2-automatic sequences, Lyapunov exponents and a dynamical analogue of Lehmer's Mahler measure problem

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Time: Wednesday, May 14th, 3:00 - 3:30
Room: ASB 10900

Title: Stern polynomials and continued fractions

Abstract: We derive new identities for a polynomial analogue of the Stern sequence and define two subsequences of these polynomials. We obtain various properties for these two interrelated subsequences which have 0-1 coefficients and can be seen as extensions or analogues of the Fibonacci numbers. We also define two analytic functions as limits of these sequences. As an application we obtain evaluations of certain finite and infinite continued fractions whose partial quotients are doubly exponential. Finally we prove transcendence results for some of the infinite continued fractions. (Joint work with K.B. Stolarsky).

Karl Dilcher and Kenneth B. Stolarsky,
Stern polynomials and double–limit continued fractions,
Acta Arith. **140** (2009), 119–134.

k-regular sequences

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Recall that an integer sequence $\mathbf{f}$ is linear recurrent provided there is a $d \times d$ square matrix $\mathbf{A}$ and $d \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$f(n) = \mathbf{u}^T \mathbf{A}^n \mathbf{v}.$$

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In the generalisation, one doesn't consider the value of n specifically, but its base-k expansion $(n)_k = i_s \cdots i_0$.

In this setting, the sequence $\mathbf{f}$ is k-regular if and only if there exist $d \times d$ square matrices $\mathbf{A}_0, \ldots, \mathbf{A}_{k-1}$ and $d \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$f(n) = \mathbf{u}^T \mathbf{A}_w \mathbf{v} = \mathbf{u}^T \mathbf{A}_{i_s} \cdots \mathbf{A}_{i_0} \mathbf{v},$$

where $w = i_s \cdots i_0$ is the word corresponding to the base-k expansion.

Growth of k-regular sequences

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We define the *growth exponent of* f, denoted $\operatorname{GrExp}(f)$, by

$$\operatorname{GrExp}(f) := \limsup_{\substack{n\to\infty \\ f(n)\neq 0}} \frac{\log|f(n)|}{\log n}.$$

The *joint spectral radius* of a finite set of matrices $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}\}$, denoted $\rho(\mathcal{A})$, is defined as the real number

$$\rho(\mathcal{A}) = \limsup_{n\to\infty} \max_{0\leqslant i_0, i_1, \ldots, i_{n-1}\leqslant k-1} \left\| \mathbf{A}_{i_0}\mathbf{A}_{i_1}\cdots\mathbf{A}_{i_{n-1}} \right\|^{1/n},$$

where $\|\cdot\|$ is any (submultiplicative) matrix norm.

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Theorem. *Let $k \geqslant 1$ and $d \geqslant 1$ be integers and $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{K}$ be a (not eventually zero) k-regular sequence. If $\mathcal{A}_f$ is any collection of k integer matrices associated to a basis of the $\mathbb{K}$-vector space $\langle \mathrm{Ker}_k(f)\rangle_{\mathbb{K}}$, then*

$$\log_k \rho(\mathcal{A}_f) = \mathrm{GrExp}(f),$$

where $\log_k$ denotes the base-k logarithm.

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For example, the *Stern sequence*, $\{s(n)\}_{n\geqslant 0}$ is given by $s(0)=0$, $s(1)=1$, and when $n\geqslant 1$, by the pair of recurrences

$$s(2n)=s(n)\quad\text{and}\quad s(2n+1)=s(n)+s(n+1).$$

These recurrences are determined by the matrix representation of the Stern sequence, where

$$\{\mathbf{A}_0,\mathbf{A}_1\}=\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right\}.$$

