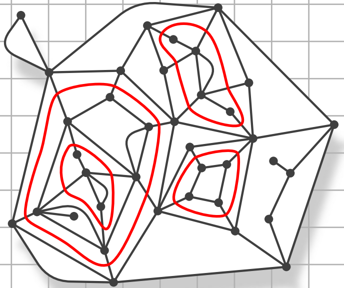
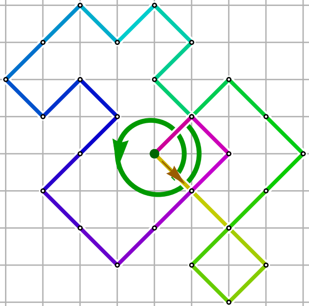
Computations & Proofs seminar, INRIA, Polytechnique
June 6th, 2017

Elliptic functions count walks on the square lattice with winding

Timothy Budd





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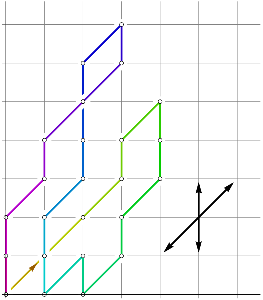
Introduction: Gessel sequence

- In 2001 Ira Gessel conjectured the number of walks with $2n$ steps $\in \{N, S, SW, NE\}$ in the quadrant starting and ending at 0 to be

$$16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} = 2, 11, 85, 782, \ldots$$



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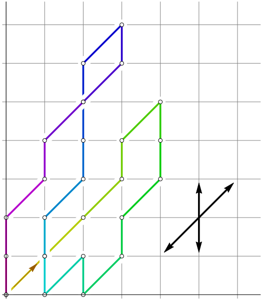

- Proving this turned out to be hard, but by now we have...
  - ...a computer-aided proof. [Kauers, Koutschan, Zeilberger, '08]
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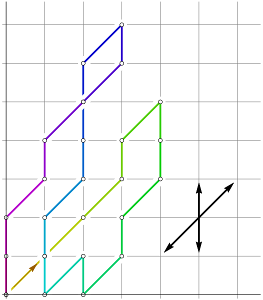

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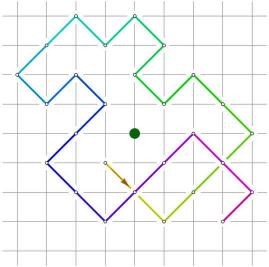
- As we will see, counting walks by winding angle provides a natural alternative route.

Introduction: Winding angle of a walk

- To a walk w on $\mathbb{Z}^2$ avoiding 0 we can naturally associate a winding angle

$$\theta_w := \sum_{i=1}^{|w|} \measuredangle(w_{i-1}, 0, w_i).$$



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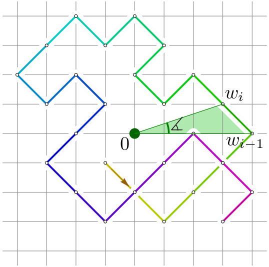

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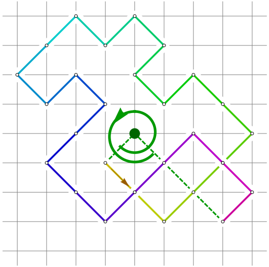

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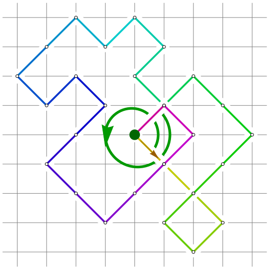

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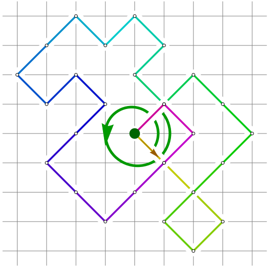

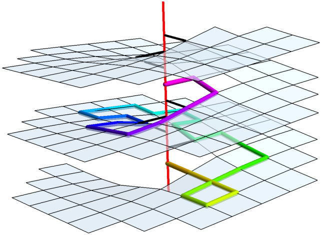

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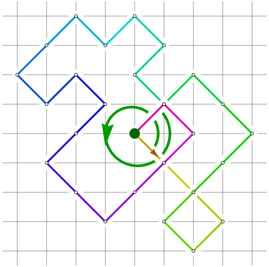
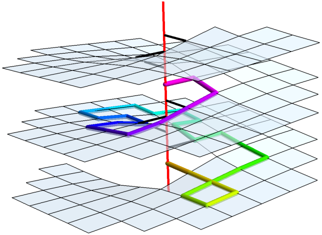
- Extends unambiguously to excursions from the origin.
- Natural interpretation as walks in the universal cover of $\mathbb{Z}^2 \setminus \{0\}$.
- First goal today is to determine the GF for simple excursions from origin

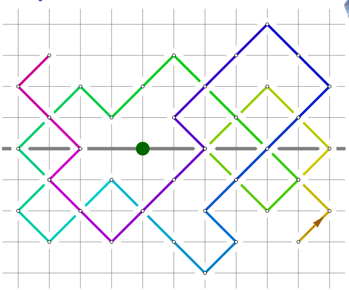
$$F(t,b) := \sum_w t^{|w|} e^{ib\,\theta_w}$$
$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \ldots$$





Decomposing into walks on the slit plane

- The general idea: decompose into a sequence of walks on the slit plane.



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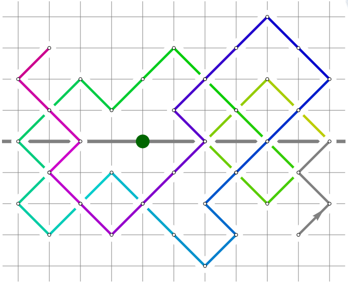

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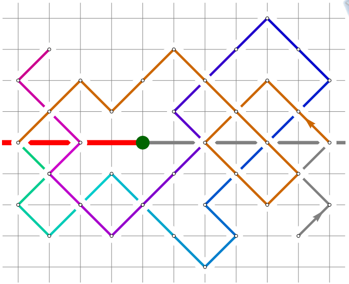

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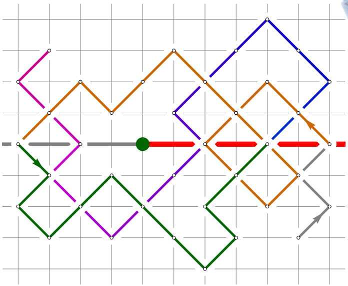

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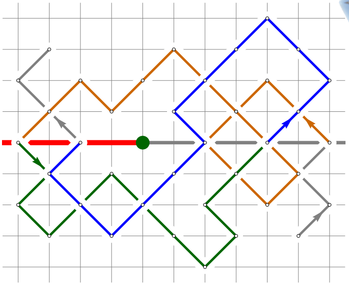

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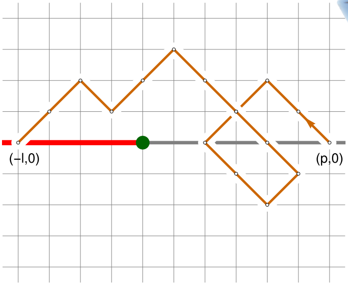
- Denote by $H^{(p,l)}(t)$ the GF for walks $(p,0) \to (-l,0)$ that hit the slit from above (counted by t^{length}).



