begin{matrix} \frac{1}{9} & \frac{4}{9} & \frac{5}{9} \\ & \frac{1}{3} & 1 \end{matrix} \middle| \, 729\, t\right)$ a diagonal?

$$f(t) = 1 + 60\,t + 20475\,t^2 + 9373650\,t^3 + 4881796920\,t^4 + 2734407111744\,t^5 + \cdots$$

- Asymptotics
$$a_n \sim \frac{2\,\sin(4\pi/9)}{\sqrt{3}\,\Gamma(1/9)\Gamma(2/3)}\,729^n\,n^{-11/9}$$
- f is not p-Lucas for $p > 3$
- $L_f^{\min} = 3(729t-1)t^2\partial_t^3 + t(8991t-7)\partial_t^2 + (5400t-1)\partial_t + 60$ has a nilpotent, but non-zero, p-curvature, for $p > 3$
- $L_f^{\min}$ is irreducible and has a log singularity at $t = 0$

Rodriguez-Villegas' example

Chebychev in his work on the distribution of primes numbers used the following fact

$$u_n := \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z}, \qquad n = 0, 1, 2, \dots$$

This is not immediately obvious (for example, this ratio of factorials is not a product of multinomial coefficients) but it is not hard to prove. The only proof I know proceeds by checking that the valuations $v_p(u_n)$ are non-negative for every prime p; an interpretation of u_n as counting natural objects or being dimensions of natural vector spaces is far from clear.

Theorem [Rodriguez-Villegas, 2005]

$$f(t) = \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n \quad \text{is algebraic of degree } 483,840 \text{ (!)}$$

- Asymptotics

$$u_n \sim \frac{1}{2\sqrt{15\,\pi}} \left(2^{14} \times 3^9 \times 5^5\right)^n n^{-1/2}$$

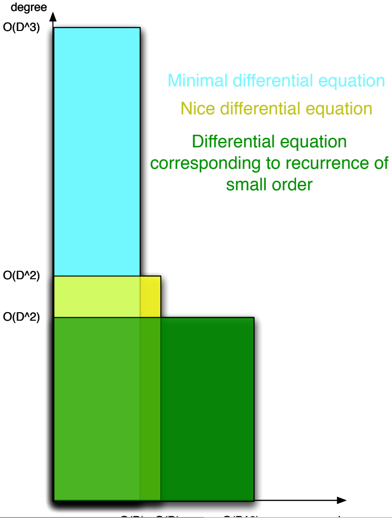
- f is not p-Lucas for $p > 5$
- The p-curvatures of $L_f^{\min}$ are zero for all $p \geq 29$.
- $L_f^{\min}$ is irreducible and only has algebraic singularities

Properties of algebraic series

Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Sizes (order, degree) of differential equations [B.-Chyzak-Lecerf-Salvy-Schost'07]



Asymptotics

Theorem [Flajolet, 1987]

If $f(t) = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then a_n has an asymptotic equivalent

$$a_n = \frac{\rho^n n^\alpha}{\Gamma(\alpha+1)} \cdot \sum_{i=0}^{m} C_i \omega_i^n + O(\rho^n n^\beta),$$

where $\alpha \in \mathbb{Q} \setminus \{-1, -2, -3, \ldots\}$; $\beta < \alpha$; $\rho \in \overline{\mathbb{Q}}_{>0}$; $C_i, \omega_i \in \overline{\mathbb{Q}}$ and $|\omega_i| = 1$

▷ Consequence of Newton-Puiseux, transfer based on Cauchy's formula (from local behaviour at singularities to asymptotics of coefficients), and

$$[t^n](1-t)^d = \binom{n+d-1}{d-1} \sim \frac{n^{d-1}}{\Gamma(d)}$$

Corollary

If $a_n \sim \gamma \rho^n n^\alpha$ and either

(i) $\alpha \in \mathbb{Z}_{<0}$; (ii) $\alpha \notin \mathbb{Q}$; (iii) $\rho \notin \overline{\mathbb{Q}}$; (iv) $\gamma \cdot \Gamma(\alpha+1) \notin \overline{\mathbb{Q}}$

then f is transcendental.

Transcendence using asymptotics

▷ $\sum_n a_n t^n = \mathrm{Diag}\left(\frac{1}{1-x-y-z}\right)$ is transcendental: $a_n = \frac{(3n)!}{n!^3} \sim 3^{3n}\frac{\sqrt{3}}{2\pi n}$

▷ $f = \sum_n p_n t^n$ is transcendental by the prime number theorem $p_n \sim n \log n$.

▷ The Apéry series $\sum a_n t^n$ with $a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ is transcendental, since

$$a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}, \quad \text{and} \quad \frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \quad \text{is transcendental}$$

▷ If $a_0 = 0, a_1 = 1, (2n+1)a_{n+2} - (7n+11)a_{n+1} + (2n+1)a_n = 0$, then $f = \sum_n a_n t^n$ is transcendental, since $a_n \sim C\left(\frac{7+\sqrt{33}}{4}\right)^n n^{\sqrt{75/44}}$ with $C \approx 0.56$.