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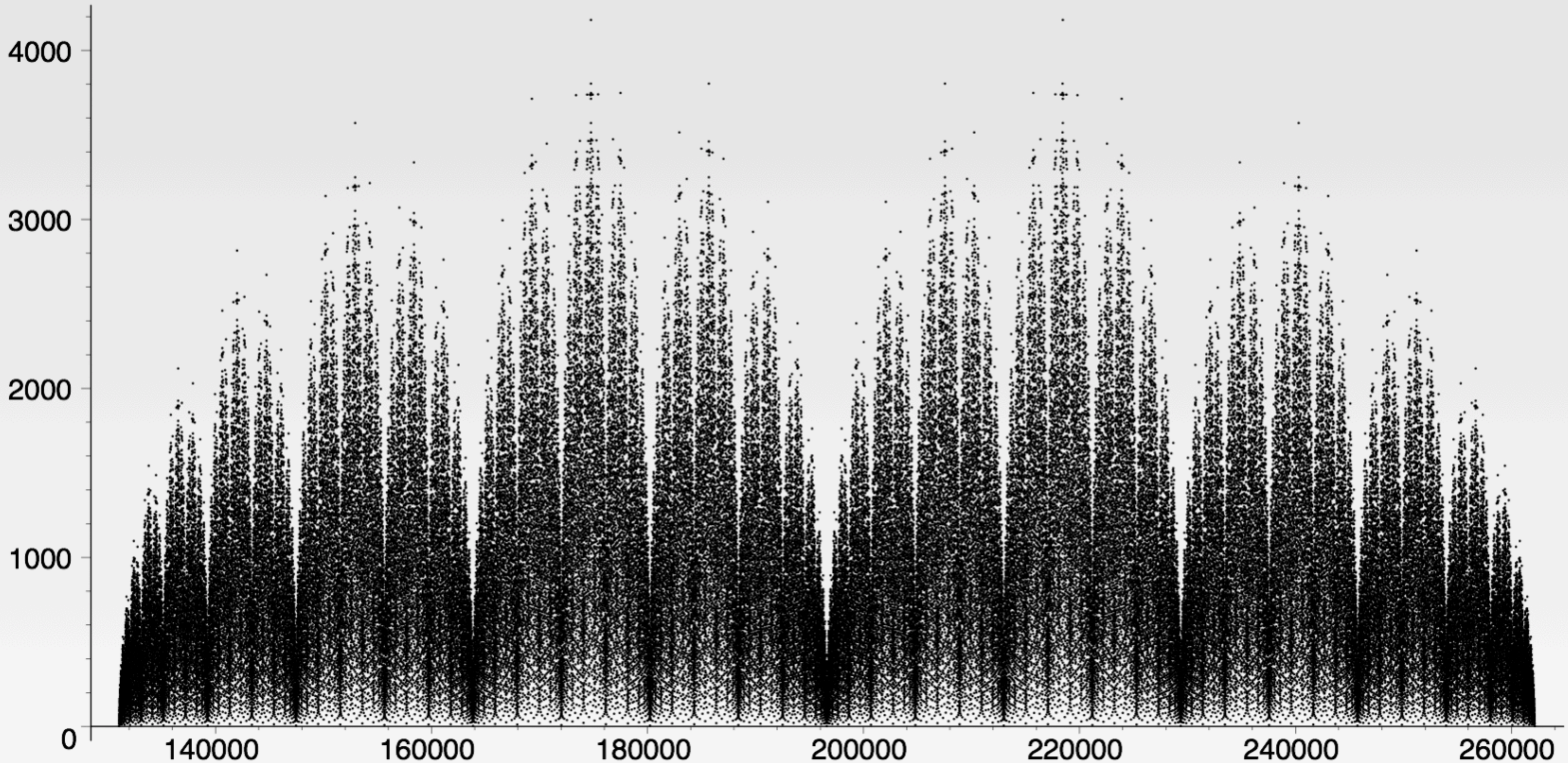
$$\{\mathbf{A}_0, \mathbf{A}_1\} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

THEOREM 1.1. *Let $\{a(n)\}_{n\geq 0}$ denote the Stern sequence. Then*

$$\limsup_{n\to\infty} \frac{a(n)}{n^{\log_2 \varphi}} = \frac{\varphi}{\sqrt{5}} \left(\frac{3}{2}\right)^{\log_2 \varphi} = \frac{\varphi^{\log_2 3}}{\sqrt{5}} = 0.9588541900\cdots.$$



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Joint spectral radius, dilation equations, and asymptotic behavior of radix-rational sequences

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ABSTRACT

Radix-rational sequences are solutions of systems of recurrence equations based on the radix representation of the index. For each radix-rational sequence with complex values we provide an asymptotic expansion, essentially in the scale $N^\alpha \log^\ell N$. The precision of the asymptotic expansion depends on the joint spectral radius of the linear representation of the sequence of first-order differences. The coefficients are Hölderian functions obtained through some dilation equations, which are usual in the domains of wavelets and refinement schemes. The proofs are ultimately based on elementary linear algebra.

Regular to Mahler

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Definition. A function $F(z) \in \mathbb{Z}[z]$ is called a *k-Mahler function* provided there are integers $k \geqslant 2$ and $d \geqslant 1$ such that

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0,$$

for some polynomials $a_0(z), \ldots, a_d(z) \in \mathbb{Z}[z]$.