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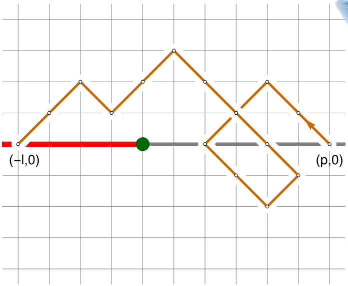

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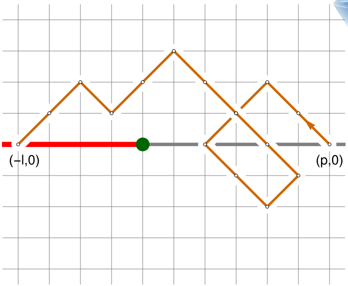

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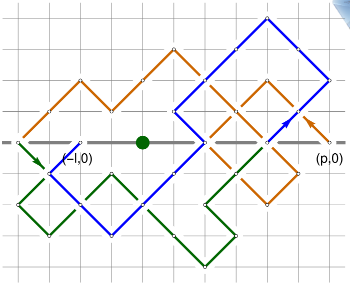

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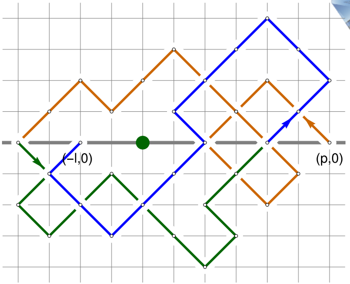

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$$\sum_{\text{such walks}} t^{|w|} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2\cos(\pi b))^N \left(\mathcal{H}^N\right)_{pl}$$

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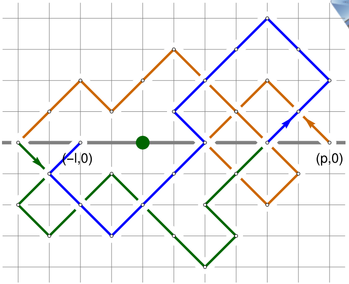

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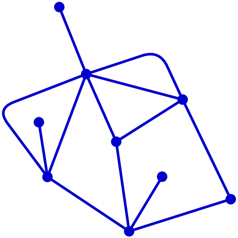
Relation with planar maps

- Planar map = a multigraph properly embedded in the plane up to homeomorphism. Take it to be rooted on the outer face.



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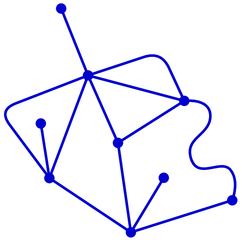

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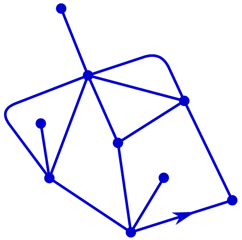

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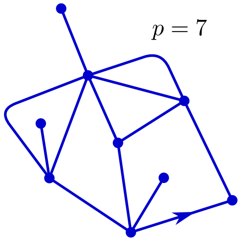

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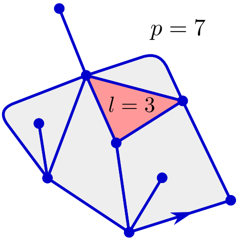

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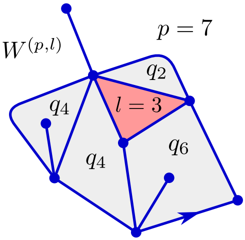

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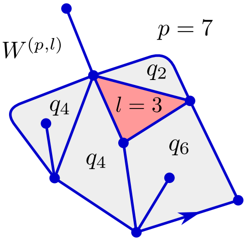
- For quasi-bipartite maps ($q_1 = q_3 = \cdots = 0$) it takes a universal form (see e.g. [Collet, Fusy, '12])

$$W^{(p,l)} = \frac{1}{l}\frac{2}{p+l}\alpha(l)\alpha(p)\left(\frac{\rho_{\mathbf{q}}}{4}\right)^{(p+l)/2} \qquad \alpha(p) := \frac{p!}{\lfloor\frac{p}{2}\rfloor!\lfloor\frac{p-1}{2}\rfloor!}$$



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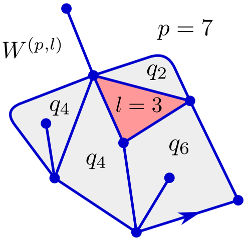
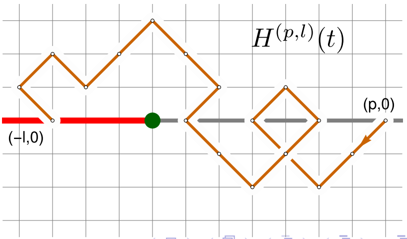
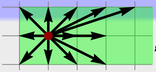
- Remarkably $H^{(p,l)}(t) = W^{(p,l)}\Big|_{\rho_{\mathbf{q}} \to \rho(t) := \frac{1-\sqrt{1-16t^2}}{8t^2} - 1}$





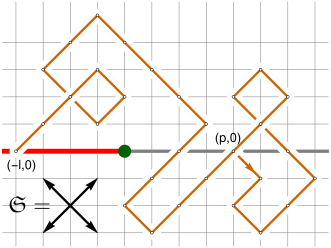
A bijective explanation

Proposition

For any step set $\mathfrak{S} \subset$ , *there exists a bijection*

$$\Phi^{(p,l)} : \{\mathfrak{S}\text{-}\textit{walks } (p,0) \to (-l,0) \textit{ hitting slit from above}\}$$

$$\longrightarrow \left\{ \begin{array}{l} \textit{“}\mathfrak{S}\textit{-walk-decorated maps” with root face degree } p \\ \textit{and marked face degree } l \end{array} \right\}$$



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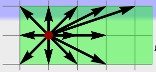
For any step set $\mathfrak{S} \subset$  *, there exists a bijection*

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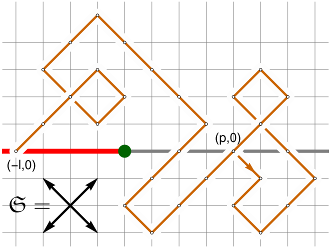

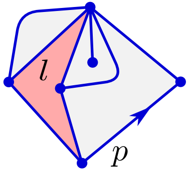

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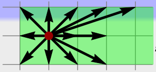
For any step set $\mathfrak{S} \subset$  *, there exists a bijection*

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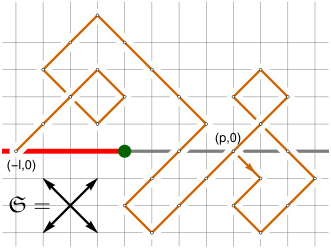
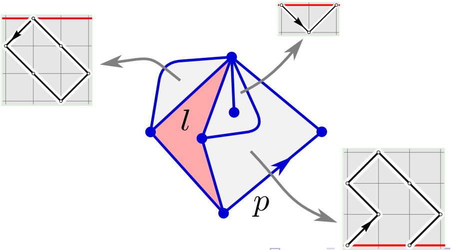
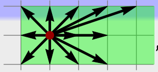
- A $\mathfrak{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree k an excursion $(0,0) \to (k-2,0)$ above or below x-axis.





A bijective explanation

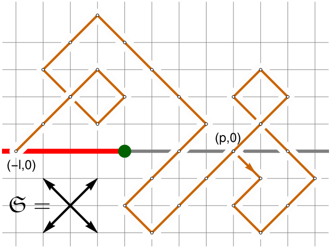
Proposition

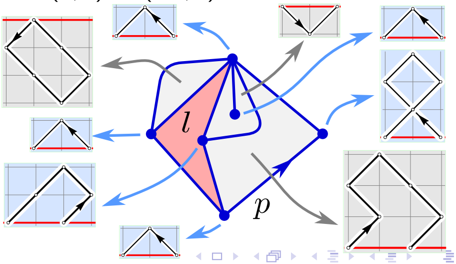
For any step set $\mathfrak{S} \subset$ , *there exists a bijection*

$$\Phi^{(p,l)} : \{\mathfrak{S}\text{-walks } (p,0) \to (-l,0) \text{ hitting slit from above}\}$$

$$\longrightarrow \left\{ \begin{array}{l} \text{“}\mathfrak{S}\text{-walk-decorated maps” with root face degree } p \\ \text{and marked face degree } l \end{array} \right\}$$

- A $\mathfrak{S}$-walk-decorated map is a rooted planar map with a marked face together with...
  - for each face (except root or marked) of degree k an excursion $(0,0) \to (k-2,0)$ above or below x-axis.
  - for each vertex an excursion $(0,0) \to (-2,0)$ above x-axis





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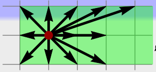
For any step set $\mathfrak{S} \subset$ *, there exists a bijection*