Algebraic series have almost integer coefficients

Theorem [Eisenstein, 1852], [Heine, 1853]

Any algebraic power series $f = \sum_{n \geq 0} a_n t^n$ in $\mathbb{Q}[[t]]$ is globally bounded: there exists an integer $C > 0$ such that $a_n C^n$ is an integer for all $n \geq 1$.

▷ The smallest possible constant C is called *Eisenstein constant* of f.

▷ Best current bound [Dwork, van der Poorten 1992]

$$C \leq 4.8 \left(8\, e^{-3}\, D^{4+2.74 \log D}\, e^{1.22 D} \right)^{D} \cdot H^{2D-1} \quad = e^{O(D^2)} \cdot H^{2D-1}$$

where D is the algebraicity degree of f, and H is its height.

▷ Research problems:

- Is this bound (asymptotically) tight?
- Find a (fast) algorithm for computing C.

Algebraic series with p-Lucas coefficients

A sequence $(a_n)_n$ of rational numbers is called p-Lucas (p prime number) if

- all the denominators of the a_n's are prime to p;
- $a_{pi+j} \equiv a_i a_j \bmod p$ for all $i \geq 0$ and $0 \leq j < p$.

Theorem [Allouche, Gouyou-Beauchamps, Skordev, 1998]

For $f = \sum_n a_n t^n$ in $\mathbb{Q}[[t]] \setminus \{0\}$, the following conditions are equivalent:

1. f is algebraic and (a_n) has the p-Lucas property for all large primes p;
2. $f = \frac{1}{\sqrt{P(t)}}$ for some $P \in \mathbb{Q}[t]$ of degree at most 2, with $P(0) = 1$.

▷ Corollary: if $r_1, \ldots, r_m$ are positive integers, then

$$f = \sum_{n \geq 0} \binom{2n}{n}^{r_1} \binom{3n}{n}^{r_2} \cdots \binom{(m+1)n}{n}^{r_m} t^n$$

is algebraic if and only if $m = 1$ and $r_1 = 1$.

Theorem [Rowland, Yassawi, 2015]

If $P(x_1,\dots,x_d) \in \mathbb{Q}[x_1,\dots,x_d]$ has degree at most 1 in each x_i, and if $P(0,\dots,0) = 1$, then the diagonal sequence

$$a_n = [x_1^n \cdots x_d^n]\ \frac{1}{P}$$

is p-Lucas for any prime p.

▷ $\sum_n \binom{n}{k}^2 \binom{n+k}{k}^2$ is the diagonal sequence of $\dfrac{1}{(1-x-y)(1-z-t)-xyzt}$

▷ $\sum_n \binom{n}{k}^d$ is the diag. seq. of $\dfrac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$

▷ $\sum_n \binom{n}{k}^2 \binom{n+k}{k}^3$ is the diag. seq. of $\dfrac{1}{1-(xyz+xy+xz+yz+z)(uv+u+v)}$

The Grothendieck–Katz p-curvatures conjecture

Conjecture [Grothendieck, 1960's, unpublished; Katz, 1972]

Let $A \in \mathbb{Q}(t)^{r \times r}$ and $(\mathsf{S}) : y' = Ay$. The following assertions are equivalent:

- (S) has a basis of algebraic solutions
- $(\mathsf{S}_p) : y' = (A \bmod p)y$ has a basis of algebraic solutions over $\mathbb{F}_p(t)$ for all primes $p \gg 0$,
- $A_p = 0 \bmod p$ for all primes $p \gg 0$, where A_p = p-curvature of (S):

$$A_0 = I_r, \quad \text{and} \quad A_{\ell+1} = A_\ell' + A_\ell A \quad \text{for} \quad \ell \geq 0.$$

▷ Proved by [Katz, 1982] for *Picard-Fuchs systems*, but still open in general

▷ For each p, the last condition can be checked algorithmically

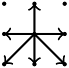
▷ [B., Caruso, Schost, 2015] Fast algorithms for the p-curvature

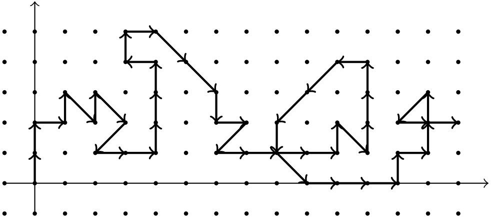
Combinatorial examples

▷ Small step walks in the quarter plane: walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



Lattice walks with small steps in the quarter plane

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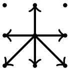
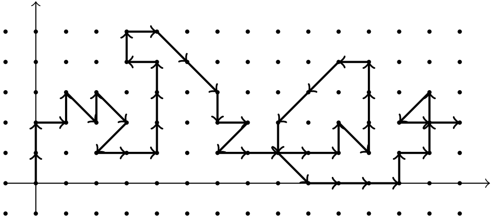
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i, j).

