Irrationality exponents and Hankel determinants

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Irrationality exponent

Let ξ be an irrational, real number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q.

Irrationality exponent

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- Very difficult to determine $\mu(\xi)$ of a given transcendental real number ξ:

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$\mu(\log(2)) \leq 3.57455391$	Marcovecchio (2009)

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Today:

- $\mu(\xi) = 2$ for some families of transcendental numbers

Thue-Morse sequence

an infinite sequence $\mathbf{t} = (e_0, e_1, e_2, \ldots)$ on $\{1, -1\}$, defined by:

- Generating function

$$\prod_{k=0}^{\infty}(1 - x^{2^k}) = \sum_{n=0}^{\infty} e_n x^n = 1 - x - x^2 + x^3 - x^4 + x^5 + \cdots$$

$$\mathbf{t} = (1, -1, -1, 1, -1, 1, 1, -1, \ldots)$$

- Recurrence relation

$$e_0 = 1$$
$$e_{2n} = e_n$$
$$e_{2n+1} = -e_n$$

Bugeaud's Result

Let

$$P_2(x) = \prod_{k=0}^{\infty} (1 - x^{2^k}).$$

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$$\mu(P_2(1/m)) = 2.$$

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Proof. Using Theorem APWW on Hankel determinant, ...

Hankel determinant

We identify a sequence

$$\mathbf{a} = (a_0, a_1, a_2, \ldots)$$

and its generating function

$$f = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Hankel determinant

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$$\mathbf{a} = (a_0, a_1, a_2, \ldots)$$

and its generating function

$$f = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$H_n^{(k)}(\mathbf{a}) = H_n^{(k)}(f) := \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+2n-2} \end{vmatrix}$$

(constant skew-diagonals)

Hankel determinant

Two notations. Using sequence and generating function

$$H_n^{(0)}\Big((1,1,1,1,1,\ldots)\Big) = H_n^{(0)}\Big(\frac{1}{1-x}\Big)$$

Special case $k=0$:

$$H_n(f) = H_n^{(0)}(f) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-2} \end{vmatrix}$$

Main Definition

An infinite ± 1-sequence $\mathbf{c} = (c_k)_{k\geq 0}$ is called *Apwenian* if its Hankel determinant of order n divided by 2^{n-1} is an odd number, i.e.,

$$\frac{H_n(\mathbf{c})}{2^{n-1}} \equiv 1 \pmod 2,$$

for all positive integer n.

- APWEN: to honor the four authors Allouche, Peyrière, Wen, Wen

- Apwenian sequences are rather precious!

1998

Allouche, Peyrière, Wen, Wen proved:

Theorem [APWW]. The Thue–Morse sequence on $\{1, -1\}$ is Apwenian.

- Thue–Morse sequence:

$$P_2(x) = \prod_{k=0}^{\infty} (1 - x^{2^k}).$$

- $H_n(P_2(x)) \neq 0$ for every positive integer n.

First proof [Allouche, Peyrière, Wen, Wen]

- “Sudoku method”
- Sixteen recurrence relations between determinants
- 12 pages

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Sudoku method

...

$$= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} - 2\mathcal{E}^p_{n+1} & -\mathcal{E}^p_{n+1} & \mathbb{O}_{n+1,1} \\ -\mathcal{E}^p_{n+1} & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ \mathbb{O}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} \mathbf{1}_{n+1,n+1} & -\mathcal{E}^p_{n+1} & -\mathbf{1}_{n+1,1} \\ -\mathcal{E}^p_{n+1} & \mathbb{O}_{n+1,n+1} & \mathbf{1}_{n+1,1} \\ -\mathbf{1}_{1,n+1} & \mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} -\mathcal{E}^p_{n+1} & \mathbf{1}_{n+1,n+1} & -\mathbf{1}_{n+1,1} \\ \mathbb{O}_{n+1,n+1} & -\mathcal{E}^p_{n+1} & \mathbf{1}_{n+1,1} \\ \mathbf{1}_{1,n+1} & -\mathbf{1}_{1,n+1} & 0 \end{vmatrix}$$

...

Sixteen relations

...