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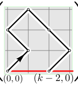
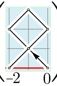
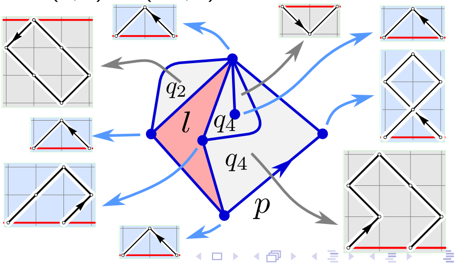
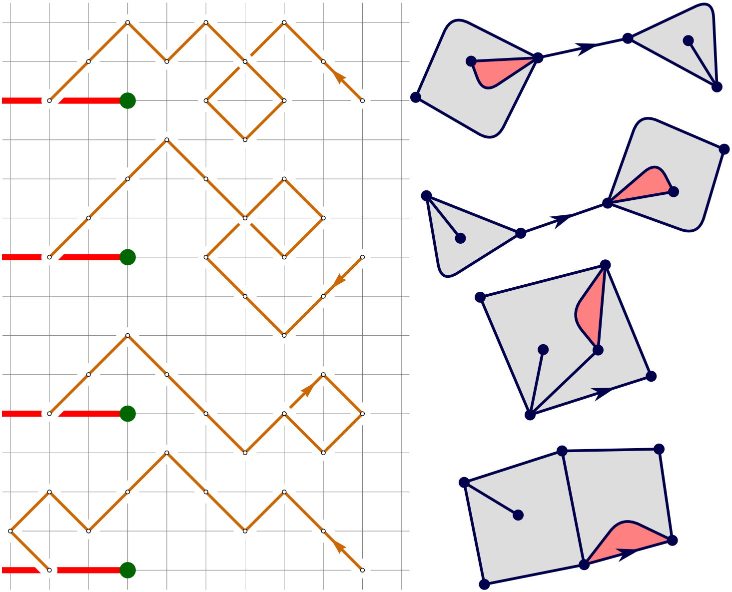
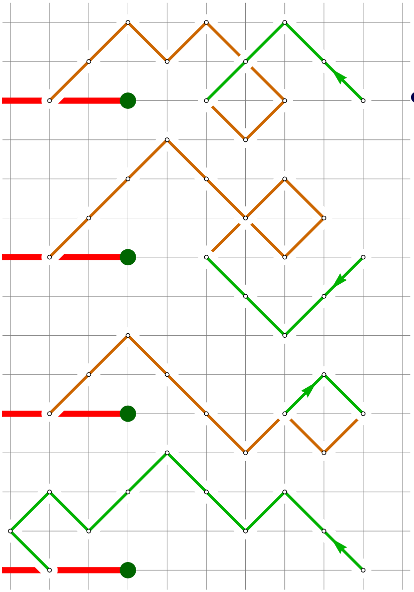
- Substituting in $W^{(p,l)}(q_i)$ the GFs

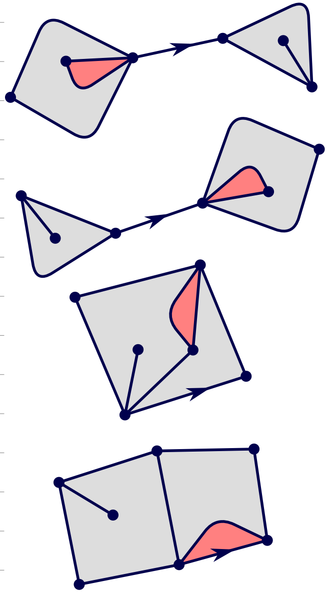
  $q_k \to \Bigg($  $\Bigg) \cdot \Bigg($  $\Bigg)^{\frac{k-2}{2}}$

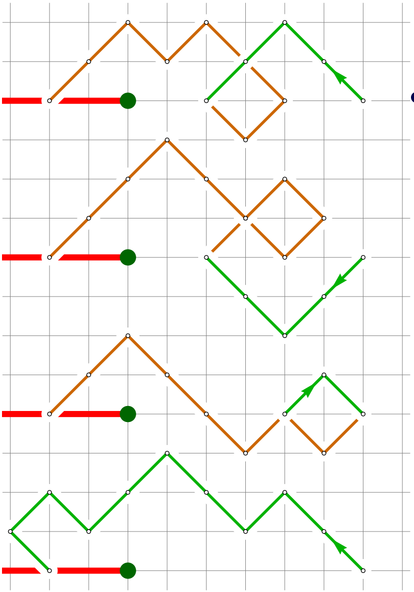
  leads to $H^{(p,l)}(t)$.

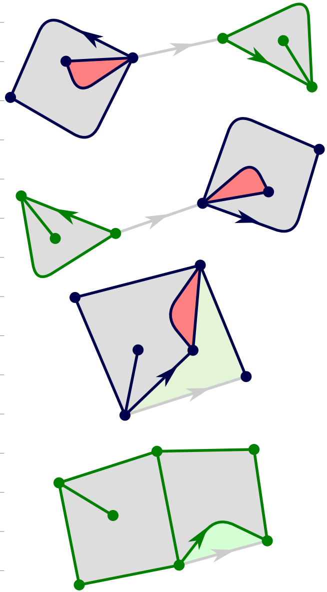




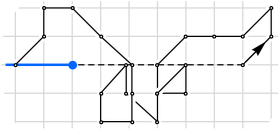








An example



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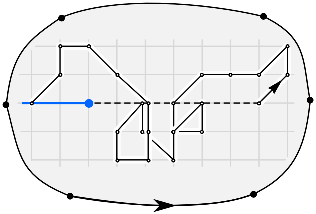

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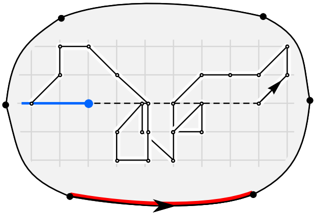

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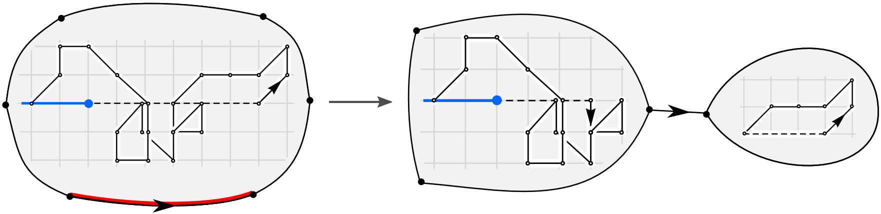

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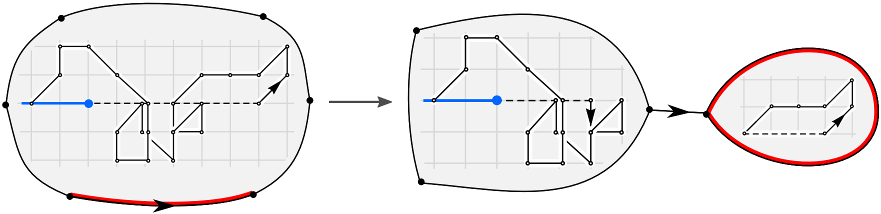

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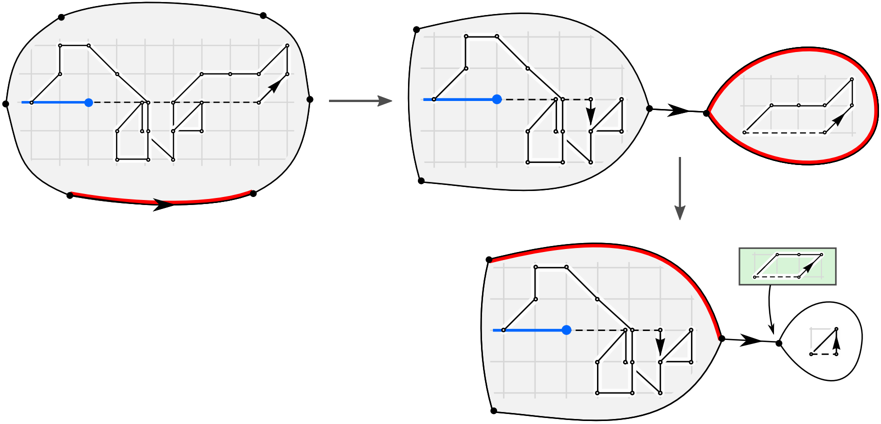