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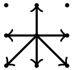
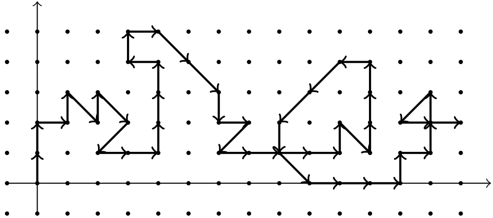
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Nature of generating functions

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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▷ Specializations:

- Walks returning to the origin ("excursions"): $F(t;0,0)$;
- Walks with prescribed length: $F(t;1,1) = \sum_{n \geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t;1,0)$;
- Walks ending on the diagonal: "$F(t;0,\infty)$" $:= \left[x^0\right] F(t;x,1/x)$.

Nature of generating functions

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \;\in \mathbb{Q}[x,y][[t]].$$

▷ Specializations:

- Walks returning to the origin ("excursions"): $F(t;0,0)$;
- Walks with prescribed length: $F(t;1,1) = \sum_{n\geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t;1,0)$;
- Walks ending on the diagonal: "$F(t;0,\infty)$" $:= \left[x^0\right] F(t;x,1/x)$.

Question:

Given $\mathfrak{S}$, what can be said about $F(t;x,y)$ and its specializations?

Are they algebraic or transcendental?

Small-step models of interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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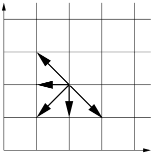

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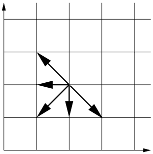

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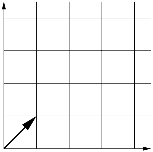

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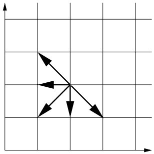

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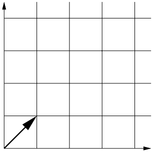

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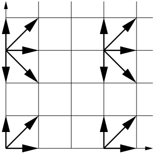
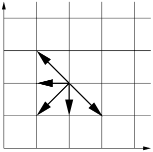
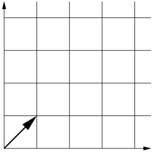


intrinsic to the half plane,

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



trivial,



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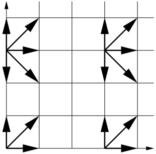

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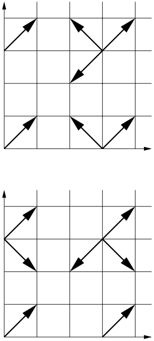
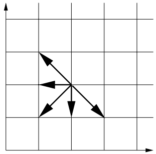
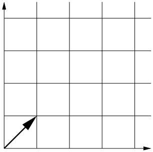


symmetrical.

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



trivial,



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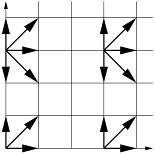

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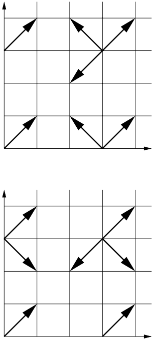
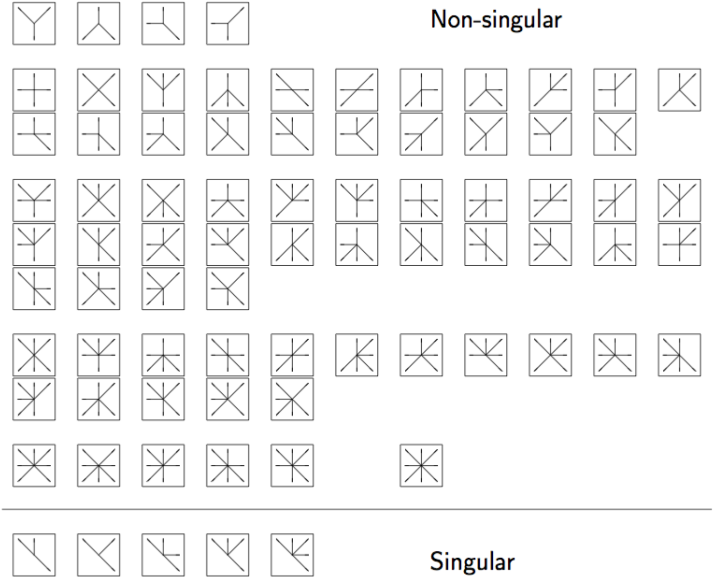


symmetrical.