13) $|\Delta_{2n}^{2p+1}| \equiv |\Delta_n^p| \cdot |\Delta_n^{p+1}| + |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$

14) $|\overline{\Delta_{2n}^{2p+1}}| \equiv |\Delta_n^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_n^p}| \cdot |\Delta_n^{p+1}|,$

15) $|\Delta_{2n+1}^{2p+1}| \equiv |\Delta_{n+1}^p| \cdot |\Delta_n^{p+1}| + |\Delta_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_n^{p+1}|,$

16) $|\overline{\Delta_{2n+1}^{2p+1}}| \equiv |\Delta_{n+1}^p| \cdot |\overline{\Delta_n^{p+1}}| + |\overline{\Delta_{n+1}^p}| \cdot |\Delta_n^{p+1}|.$

2011

Coons

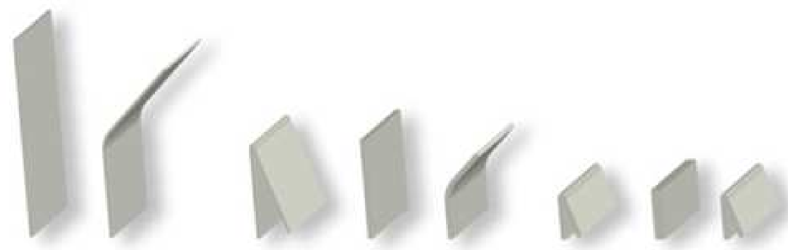
Let

$$S_2 = S_2(x) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 + x^{2^n}}.$$

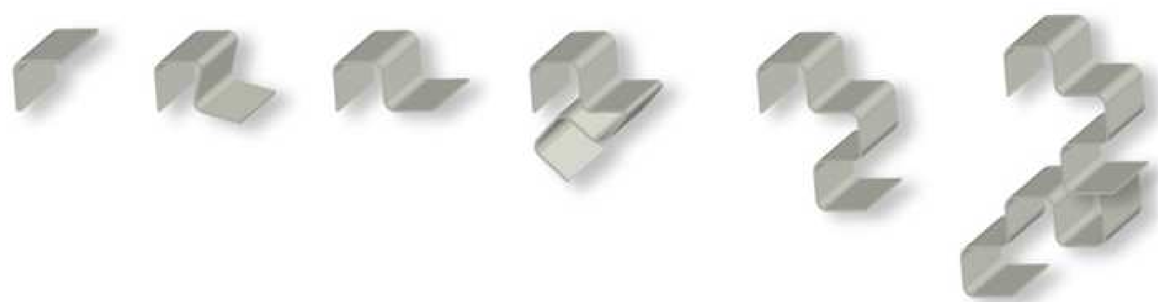
Then $H_n(S_2) \equiv 1 \pmod 2$.

- Same proof than APWW.
- The Gros sequence (1872)

Regular paperfolding sequence



(Source: Wikipedia)



1= Left turn, 0=Right turn

$$\mathbf{r} = (1,1,0,1,1,0,0,1,1,1,0,0,1,0,0,1,1,1,0,1,\ldots)$$

- Generating function

$$G_{0,2}(x) = \sum_{n \geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n - 1}}{1 - x^{2^{n+2}}}.$$

- Recurrence relations:

$$r_{4n} = 1, \quad r_{4n+2} = 0, \quad r_{2n+1} = r_n$$

Coons and Vrbik conjectured (2012) and Guo, Wu and Wen proved

Theorem GWW (2014).

The parities of the Hankel determinants of the regular paper-folding sequence $\mathbf{r}$ are periodic of period 10

$$(H_k(\mathbf{r}))_{k=0,1,\ldots} \equiv (1,1,1,0,0,1,0,0,1,1)^* \pmod 2.$$

Proof.

$$
\begin{aligned}
\left|\begin{array}{cc} A_n & \mathbf{f}_n^0 \\ \mathbf{f}_n^0 & B_n \end{array}\right|
&= \left|\begin{array}{cc|cccc}
A_n & \mathbf{f}_n^0 & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} \\
\mathbf{f}_n^0 & B_n & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} \\
\hline
\alpha(n) & \mathbf{0}_{1,n} & 1 & 0 & 0 & 0 \\
\beta(n) & \mathbf{0}_{1,n} & 0 & 1 & 0 & 0 \\
\mathbf{0}_{1,n} & \beta(n) & 0 & 0 & 1 & 0 \\
\mathbf{0}_{1,n} & \alpha(n) & 0 & 0 & 0 & 1
\end{array}\right| \\
&= \left|\begin{array}{cc|cccc}
\mathbf{0}_n & \mathbf{f}_n^0 & -\alpha(n)^t & -\beta(n)^t & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} \\
\mathbf{f}_n^0 & \mathbf{0}_n & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & -\alpha(n)^t & -\beta(n)^t \\
\hline
\alpha(n) & \mathbf{0}_{1,n} & 1 & 0 & 0 & 0 \\
\beta(n) & \mathbf{0}_{1,n} & 0 & 1 & 0 & 0 \\
\mathbf{0}_{1,n} & \beta(n) & 0 & 0 & 1 & 0 \\
\mathbf{0}_{1,n} & \alpha(n) & 0 & 0 & 0 & 1
\end{array}\right| \\
&= \left|\begin{array}{ccc|ccc}
\mathbf{0}_n & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} & \mathbf{f}_n^0 & -\alpha(n)^t & -\beta(n)^t \\
\mathbf{0}_{1,n} & 1 & 0 & \beta(n) & 0 & 0 \\
\mathbf{0}_{1,n} & 0 & 1 & \alpha(n) & 0 & 0 \\
\hline
\mathbf{f}_n^0 & -\alpha(n)^t & -\beta(n)^t & \mathbf{0}_n & \mathbf{0}_{n,1} & \mathbf{0}_{n,1} \\
\alpha(n) & 0 & 0 & \mathbf{0}_{1,n} & 1 & 0 \\
\beta(n) & 0 & 0 & \mathbf{0}_{1,n} & 0 & 1
\end{array}\right| .
\end{aligned}
$$