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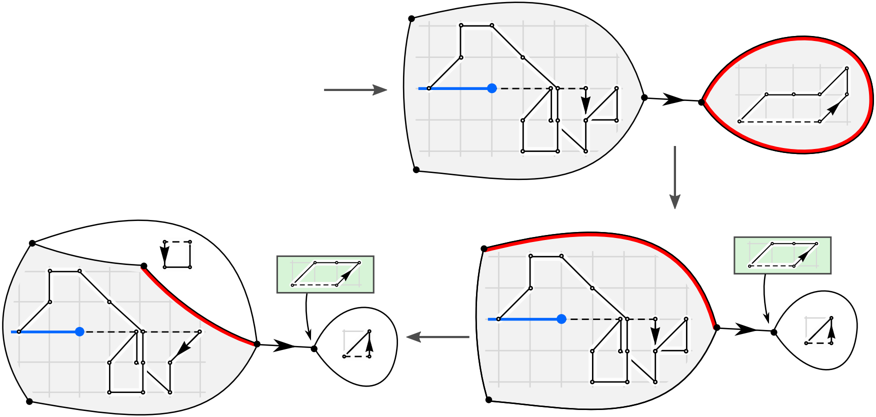

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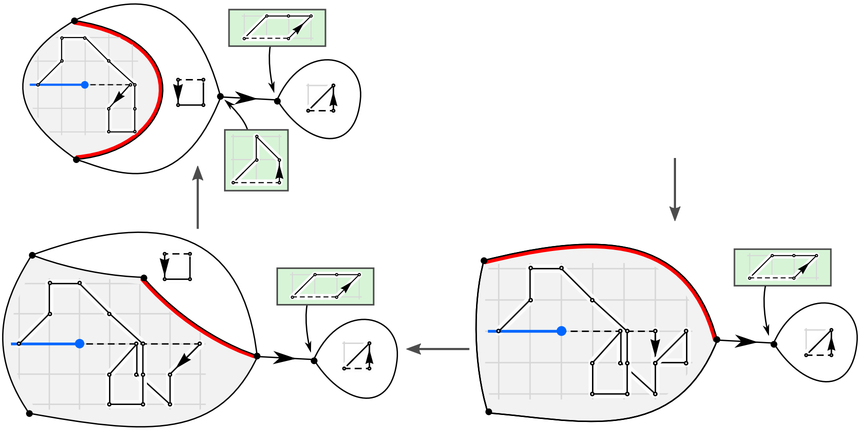

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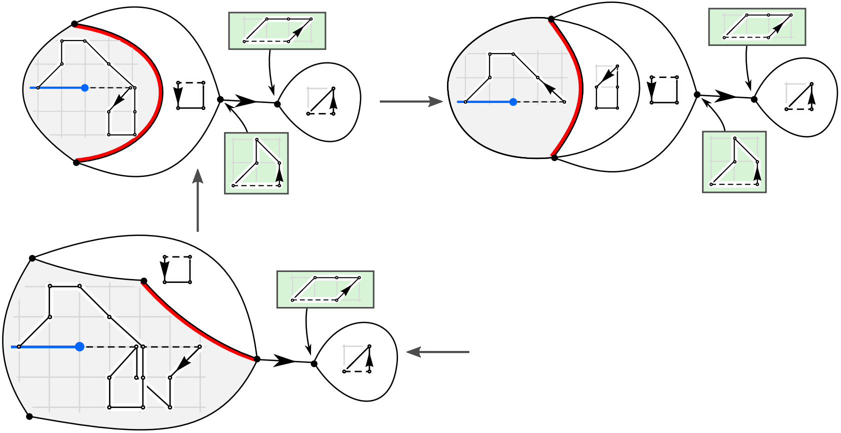

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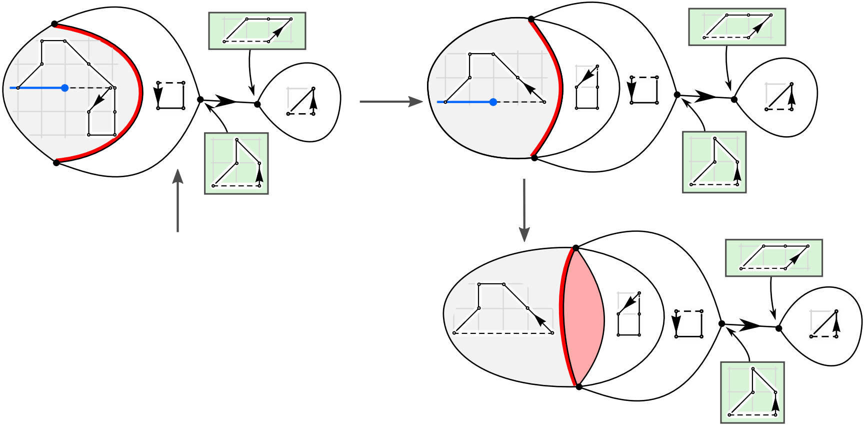

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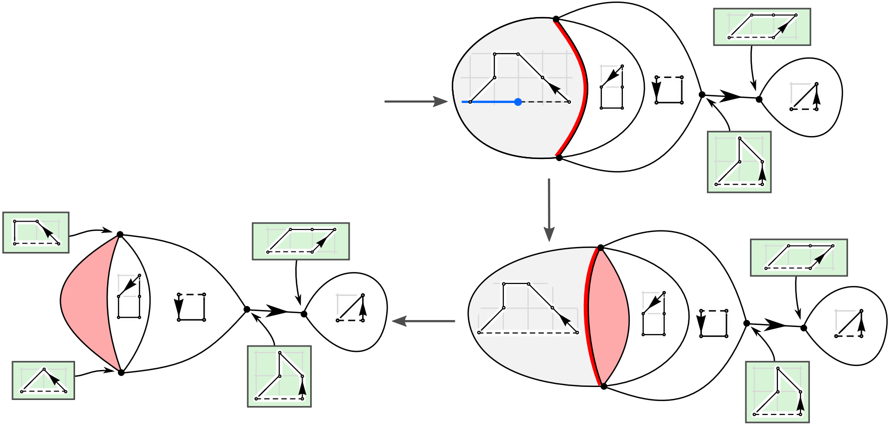
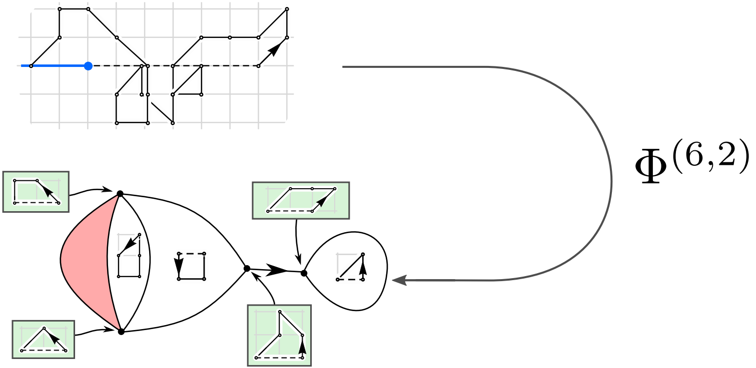
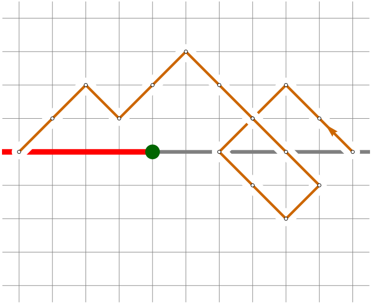
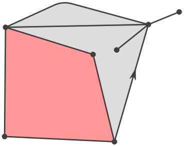