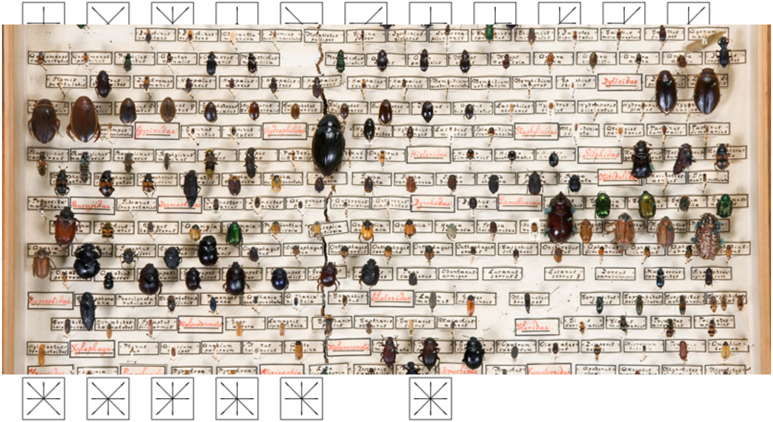
One is left with 79 interesting distinct models.

The 79 models



The 79 models

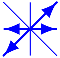
Non-singular

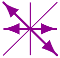


Singular

Pólya: 

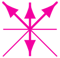
Kreweras: 

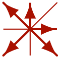
Gessel: 

Gouyou-Beauchamps: 

King: 

Tandem: 

Trident: 

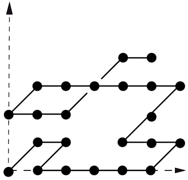
Scarecrow: 

A difficult model: Gessel's question

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g_{n;i,j}$ = number of walks from $(0,0)$ to (i,j) with n steps in $\mathfrak{S}$

Question: Find the nature of the generating function

$$G(t;x,y) = \sum_{i,j,n=0}^{\infty} g_{n;i,j}\, x^i y^j t^n \in \mathbb{Q}[[x,y,t]]$$



Theorem [B. & Kauers 2010] $G(x,y,t)$ is an algebraic power series†.

→ Effective, computer-driven discovery and proof
→ Key step in discovery: p-curvature computation of two 11th order (guessed) differential operators for $G(t;x,0)$, and $G(t;0,y)$

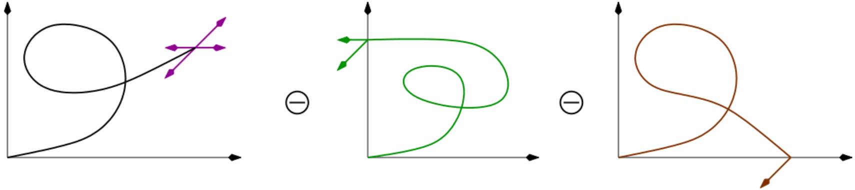
† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$G(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(t;x,y) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)$$



Algebraic reformulation: solving a functional equation

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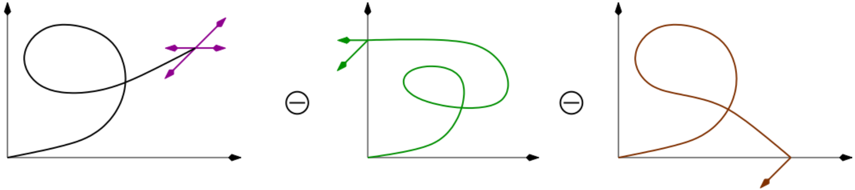

Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

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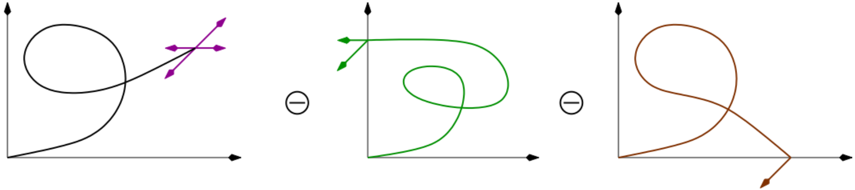
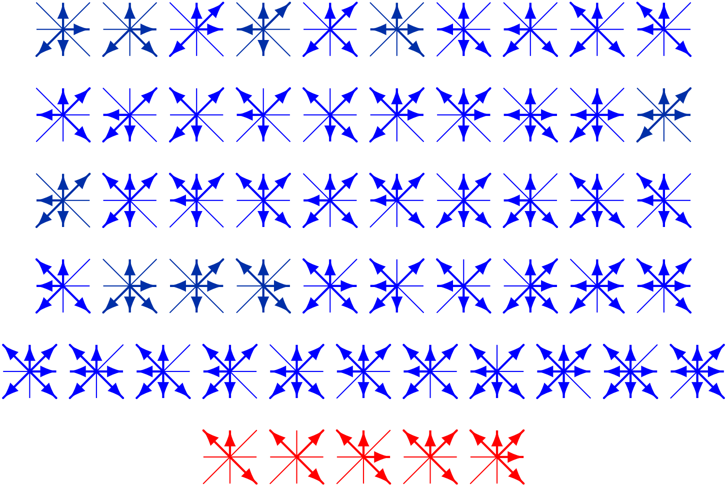


Task: For the other models: solve 78 similar equations!