10 pages

plus

- $h_{10k+3} \equiv g_{5k+1}(c_{5k+2}+e_{5k+2})+h_{5k+1}(d_{5k+2}+e_{5k+2})+b_{5k+2}(x_{5k+1}+y_{5k+1})$. If $k=2l$, by the hypothesis, then $h_{10k+3} \equiv g_{10l+1}(c_{10l+2}+e_{10l+2})+h_{10l+1}(d_{10l+2}+e_{10l+2})+b_{10l+2}(x_{10l+1}+y_{10l+1}) \equiv 1\times(0+1)+1\times(1+1)+1\times(1+0) \equiv 0$. If $k=2l+1$, by the hypothesis, then $h_{10k+3} \equiv g_{10l+6}(c_{10l+7}+e_{10l+7})+h_{10l+6}(d_{10l+7}+e_{10l+7})+b_{10l+7}(x_{10l+6}+y_{10l+6}) \equiv 0\times(0+0)+0\times(1+0)+0\times(0+1) \equiv 0$.
- $h_{10k+4} \equiv g_{5k+2}(c_{5k+2}+e_{5k+2})+h_{5k+2}(d_{5k+2}+e_{5k+2})+b_{5k+2}(x_{5k+2}+y_{5k+2})$. If $k=2l$, by the hypothesis, then $h_{10k+4} \equiv g_{10l+2}(c_{10l+2}+e_{10l+2})+h_{10l+2}(d_{10l+2}+e_{10l+2})+b_{10l+2}(x_{10l+2}+y_{10l+2}) \equiv 0\times(0+1)+1\times(1+1)+1\times(0+1) \equiv 1$. If $k=2l+1$, by the hypothesis, then $h_{10k+4} \equiv g_{10l+7}(c_{10l+7}+e_{10l+7})+h_{10l+7}(d_{10l+7}+e_{10l+7})+b_{10l+7}(x_{10l+7}+y_{10l+7}) \equiv 0\times(0+0)+0\times(1+0)+1\times(0+1) \equiv 1$.
- $h_{10k+5} \equiv g_{5k+2}(c_{5k+3}+e_{5k+3})+h_{5k+2}(d_{5k+3}+e_{5k+3})+b_{5k+3}(x_{5k+2}+y_{5k+2})$. If $k=2l$, by the hypothesis, then $h_{10k+5} \equiv g_{10l+2}(c_{10l+3}+e_{10l+3})+h_{10l+2}(d_{10l+3}+e_{10l+3})+b_{10l+3}(x_{10l+2}+y_{10l+2}) \equiv 0\times(0+0)+1\times(1+0)+0\times(0+1) \equiv 1$. If $k=2l+1$, by the hypothesis, then $h_{10k+5} \equiv g_{10l+7}(c_{10l+8}+e_{10l+8})+h_{10l+7}(d_{10l+8}+e_{10l+8})+b_{10l+8}(x_{10l+7}+y_{10l+7}) \equiv 0\times(0+1)+1\times(1+1)+1\times(0+1) \equiv 1$.

6 pages

Source: GWW, Lin. Algebra Appl., (2014)

New proofs for the Hankel determinants ?

An example:

$$F_3(x) = \prod_{k\geq 0}(1 - x^{3^k} - x^{2\cdot 3^k}).$$

n	1	2	3	4	5	6	7	8	9
$H_n(\mathbf{f})$	1	-2	-4	8	16	-32	-64	128	4864
$H_n(\mathbf{f})/2^{n-1}$	1	-1	-1	1	1	-1	-1	1	19
$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 2$	1	1	1	1	1	1	1	1	1
$H_n(\mathbf{f}) \pmod 3$	1	1	2	2	1	1	2	2	1
$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 6$	1	5	5	1	1	5	5	1	1

New proofs for the Hankel determinants ?