An example



From walks to (rigid) loop-decorated maps





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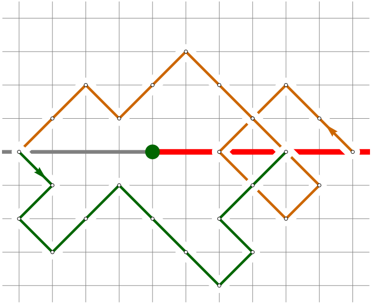

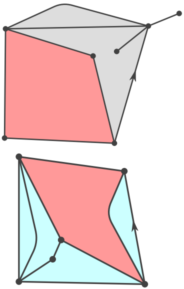

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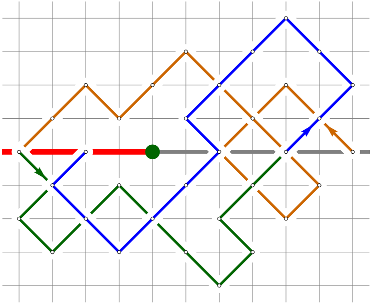

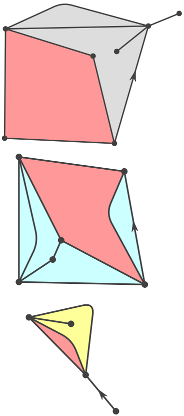

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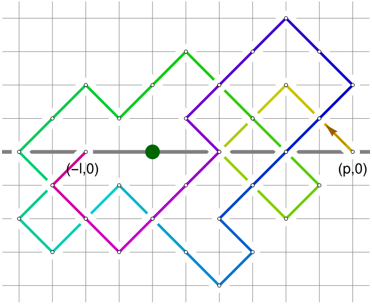

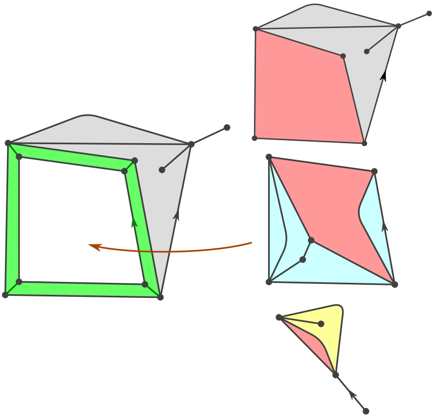

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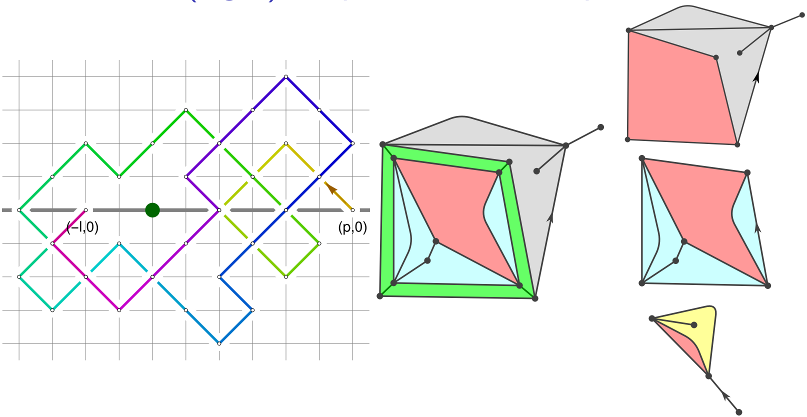

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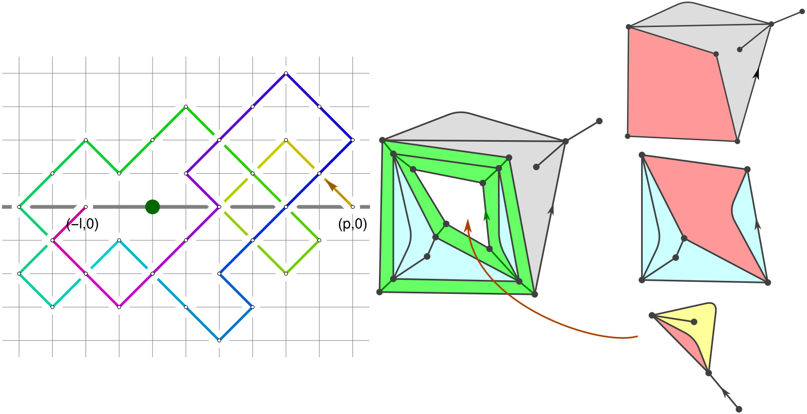

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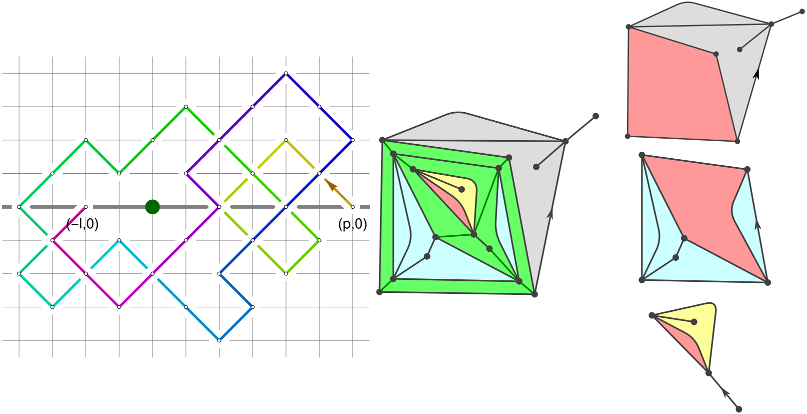

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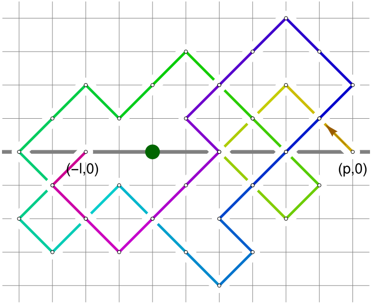

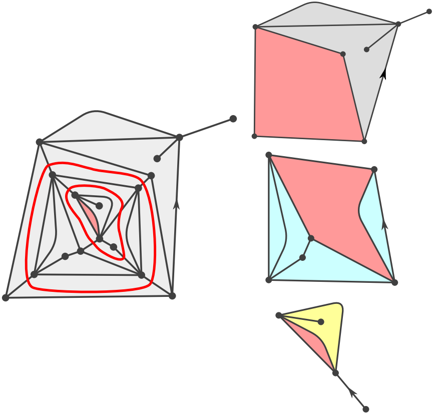

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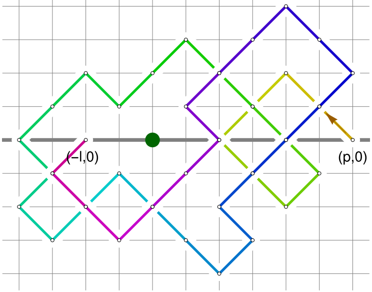

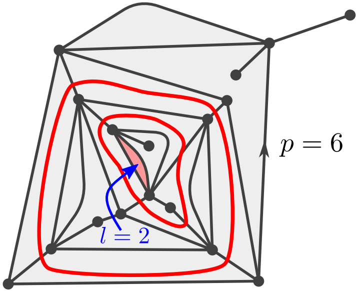

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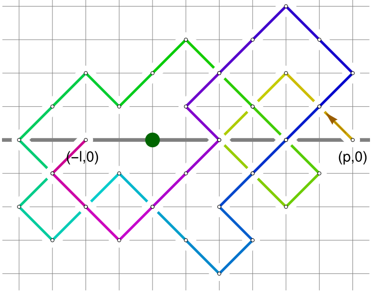