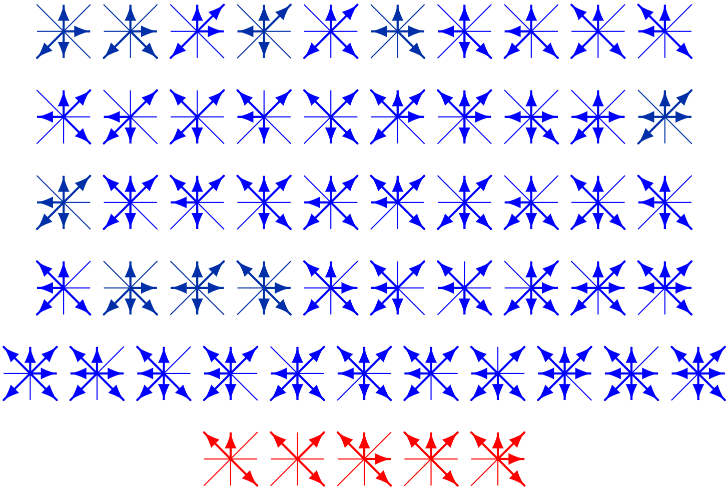
Transcendence of $79 - 23 = 56$ models



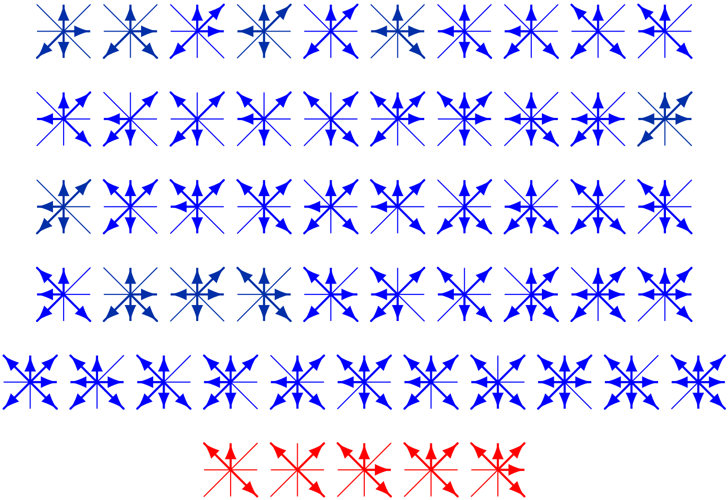
For non-singular models: $F_{\mathfrak{S}}(t;0,0)$ transcendental [B., Raschel, Salvy, 2013]

▷ Proof uses asymptotics

Transcendence of $79 - 23 = 56$ models



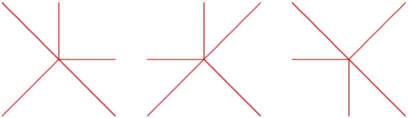
For singular models: $F_{\mathfrak{S}}(t;1,1)$ transcendental [Melczer & Mishna, 2013] ▷
Proof uses: infinite number of singularities



For all 56 cases: $F_{\mathfrak{S}}(t; x, y)$ is even non-D-finite!

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is transcendental for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

Irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being algebraic (even D-finite).

▷ Open: show that $F_{\mathfrak{S}}(t;1,1)$ is also transcendental!

	OEIS	$\mathfrak{S}$	nature		OEIS	$\mathfrak{S}$	nature
1	A005566		T	13	A151275		T
2	A018224		T	14	A151314		T
3	A151312		T	15	A151255		T
4	A151331		T	16	A151287		T
5	A151266		T	17	A001006		A
6	A151307		T	18	A129400		A
7	A151291		T	19	A005558		T
8	A151326		T				
9	A151302		T	20	A151265		A
10	A151329		T	21	A151278		A
11	A151261		T	22	A151323		A
12	A151297		T	23	A060900		A

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Transcendence (1–19) proved in [B., Chyzak, van Hoeij, Kauers & Pech '16]

Models with D-Finite $F(t;1,1)$ [B. & Kauers '09, '10], [Bousquet-Mélou & Mishna '10]

	OEIS	$\mathfrak{S}$	nature	asympt
1	A005566		T	$\frac{4}{\pi}\frac{4^n}{n}$
2	A018224		T	$\frac{2}{\pi}\frac{4^n}{n}$
3	A151312		T	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$
4	A151331		T	$\frac{8}{3\pi}\frac{8^n}{n}$
5	A151266		T	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$
6	A151307		T	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
7	A151291		T	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$
8	A151326		T	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$
9	A151302		T	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
10	A151329		T	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$
11	A151261		T	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$
12	A151297		T	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
13	A151275		T	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
14	A151314		T	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
15	A151255		T	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
16	A151287		T	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A001006		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
18	A129400		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
19	A005558		T	$\frac{8}{\pi}\frac{4^n}{n^2}$
20	A151265		A	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
21	A151278		A	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
22	A151323		A	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
23	A060900		A	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

- ▷ Transcendence (1–19) proved in [B., Chyzak, van Hoeij, Kauers & Pech '16]
- ▷ Asymptotics guessed in [B., Kauers '09], proved in [Melczer, Wilson '15]

Transcendence, and explicit expressions, for models 1–19

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let $\mathfrak{S}$ be one of the models 1-19. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1, 1)$, and $\mathfrak{S} =$  at $(1, 0), (0, 1), (1, 1)$

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Example (King walks in the quarter plane, A151331)

F 

$$F(t;1,1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\left. \begin{matrix} \frac{3}{2} \;\; \frac{3}{2} \\ 2 \end{matrix} \right| \frac{16x(1+x)}{(1+4x)^2} \right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

is transcendental.