We developed two computer assisted automatic proofs:

n	1	2	3	4	5	6	7	8	9
$H_n(\mathbf{f})$	1	-2	-4	8	16	-32	-64	128	4864

Automatic Proof 1:

$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 2$	1	1	1	1	1	1	1	1	1

Automatic Proof 2:

$H_n(\mathbf{f}) \pmod 3$	1	1	2	2	1	1	2	2	1

$\frac{H_n(\mathbf{f})}{2^{n-1}} \pmod 6$	1	5	5	1	1	5	5	1	1

Automatic proof 1
[Bugeaud, H.(2014), Fu, H.(2016)]

Basic idea: Count the number of permutations of length n modulo 2.

Theorem: The following sequence is Apwenian

$$F_5(x; 1-z-z^2-z^3+z^4) = \prod_{k\geq 0}(1-x^{5^k}-x^{2\cdot 5^k}-x^{3\cdot 5^k}+x^{4\cdot 5^k}).$$

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 4 & 3 & 2 & 7 & 15 & 0 & 13 & 1 & 11 & 10 & 14 & 8 & 12 & 6 & 5 & 9 \end{pmatrix}$$

$$w = \left(\begin{array}{cccc|ccc|ccc|ccc|ccc} 0 & 5 & 10 & 15 & 1 & 6 & 11 & 2 & 7 & 12 & 3 & 8 & 13 & 4 & 9 & 14 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a \end{array}\right)$$

Classifying permutations by types

Consider $m = \ell = 5n+1$ and the type $adbca$:

$$J^{adbca}_{5n+1,5n+1}$$

$$= \left(\begin{array}{cccc|ccc|ccc|ccc|ccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a \end{array}\right)$$

$$= \left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & a & e & e & d & d & d & c & b & c & c & b & b & a & a & a & \underline{19} \end{array}\right)$$

$$= \left(\begin{array}{cccc|ccc|ccc|ccc|cccc} 0 & \tilde{5} & 10 & 15 & 1 & 6 & 11 & 2 & \tilde{7} & 12 & \tilde{3} & 8 & 13 & 4 & 9 & 14 & 19 \\ e & \underline{19} & e & e & d & d & d & c & \underline{c} & c & \underline{b} & b & b & a & a & a & \underline{a} \end{array}\right)$$

$$= \begin{pmatrix} 0 & \tilde{5} & 10 & 15 \\ e & \underline{19} & e & e \end{pmatrix} \begin{pmatrix} 1 & 6 & 11 \\ d & d & d \end{pmatrix} \begin{pmatrix} 2 & \tilde{7} & 12 \\ c & \underline{c} & c \end{pmatrix} \begin{pmatrix} \tilde{3} & 8 & 13 \\ \underline{b} & b & b \end{pmatrix} \begin{pmatrix} 4 & 9 & 14 & 19 \\ a & a & a & \underline{a} \end{pmatrix}$$

$$= Z_{n+1} \times Y_n \times X_n \times X_n \times Z_{n+1}.$$

product of atoms

Evaluating the atoms

η	000	001	010	011	100	101	110	111
$\Psi_G(\eta)$	$\bar{X}_n$	$\bar{Y}_n$	0	$\bar{Z}_{n+1}$	$\bar{Z}_{n+1}$	0	$\bar{X}_{n+1}$	$\bar{Y}_{n+1}$
η	0000	0010	0100	0110	1000	1010	1100	1110
$\Psi_Z(\eta)$	0	$\bar{Z}_n$	0	0	$\bar{X}_n$	$\bar{Y}_n$	0	$\bar{Z}_{n+1}$
η	0001	0011	0101	0111	1001	1011	1101	1111
$\Psi_Z(\eta)$	0	$\bar{Z}_n$	0	0	$\bar{X}_n$	0	0	$\bar{Z}_{n+1}$
η	0000	0010	0100	0110	1000	1010	1100	1110
$\Psi_X(\eta)$	0	$\bar{X}_n$	0	0	0	$\bar{Z}_{n+1}$	0	$\bar{X}_{n+1}$
η	0001	0011	0101	0111	1001	1011	1101	1111
$\Psi_X(\eta)$	0	$\bar{X}_n$	0	0	0	0	0	$\bar{X}_{n+1}$