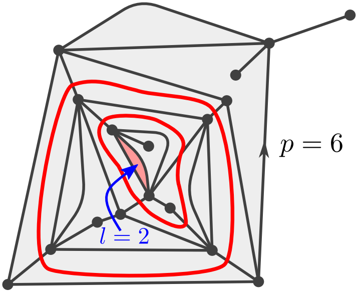

- Recall

$$\sum_{\text{such walks}} t^{|w|} e^{ib\theta_w} = \sqrt{\frac{p}{l}} \sum_{N=1}^{\infty} (2\cos(\pi b))^N \left(\mathcal{H}^N\right)_{pl} = \sqrt{\frac{p}{l}} \left(\frac{2\cos(\pi b)\mathcal{H}}{I - 2\cos(\pi b)\mathcal{H}}\right)_{pl}$$

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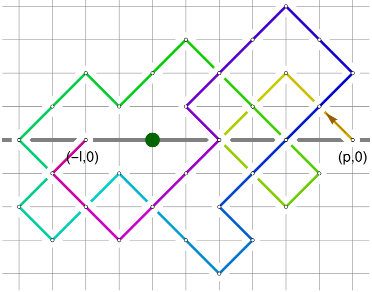

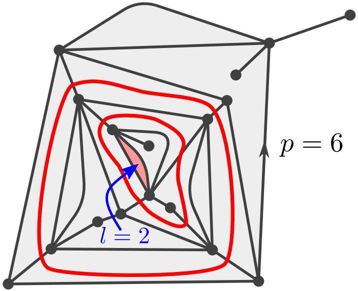

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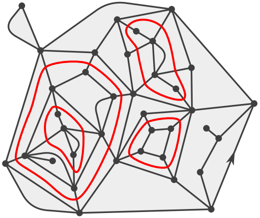
- Hence this also enumerates planar maps decorated with rigid loops with outer and marked face degrees p, l carrying a weight

$$(2\cos(\pi b))^{\#\text{loops}+1} \prod_{\text{regular faces}} q_{\text{degree}}$$

Planar maps coupled to a rigid $O(n)$ loop model

- Rigid $O(n)$ model: a planar map + disjoint loops, that intersect solely quadrangles through opposite sides. Enumerated with

$$\text{weight} \quad n^{\#\text{loops}} g^{\#\text{loop faces}} \prod_{\text{regular faces}} q_{\text{degree}}$$



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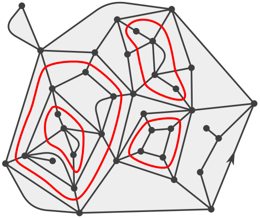

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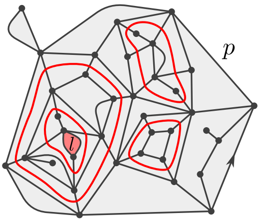

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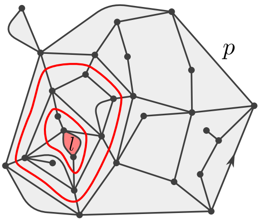

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- Recently in [Borot, Bouttier, Duplantier, '16] (for triangulations) exact statistics for the nesting of loops was obtained, i.e. distribution of # loops surrounding a marked vertex/face.
- Importantly: the form of the GF $\mathcal{G}^{(p,l)}(n, g, \mathbf{q})$ is universal and is not affected by suppressing loops that do not surround the marked face.

- We know that (with $n = 2\cos(\pi b)$ and appropriate g,$\mathbf{q}$)

$$\sqrt{\frac{p}{l}}\left(\frac{\mathcal{H}}{I - n\mathcal{H}}\right)_{pl} = \qquad \mathcal{G}^{(p,l)}(n, g, \mathbf{q})$$

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$$\sum_{p,l\geq 1} x_1^p x_2^l \sqrt{\frac{p}{l}} \left(\frac{\mathcal{H}}{I - n\mathcal{H}}\right)_{pl} = \sum_{p,l\geq 1} x_1^p x_2^l \, \mathcal{G}^{(p,l)}(n, g, \mathbf{q})$$

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- Adapting GF from [Borot, Bouttier, Duplantier, '16] and computing a series expansion:

$$= 4 \sum_{m=1}^{\infty} \frac{1}{q^m + q^{-m} - n} \frac{\cos(2\pi m \, v(x_2)) \; x_1 \frac{\partial}{\partial x_1} \cos(2\pi m \, v(x_1))}{m(q^{-m} - q^m)}$$

where $q = q(4t) = t^2 + 8t^4 + \cdots$ is the nome of modulus $4t$ and

$$v(x) := \mathrm{cd}^{-1}(-x/\sqrt{\rho}, \rho)/(4K(\rho)), \quad \rho(t) = \frac{1 - \sqrt{1 - 16t^2}}{8t^2} - 1$$

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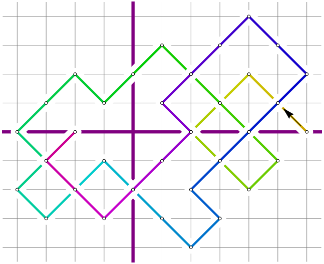
Proposition (Diagonalization of $\mathcal{H}$)

$\mathcal{H} = U^T \cdot \Lambda_q \cdot U$ in the sense of operators on $\ell^2(\mathbb{R})$ with

$$\Lambda_q = diag\left(\frac{1}{q^m + q^{-m}}\right)_{m\geq 1}, \quad U_{mp} = \sqrt{\frac{4p}{m(q^{-m} - q^m)}}\,[x^p]\cos(2\pi m\, v(x))$$

Refinement: increase winding angle resolution

- Up to now: decomposed walk into sequence of walks on slit plane, each numerated by $\sqrt{\frac{p}{l}}\mathcal{H}_{pl}$.



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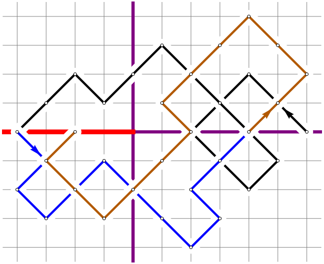

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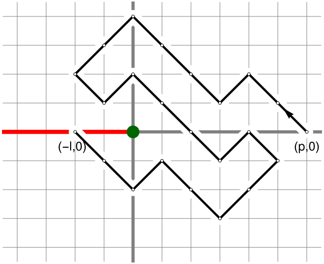
- Why not decompose into walks on half plane?



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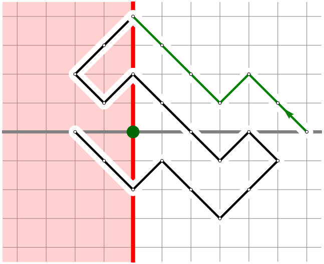

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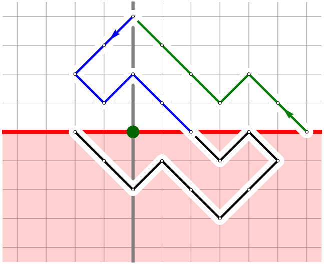

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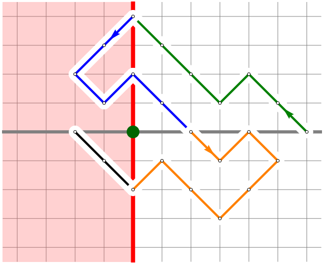

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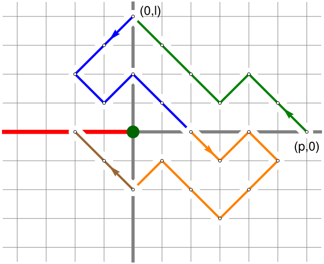

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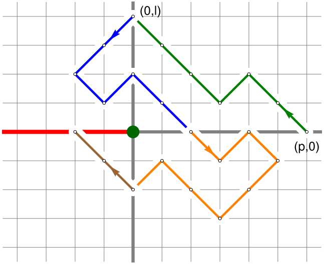


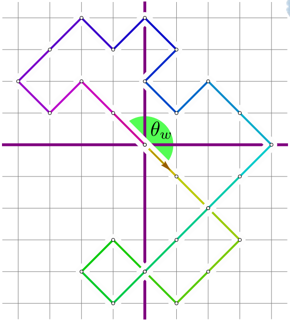
- Hence $\mathcal{J}$ has same eigenmodes as $\mathcal{H}$ but eigenvalues are $\frac{1}{q^{m/2}+q^{-m/2}}$ instead of $\frac{1}{q^m+q^{-m}}$. Such an operation $q \to \sqrt{q}$ on elliptic functions is called a "Landen transformation" and is thus connected to angle doubling.