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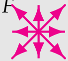
Example (King walks in the quarter plane, A151331)

F  $(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ & 2 \end{matrix} \middle| \frac{16x(1+x)}{(1+4x)^2} \right) dx$

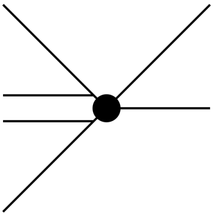
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

is transcendental.

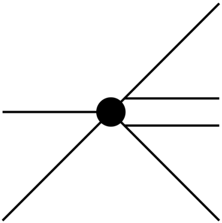
▷ Computer-driven discovery and proof; no human proof yet

▷ Proof uses creative telescoping, ODE factorization & solving, Kovacic algo

Two very interesting models with repeated steps



Case A



Case B

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

- $F_A(t;x,y)$ is D-finite and transcendental.
- $F_B(t;x,y)$ is algebraic.

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses Guess'n'Prove and new algorithm for transcendence.

Existing Algorithms

Recurrences of order 1

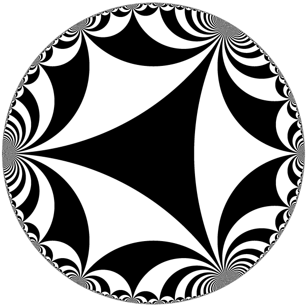
Theorem [Schwarz, 1873]

Let $a, b, c \in \mathbb{Q}$, s.t. $a, b, c-a, c-b \notin \mathbb{Z}$. Set $(\lambda, \mu, \nu) = (1-c, c-a-b, b-a)$. Up to permutations and sign changes of λ, μ, ν, and addition to (λ, μ, ν) of $(\ell, m, n) \in \mathbb{Z}^3$ with $\ell + m + n$ even, a table gives all algebraic ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| \, t\right)$'s.

Tabelle

enthaltend, abgesehen vom gemeinsamen Factor π, die Bogenzahlen der Winkel und den Flächeninhalt der reducirten sphärischen Dreiecke, welche auf einer Kugeloberfläche vom Radius 1 durch die Symmetrieebenen einer concentrischen regelmässigen Doppelpyramide oder eines concentrischen regelmässigen Polyeders bestimmt werden.

No.	λ''	μ''	ν''	$\frac{\text{Inhalt}}{\pi}$	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6} = A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} = 2A$	
IV.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12} = B$	Würfel und Oktaeder
V.	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6} = 2B$	
VI.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{30} = C$	Dodekaeder und Ikosaeder
VII.	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{15} = 2C$	
VIII.	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{15} = 2C$	
IX.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10} = 3C$	
X.	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15} = 4C$	
XI.	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5} = 6C$	
XII.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5} = 6C$	
XIII.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5} = 6C$	
XIV.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30} = 7C$	
XV.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3} = 10C$	



▷ Proof based on geometric arguments (sphere tilings by spherical triangles)

▷ Basic case: ${}_2F_1\left(\begin{matrix} r & 1-r \\ & \frac{1}{2} \end{matrix} \middle| \, t\right) = \dfrac{\cos((1-2r)\cdot \arcsin(\sqrt{t}))}{\sqrt{1-t}}$, $r \in \mathbb{Q}$ + sporadic cases

Algebraic and Gauss hypergeometric series

Whatever the beauty of Schwarz's result, one must recognize that it is achieved through a long detour. [Kampé de Fériet, 1937]

Theorem [Landau, 1904], [Stridsberg, 1911], [Landau, 1911], [Errera, 1913]

Assume $a, b, c \in \mathbb{Q}$ such that $a, b, c-a, c-b \notin \mathbb{Z}$. Then ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| \, t\right)$ is algebraic if and only if for every r coprime with the denominators of a, b and c, either $\{ra\} \le \{rc\} < \{rb\}$ or $\{rb\} \le \{rc\} < \{ra\}$. $\quad (\{x\} \stackrel{\text{def}}{=} x - \lfloor x \rfloor)$

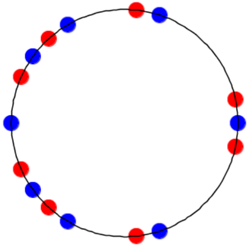
▷ Proof based on Eisenstein's theorem.