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The function $S(z) = (1/z)\sum_{n\geqslant 0} s(n)z^n$ satisfies the 2-Mahler equation

$$S(z) = (1 + z + z^2)S(z^2).$$

A transcendence test for Mahler functions

Let $k \geqslant 2$ and $d \geqslant 1$ be integers and $F(z)$ be a k-Mahler function converging inside the unit disc satisfying

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0,$$

for polynomials $a_0(z), \ldots, a_d(z) \in \mathbb{C}[z]$. Set $a_i := a_i(1)$ and form the polynomial

$$p_F(\lambda) := a_0\lambda^d + a_1\lambda^{d-1} + \cdots + a_{d-1}\lambda + a_d.$$

If $a_0 a_d \neq 0$ and $p_F(\lambda)$ has distinct roots, then the function $F(z)$ is transcendental over $\mathbb{C}(z)$ provided

- $p(k^n) \neq 0$ for all $n \in \mathbb{Z}$ or
- the eigenvalue $\lambda_F \neq k^n$ for any $n \in \mathbb{Z}$.

If $\lambda_F = k^n$ for some $n \in \mathbb{Z}$, the test is inconclusive.

Radial asymptotics of Mahler functions

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Theorem. *Let $F(z)$ be a k-Mahler function whose characteristic polynomial $p_F(\lambda)$ has distinct roots. Then there is an eigenvalue λ_F with $p_F(\lambda_F) = 0$, such that as $z \to 1^-$*

$$F(z) = \frac{C(z)}{(1-z)^{\log_k \lambda_F}}(1 + o(1)),$$

where $\log_k$ denotes the principal value of the base-k logarithm and $C(z)$ is a real-analytic nonzero oscillatory term, which on the interval $(0,1)$ is bounded away from 0 and ∞, and satisfies $C(z) = C(z^k)$.

Questions about the asymptotic behaviour of Mahler functions are quite classical, and some special cases of the theorem above are known. See, e.g., Mahler (1940), de Bruijn (1948), Dumas (1993), Dumas and Flajolet (1996), and most recently Brent, Coons, and Zudilin (2015).

Properties via matrix cocycles

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Mahler functions are coordinates of a vector $\mathbf{F}(z) := [F_1(z), \ldots, F_d(z)]^T$ such that there is a matrix of rational functions $\mathbf{A}(z)$ such that

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Properties of this cocycle can give you:

- transcendence results
- algebraic independence results
- irrationality measures

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$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

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THEOREM 1.1. *A Mahler number cannot be a Liouville number.*

Mahler measure and possible gaps

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Definition. Let $p(z) = a_0 \prod_{i=0}^{s}(z - \alpha_i) \in \mathbb{C}[z]$. The (logarithmic) Mahler measure of $p(z)$ is given by

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In **this talk** I want to show you that Lehmer's problem,
restricted to height-one polynomials,
is a property of a matrix cocycle associated to a 2-automatic sequence.

Fourier matrix of a 2-automatic sequence

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$$\varrho : \begin{cases} 0 \mapsto w_0 \\ 1 \mapsto w_1 \, , \end{cases}$$

where w_0 and w_1 are finite words over Σ_2 of equal length $|w_0| = |w_1| = L \geqslant 2$.

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For $0 \leqslant i, j \leqslant 1$, let T_{ij} be the set of all positions m where the letter i appears in w_j, and let $T := (T_{ij})_{0 \leqslant i,j \leqslant 1}$ be the resulting 2×2-matrix.

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Definition. Using T, we build a matrix of pure point measures $\delta_T = (\delta_{T_{ij}})_{0 \leqslant i,j \leqslant 1}$, where $\delta_S := \sum_{x \in S} \delta_x$ with $\delta_\varnothing = 0$. This gives rise to an analytic matrix-valued function via

$$B(k) := \overline{\widehat{\delta_T}(k)},$$

which we call the *Fourier matrix* of ϱ.