Winding angle of excursions

- Wish to enumerate excursions from origin by length and winding angle:

$$F(t,b) := \sum_{w} t^{|w|} e^{ib\,\theta_w}$$

$$= 4t^2 + (12 + 4e^{-ib\frac{\pi}{2}} + 4e^{ib\frac{\pi}{2}})t^4 + \dots$$



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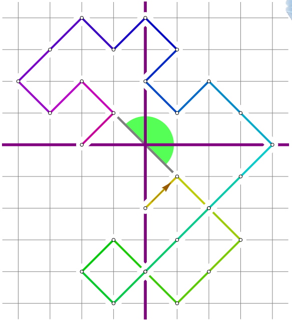

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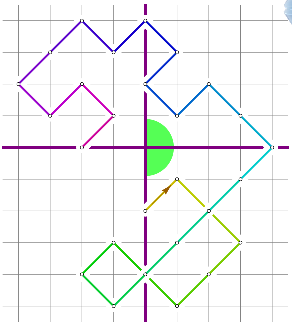

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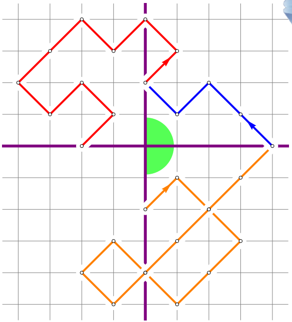

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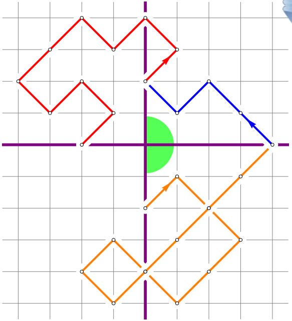

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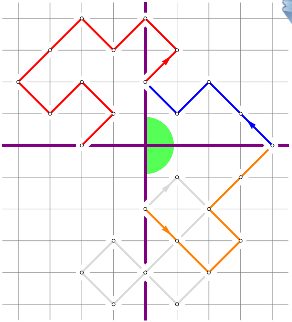

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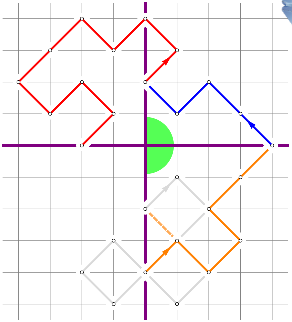

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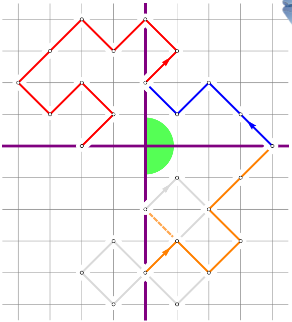

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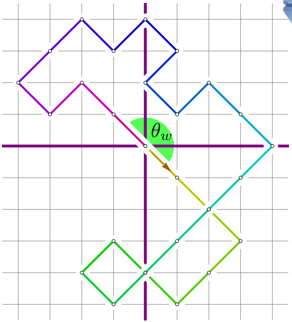

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$$= \sec\left(\tfrac{\pi b}{2}\right)\left[1 - \frac{\pi\tan\left(\frac{\pi b}{4}\right)}{2K(4t)}\,\frac{\theta_1'(\frac{\pi b}{4},\sqrt{q})}{\theta_1(\frac{\pi b}{4},\sqrt{q})}\right]$$

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Theorem (Excursions in the $\frac{n\pi}{4}$-cone.)

For any set of integers $-n < m - n < p < m < n$ the generating function $F_{n,m,p}(t)$ for excursions from the origin with winding angle $\frac{p\pi}{2}$ staying strictly inside angular region $(\frac{p+m-n}{4}\pi, \frac{p+m}{4}\pi)$ is given by

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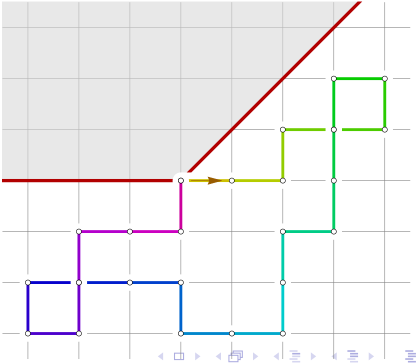

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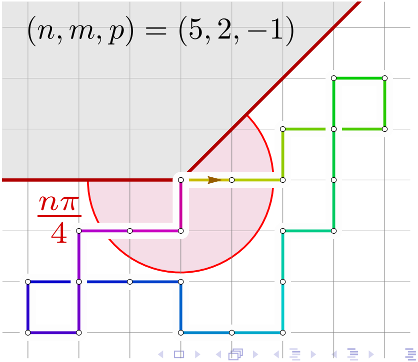

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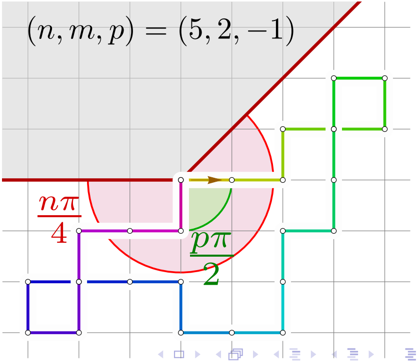

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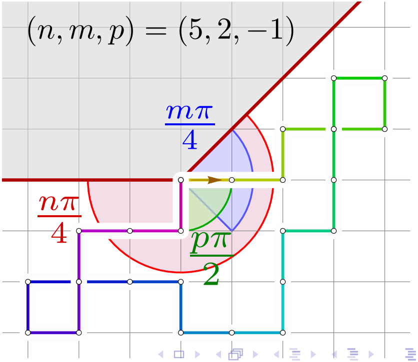

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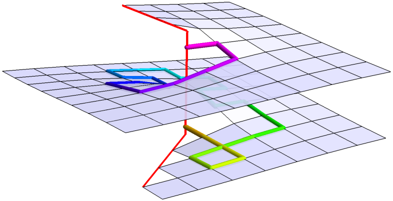

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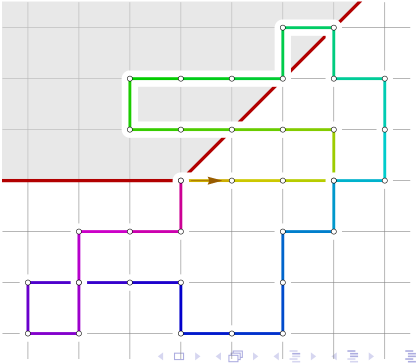

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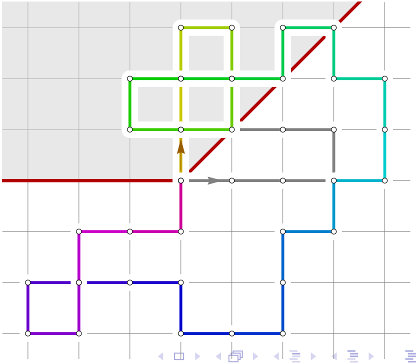

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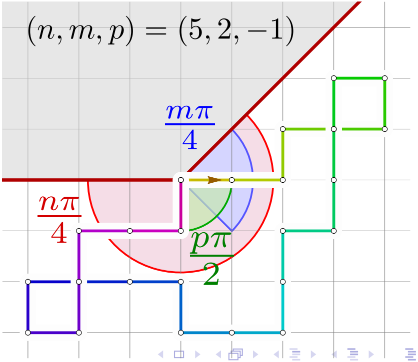

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which is algebraic, i.e. $P(t, F_{n,m,p}(t)) = 0$ for some $P(t,x) \in \mathbb{Z}[t,x]$.