▷ $$\frac{{}_2F_1\left(\begin{matrix} -\frac{1}{2} & -\frac{1}{6} \\ & \frac{2}{3} \end{matrix} \middle| \, 16\,t\right) - 1}{2t} = 1 + 2\,t + 11\,t^2 + 85\,t^3 + 782\,t^4 + \cdots \text{ is algebraic}$$

▷ $${}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \middle| \, 1728\,t\right) = 1 + 60\,t + 39780\,t^2 + 38454000\,t^3 + \cdots \text{ not algebraic}$$

Theorem [Beukers, Heckman, 1989]

Let $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_{k-1}, b_k = 1\}$ be two sets of rational parameters, assumed disjoint modulo $\mathbb{Z}$. Let D be their common denominator. Then ${}_kF_{k-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_k \\ b_1 & \cdots & b_{k-1} & \end{matrix} \,\middle|\, t\right)$ is algebraic iff $\{e^{2i\pi r a_j}, j \leq k\}$ and $\{e^{2i\pi r b_j}, j < k\}$ interlace on the unit circle for all $1 \leq r < D$ with $\gcd(r, D) = 1$.



▷ $\displaystyle\sum_n \frac{(30n)!\,n!}{(15n)!(10n)!(6n)!} t^n = {}_8F_7\left(\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{5} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{4}{5} & \end{matrix} \,\middle|\, 2^{14}\, 3^9\, 5^5\, t\right)$ is algebraic

Differential equations of order 2

Differential equations of order 2 with algebraic solutions

Problem: Decide if *all* solutions of a given equation L of order 2 are algebraic

• invariant theory: [Liouville, 1839], [Pépin, 1863, 1881], [Fuchs 1876, 1878], [Brioschi, 1877], [Singer & Ulmer 1993]
▷ Starting point: there exists a "Primform" of degree ≤ 12 whose evaluation at some solution basis of $L(y) = 0$ is some root of a rational function

• Klein pullback method: [Klein, 1876, 1877, 1913], [Baldassarri & Dwork 1979], [Baldassarri, 1980], [Berkenbosch 2004], [van Hoeij & Weil 2005]
▷ Starting point: L has only algebraic solutions iff it is a weak pullback by a rational map of an element in the Schwarz list

▷ [Marotte, 1887], [Kovacic, 1986], [Singer 1981], [Singer & Ulmer 1998], [Ulmer & Weil, 1996]: generalization to Liouvillian solutions

Singer's algorithm

Problem: Decide if *all* solutions of a given equation L of order n are algebraic

- Starting point [Jordan, 1878]: If so, then for some solution y of L, $u = y'/y$ has alg. degree at most $(49n)^{n^2}$ and satisfies a Ricatti equation of order $n-1$

Algorithm [Singer, 1979]

1. Decide if the Ricatti equation has an algebraic solution of degree at most $(49n)^{n^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given algebraic u, decide if $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

▷ [Painlevé, 1887], [Boulanger, 1898]: Same for $n = 3$ and L irreducible

▷ Impractical bound: 92236816 for $n = 2$; approx. 10^{330} for $n = 11$

▷ [Singer, 2014]: generalization to computing L^{alg}, whose solution space is spanned by the algebraic solutions of L

The new method

A first version

Input: $f(t) \in \mathbb{Q}[[t]]$, given as the generating function of an explicit binomial sum, or as the diagonal of an explicit rational function
Output: T if $f(t)$ is transcendental, A if it is algebraic

1. Compute an ODE L for $f(t)$ — Creative telescoping
2. Compute $L_f^{\min}$ — Bounds + diff. Hermite-Padé
3. Decide if $L_f^{\min}$ has only algebraic solutions; if so return A, else return T. [Singer, 1979]

▷ Steps 2 and 3 can (in principle) be replaced by:

2. Compute L^{alg} and decide if it annihilates f — [Singer, 2014]

▷ $L_f^{\min}$ and L^{alg} can (in principle) be found using ODE factorization [Schlesinger, 1897], [Singer, 1981], [Grigoriev, 1990]

▷ Astronomic degree bound [Grigoriev, 1990]: $\exp\left(\left(\text{bitsize}(L)2^n\right)^{2^n}\right)$

An efficient version

Input: $f(t) \in \mathbb{Q}[[t]]$, given as the generating function of an explicit binomial sum, or as the diagonal of an explicit rational function
Output: T if $f(t)$ is transcendental, A if it is algebraic

1. Compute an ODE L for $f(t)$ — Creative telescoping
2. Compute $L_f^{\min}$ — Bounds + diff. Hermite-Padé
3. If $L_f^{\min}$ has a logarithmic singularity, return T
4. Compute a bound B — [Dwork, van der Poorten 1992]
   Set $p :=$ nextprime(B). Repeat:
   1. $p :=$ nextprime$(2p)$
   2. if p-curvature of $L_f^{\min}$ is $\neq 0$, return T — [B., Caruso, Schost, 2015]
   3. guess $P_p(x,y) \in \mathbb{Z}[x,y]$ such that $P_p(t, f(t)) = 0 \bmod t^p$ — alg. Hermite-Padé

   until either p-curvature is $\neq 0$, or non-trivial candidate $P_p(x,y)$ found.
5. Certify the candidate and return A, or goto 4 — algeqtodiffeq