Two paradigmatic examples

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Example. Consider the Thue–Morse substitution, as given by

$$\varrho_{\mathrm{TM}} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10. \end{cases}$$

Here, one has $T_{\mathrm{TM}} = \left(\begin{smallmatrix} \{0\} & \{1\} \\ \{1\} & \{0\} \end{smallmatrix}\right)$, which gives

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Example. On the other hand, for the period doubling substitution,

$$\varrho_{\mathrm{pd}} : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 00, \end{cases}$$

the corresponding matrices are $T_{\mathrm{pd}} = \left(\begin{smallmatrix} \{0\} & \{0,1\} \\ \{1\} & \varnothing \end{smallmatrix}\right)$ together with

$$\delta_{T_{\mathrm{pd}}} = \begin{pmatrix} \delta_0 & \delta_0 + \delta_1 \\ \delta_1 & 0 \end{pmatrix} \quad \text{and} \quad B_{\mathrm{pd}}(k) = \begin{pmatrix} 1 & 1 + e^{2\pi i k} \\ e^{2\pi i k} & 0 \end{pmatrix}.$$

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We use the ergodic transformation $k \mapsto Lk \bmod 1$ defined on the 1-torus and the Fourier matrix to build the matrix cocycle

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- If $v \in \mathbb{C}^2$ is any (fixed) row vector, the values

$$\chi^B(v,k) := \lim_{n\to\infty} \frac{1}{n} \log \|vB^{(n)}(k)\|$$

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- As a function of v, they take at most two values in this case.
- A vector v from the Oseledec subspace $\mathcal{V}_{i+1} \setminus \mathcal{V}_i$ satisfies the property that, for almost every $k \in \mathbb{R}$, the norm $\|vB^{(n)}(k)\|$ grows like $e^{n\chi^B_{i+1}}$ as $n \to \infty$.

- For invertible cocycles, these exponents have v-independent forms

$$\chi^B_{\max}(k) := \lim_{n\to\infty} \frac{1}{n} \log\left\| B^{(n)}(k) \right\| \quad \text{and}$$

$$\chi^B_{\min}(k) := -\lim_{n\to\infty} \frac{1}{n} \log\left\| \left(B^{(n)}(k)\right)^{-1} \right\|.$$

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Theorem (Baake, C. and Mañibo). *For any primitive, 2-automatic sequence ϱ, the extremal Lyapunov exponents are explicitly given by*

$$\chi^B_{\min} = 0 \quad \textit{and} \quad \chi^B_{\max} = \mathfrak{m}(Q-R),$$

with Q and R defined in the next lemma.

Lemma. *Consider the sets*

$$P_a := \left\{ m \mid \mathcal{C}_m = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \right\} \quad \text{and} \quad P_b := \left\{ m \mid \mathcal{C}_m = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] \right\},$$

which collect bijective positions of equal type. Further, let $z = e^{2\pi i k}$ *and set*

$$Q(z) := \overline{\widehat{\delta_{P_a}}(k)} \quad \text{and} \quad R(z) := \overline{\widehat{\delta_{P_b}}(k)}.$$

Then, $\det\big(B(k)\big) = p_L(z) \cdot \big(Q - R\big)(z)$, *where* $p_L(z) = 1 + z + \ldots + z^{L-1}$.

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Proof. Similar to the definitions of Q and R above, define $P_0 := \{m \mid \mathcal{C}_m = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\}$ and $P_1 := \{m \mid \mathcal{C}_m = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$, and let

$$S_0(z) := \overline{\widehat{\delta_{P_0}}(k)} \quad \text{and} \quad S_1(z) := \overline{\widehat{\delta_{P_1}}(k)}.$$

In general, the Fourier matrix of ϱ satisfies

$$B(k) = \begin{pmatrix} \bigl(S_0 + Q\bigr)(z) & \bigl(S_0 + R\bigr)(z) \\ \bigl(S_1 + R\bigr)(z) & \bigl(S_1 + Q\bigr)(z) \end{pmatrix} \quad \text{with } z = e^{2\pi i k}.$$

Since there are only four distinct column types, we see that $S_0+S_1+Q+R = p_L$. One can now verify the lemma by direct computation. $\square$

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Proof. By the ergodic theorem, multiplicativity of the determinant, and additivity of the logarithm, for almost every $k \in \mathbb{R}$

$$\begin{aligned}\chi^B_{\min}(k) + \chi^B_{\max}(k) &= \lim_{n\to\infty} \frac{1}{n} \log\left|\det\left(B^{(n)}(k)\right)\right| \\ &= \int_0^1 \log\left|(Q-R)\left(e^{2\pi i t}\right)\right| dt = \mathfrak{m}(Q-R).\end{aligned}$$

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The row vector $(1,1)$ is a left eigenvector of $B(k)$, for all k, with eigenvalue $p_L\bigl(e^{2\pi i k}\bigr)$. Hence, we get one of the exponents to be $\chi^B_1 = \mathfrak{m}(p_L) = 0$. From the sum above, and from the non-negativity of the Mahler measure, we then get that the exponent corresponding to this invariant subspace is the minimal one, $\chi^B_1 = \chi^B_{\min}$, the other being $\chi^B_{\max} = \mathfrak{m}(Q-R)$. $\square$

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- One can easily bound heights
- bound degrees of the entries,
- and see how they are related.