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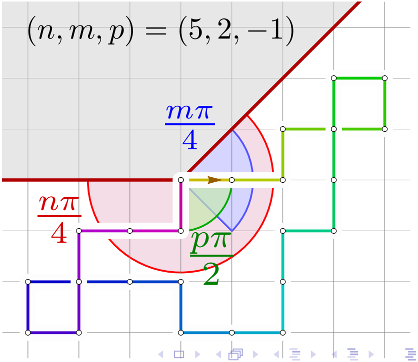

Application: walks in cones (Gessel case)

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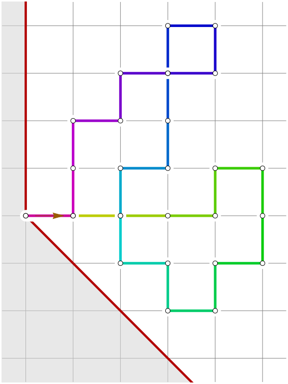

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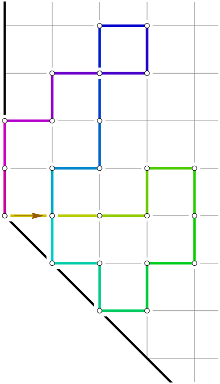

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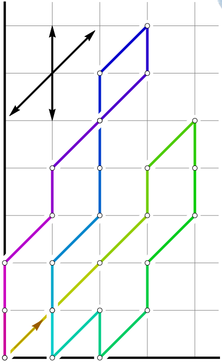

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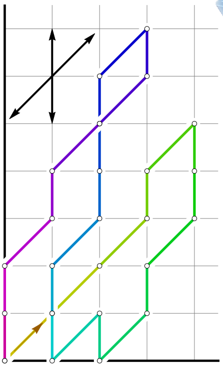

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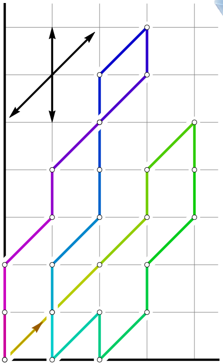

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- Can reproduce the known formula

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  by checking that both solve same algebraic equation...

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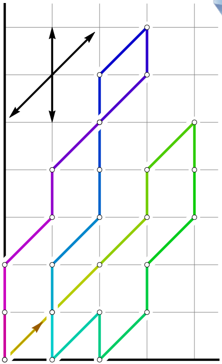

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$$\begin{aligned}\frac{1}{t^2} F_{3,2,0}(t) &= \frac{1}{4t^2} F(t, \tfrac{4}{3}) \\ &= \frac{1}{2t^2}\left[\frac{\sqrt{3}\pi}{2K(4t)} \frac{\theta_1'(\frac{\pi}{3}, \sqrt{q})}{\theta_1(\frac{\pi}{3}, \sqrt{q})} - 1\right] \\ &= 1 + 2t^2 + 11t^4 + 85t^6 + \cdots,\end{aligned}$$

which is an algebraic series.

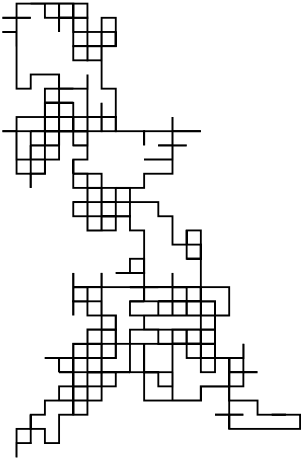
- Can reproduce the known formula

$$\sum_{n=0}^{\infty} t^{2n}\, 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} = \frac{1}{2t^2}\left[{}_2F_1\Big(-\tfrac{1}{2}, -\tfrac{1}{6}; \tfrac{2}{3}; (4t)^2 \Big) - 1\right].$$

by checking that both solve same algebraic equation. . . or by comparing modular properties of both as suggested by Alin Bostan.

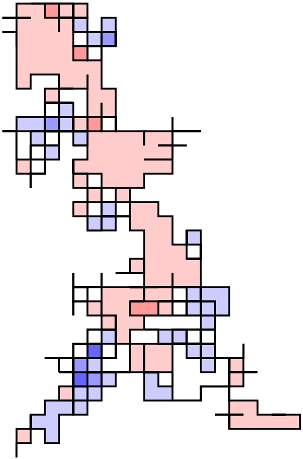
Application: winding field of a random loop

- Consider a uniform loop of length $2n$ on $\mathbb{Z}^2$.



Application: winding field of a random loop

- Consider a uniform loop of length $2n$ on $\mathbb{Z}^2$.
- One may color each square according to the total winding angle of the loop around it.



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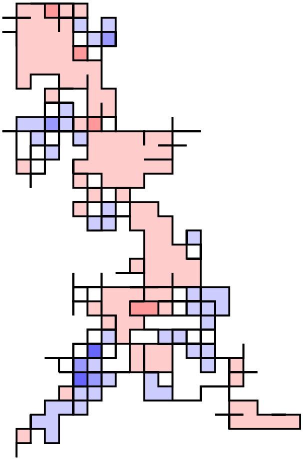

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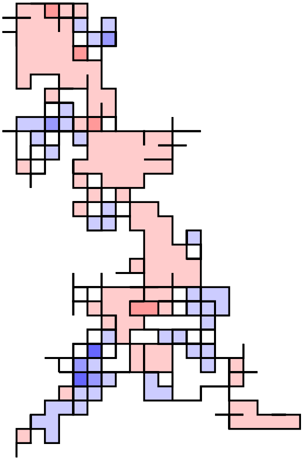

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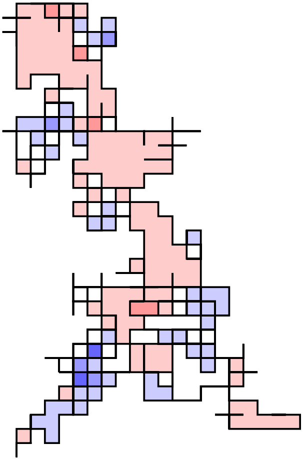
$$\frac{2n}{k\binom{2n}{n}^2}[t^{2n}]\frac{1}{q^{-2k}-q^{2k}} \sim \frac{n}{4\pi k^2}$$

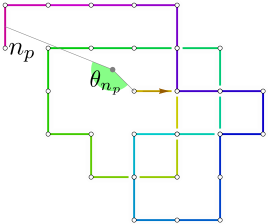
- The $n \to \infty$ asymptotics reproduces result of Brownian motion.

[Garban, Ferreras,'06]



Jacobi elliptic functions are characteristic functions

- Let $n_p \geq 1$ be a geometric random variable with parameter $p \in (0,1)$.
- Let θ_{n_p} be the winding angle around $(-\frac{1}{2}, \frac{1}{2})$ of an SSRW at time $n_p - \frac{1}{2}$.
- Denote by $[\,\cdot\,]_{\pi\mathbb{Z}}$ resp. $[\,\cdot\,]_{\pi(\mathbb{Z}+\frac{1}{2})}$ rounding to nearest integer resp. half-integer multiple of π.



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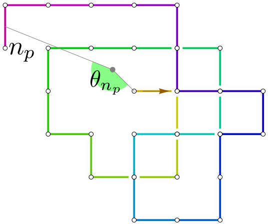

- Then

$$\mathbb{E}\exp\left(ib\,[\theta_{n_p}]_{\pi(\mathbb{Z}+\frac{1}{2})}\right) = \mathrm{cn}(u;p), \qquad u := K(p)b$$
$$\mathbb{E}\exp\left(ib\,[\theta_{n_p-1}]_{\pi\mathbb{Z}}\right) = \mathrm{dn}(u;p),$$

with cn, dn Jacobi elliptic functions.

Concluding remarks

- It is still mysterious why some of the generating functions are so simple.
  - Is there a combinatorial explanation of: Landen transformation $\leftrightarrow$ angle doubling?
- Counting by winding angle is possible for other walks with small steps ($\in \{-1, 0, 1\}^2$).
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Thanks for your attention!