▷ Termination ensured by Grothendieck-Katz for diagonals
▷ Conjecture: Steps 4–5 are not necessary

Computing $L_f^{\min}$

▷ Strategy (inspired by the approach in [van Hoeij, 1997], itself based on ideas from [Chudnovsky, 1980], [Bertrand & Beukers, 1982], [Ohtsuki, 1982])

1. $L_f^{\min}$ is Fuchsian
2. $L_f^{\min}$ can be written

$$L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)} \partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \leq \operatorname{ord}(L)$$

   with $A(t)$ squarefree and $\deg(a_{n-i}) \leq \deg(A^i) - i$.
3. $\deg(A)$ can be bounded in terms of n and of local information of L (via *apparent singularities* and *Fuchs' relation*)
4. Guess and Prove: For $n = 1, 2, \ldots,$
   1. Guess differential equation of order n for f (use bounds and differential Hermite-Padé)
   2. Once found a nontrivial candidate, certify it, or go to previous step.

Bounds for $L_f^{\min}$

$$L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)}\partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \le \mathrm{ord}(L)$$

Task: get a bound on $\deg(A)$ in terms of n and of local information of L

- $A(t) = A_{\mathsf{sing}}(t)A_{\mathsf{app}}(t)$, where the roots of A_{sing}, resp. of A_{app}, are the finite *true* singular points, resp. the finite *apparent* singular points, of $L_f^{\min}$.
- Trivial: $\deg(A_{\mathsf{sing}}) \le \#\{\text{finite true singularities of } L\}$
- Fuchs' relation

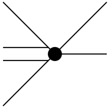
$$\sum_{p\in\mathbb{C}\cup\{\infty\}} S_p(L_f^{\min}) = \sum_{p \text{ singularity of } L_f^{\min}} S_p(L_f^{\min}) = -n(n-1),$$

with $S_p(L_f^{\min}) = (\text{sum of local exponents of } L_f^{\min} \text{ at } p) - (0+1+\cdots+(n-1))$

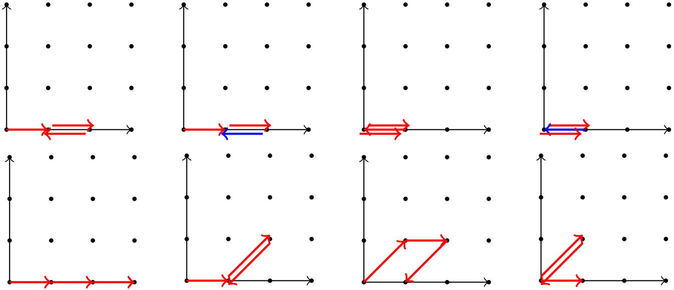
- Main point: If p is an apparent singularity of $L_f^{\min}$ then $S_p(L_f^{\min}) \ge 1$, thus:

$$\deg(A_{\mathsf{app}}) \le -n(n-1) - \sum_{p \text{ true singularity of } L} \min(0, S_p^{(n)}(L)),$$

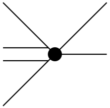
where $S_p^{(n)}(L) :=$ (sum of the smallest n exponents of L at p) $-\binom{n}{2}$



Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]
$F_A(t;1,0) = 1 + t + 4t^2 + 8t^3 + 39t^4 + 98t^5 + 520t^6 + \cdots$ is transcendental.



Example: a difficult quadrant model with repeated steps



Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]
$F_A(t;1,0) = 1 + t + 4t^2 + 8t^3 + 39t^4 + 98t^5 + 520t^6 + \cdots$ is transcendental.

1. $F_A(t;x,y)$ is D-finite in its three variables High-tech Guess'n'Prove
   (+ kernel method, non-commutative Gröbner bases, desingularisation.)
   ▷ Discovers and proves a differential equation L for $f(t) = F_A(t;1,0)$ of order 11 and degree 73
2. L is Fuchsian, 6 finite sing, 55 apparent sing., has a log sing. at $t = 0$
3. If $\mathrm{ord}(L_f^{\min}) \leq 10$, then $L_f^{\min}$ has coefficients of degrees at most 580
4. Differential Hermite-Padé approximants rule out this possibility.
5. Thus, $L_f^{\min} = L$, and so f is transcendental

▷ All other criteria fail

Summary and Questions

- Simple, efficient and robust algorithm for transcendence / algebraicity
- Basic theoretical tool: Fuchs relation
- Basic algorithmic tool: Guess'n'Prove via Hermite-Padé approximants + efficient computer algebra
- Brute-force / naive algorithms = hopeless on combinatorial examples

▷ generalization to algebraic independence of D-finite series?

▷ bounds for p-curvatures (effective Grothendieck conjecture)?

▷ transcendental diagonals with algebraic singularities?

▷ many open questions on transcendence of 2D and 3D lattice walks

Thanks for your attention!