Example: Littlewood polynomials and bijective morphisms

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Definition. A polynomial $q(z) = \sum_{m=0}^{n-1} c_m z^m$ of degree $n-1$ with coefficients $c_m \in \{-1, 1\}$ is called a *Littlewood polynomial of order* $n-1$.

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Let $\mathcal{C}_m$ be the m^{th} column of ϱ, and starting with $q(z)$, we choose $\mathcal{C}_m$ to be

$$\mathcal{C}_m = \begin{cases} \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], & \text{if } c_m = 1, \\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], & \text{if } c_m = -1, \end{cases}$$

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By construction, we have

$$\chi^B_{\max} = \mathfrak{m}(Q - R) = \mathfrak{m}(q) \quad \text{and} \quad B(k) = \begin{pmatrix} Q(z) & R(z) \\ R(z) & Q(z) \end{pmatrix} \text{ where } z = e^{2\pi i k}.$$

The substitution corresponding to $q = Q - R$ is essentially unique, up to the obvious freedom that emerges from the relation $\mathfrak{m}(-q) = \mathfrak{m}(q)$, that is, from changing all signs. This is the case precisely because a given sequence of signs uniquely specifies the columns of ϱ. For example, let us consider the polynomial

$$q(z) = -1 - z + z^2 - z^3 + z^4,$$

whence we get the substitutions

$$\varrho_q : \begin{cases} 0 \mapsto 11010 \\ 1 \mapsto 00101 \end{cases} \qquad \text{and} \qquad \varrho_{-q} : \begin{cases} 0 \mapsto 00101 \\ 1 \mapsto 11010 \end{cases}$$

with associated Fourier matrices

$$B_q(k) = \begin{pmatrix} e^{4\pi ik} + e^{8\pi ik} & 1 + e^{2\pi ik} + e^{6\pi ik} \\ 1 + e^{2\pi ik} + e^{6\pi ik} & e^{4\pi ik} + e^{8\pi ik} \end{pmatrix} \; \text{and} \; B_{-q}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_q(k).$$

Both induce a cocycle whose maximum Lyapunov exponent is

$$\chi^B_{\max} = \mathfrak{m}(q) \approx 0.656256$$

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If there is only one such column, one can still construct a primitive substitution by recalling that $\mathfrak{m}(-q) = \mathfrak{m}(q)$, so one only needs to exchange the two bijective column types.

As a concrete example, consider $q(z) = 1 + z^2$. The two standard choices

$$\varrho_q : \begin{cases} 0 \mapsto 000 \\ 1 \mapsto 101 \end{cases} \qquad \text{and} \qquad \varrho_{q'} : \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 111 \end{cases}$$

both give non-primitive substitutions; in fact, their substitution matrices are not even irreducible. However,

$$\varrho_{-q} : \begin{cases} 0 \mapsto 101 \\ 1 \mapsto 000 \end{cases} \qquad \text{and} \qquad \varrho_{-q'} : \begin{cases} 0 \mapsto 111 \\ 1 \mapsto 010 \end{cases}$$

are both primitive and aperiodic, and have $\chi^B_{\max} = \mathfrak{m}(q)$. One can see in this example that replacing q by $-q$ really means an exchange of w_0 and w_1 in the definition of ϱ.

Lehmer's polynomial, morphism, and dynamical analogue

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Recall further that ℓ_{L} is the polynomial with the smallest known positive logarithmic Mahler measure, $\mathfrak{m}(\ell_{\mathrm{L}}) \approx \log(1.176281)$. Here,

$$\varrho_{\ell_{\mathrm{L}}} : \begin{cases} 0 \mapsto 00111111000 \\ 1 \mapsto 11100000011 \end{cases}$$

is one of the eight substitutions that correspond to the polynomial ℓ_{L}.

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Lehmer's problem (dynamical analogue). Does there exist a primitive, 2-automatic sequence ϱ with maximal Lyapunov exponent

$$0 < \chi^{B}_{\max} < \mathfrak{m}(\ell_{\mathrm{L}}) \approx \log(1.176280818)\,?$$

End (fin)