Output

```
v= [1, -1, -1, -1, 1]
1 ecbaed:  Xn:0010 Xn:000 Xn:000 Xn:000 Xn:000
2 ecdabe:  Xn:0011 Xn:000 Xn:000 Xn:000 Xn:000
...
---------
167 acdba:  Zm:100 Xn:000 Xn:000 Yn:001 Zm:011
168 adbca:  Zm:100 Yn:001 Xn:000 Xn:000 Zm:011
169 adcba:  Zm:100 Yn:001 Yn:001 Yn:001 Zm:011
170 dcbaa:  Zm:100 Xn:000 Xn:000 Xn:000 Zm:011
Y(5n+1) = Xn Zm + Yn Zm
---------

...
224 adcbab:  Zm:100 Ym:111 Ym:111 Zm:Z1111 Zm:011
225 dcbbaa:  Zm:100 Xm:110 Xm:110 Zm:Z1110 Zm:011
Z(5n+4) = Xm Ym Zm + Xm Zm + Ym Zm
```

Proof

By the output of the program, for each $n \geq 1$ we have

$$\begin{aligned}
X_{5n+0} &\equiv X_n, \\
Y_{5n+0} &\equiv Y_n, \\
X_{5n+1} &\equiv Z_{n+1} Y_n, \\
Y_{5n+1} &\equiv Z_{n+1}(X_n + Y_n), \\
X_{5n+2} &\equiv Z_{n+1}(X_n + Y_n), \\
Y_{5n+2} &\equiv Z_{n+1} X_n, \\
X_{5n+3} &\equiv Z_{n+1}(X_{n+1} + Y_{n+1}), \\
Y_{5n+3} &\equiv Z_{n+1} X_{n+1}, \\
X_{5n+4} &\equiv Z_{n+1} Y_{n+1}, \\
Y_{5n+4} &\equiv Z_{n+1}(X_{n+1} + Y_{n+1}),
\end{aligned}$$

$$
\begin{aligned}
Z_{5n+0} &\equiv Z_n(X_n + X_nY_n + Y_n),\\
Z_{5n+1} &\equiv Z_{n+1}(X_n + X_nY_n + Y_n),\\
Z_{5n+2} &\equiv Z_{n+1}(X_n + X_nY_n + Y_n),\\
Z_{5n+3} &\equiv Z_{n+1},\\
Z_{5n+4} &\equiv Z_{n+1}(X_{n+1} + X_{n+1}Y_{n+1} + Y_{n+1}).
\end{aligned}
$$

From these relations, we can show that

$$\frac{H_n(F_5)}{2^{n-1}} \equiv Z_n \equiv 1 \pmod 2.$$

Results from automatic proof 1

Let d be a positive integer and $\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_{d-1})$ be a $\{1, -1\}$-sequence of length $d-1$. We define a power series associated with ϵ by

$$\Phi(1 + \sum_{j=1}^{d-1} \epsilon_j z^j) := \prod_{k \geq 0} \Big(1 + \sum_{j=1}^{d-1} \epsilon_j x^{j \cdot d^k}\Big).$$

For example,

$$\Phi(1 - z - z^2 - z^3 + z^4) = \prod_{k \geq 0} (1 - x^{5^k} - x^{2 \cdot 5^k} - x^{3 \cdot 5^k} + x^{4 \cdot 5^k}).$$

Results from automatic proof 1

The following power series are all Apwenian:

$$\begin{aligned}
F_2(x) &= \Phi(1-x), \qquad [APWW, 1998]\\
F_3(x) &= \Phi(1-x-x^2),\\
F_5(x) &= \Phi(1-x-x^2-x^3+x^4),\\
F_{11}(x) &= \Phi(1-x-x^2+x^3-x^4+x^5+x^6+x^7+x^8-x^9\\
&\qquad -x^{10}),\\
F_{13}(x) &= \Phi(1-x-x^2+x^3-x^4-x^5-x^6-x^7-x^8+x^9\\
&\qquad -x^{10}-x^{11}+x^{12}),\\
F_{17a}(x) &= \Phi(1-x-x^2+x^3-x^4+x^5+x^6+x^7+x^8+x^9\\
&\qquad +x^{10}+x^{11}-x^{12}+x^{13}-x^{14}-x^{15}+x^{16}),\\
F_{17b}(x) &= \Phi(1-x-x^2-x^3+x^4+x^5-x^6+x^7+x^8+x^9\\
&\qquad -x^{10}+x^{11}+x^{12}-x^{13}-x^{14}-x^{15}+x^{16}).
\end{aligned}$$

Conjecture/Problems

Conjecture 1. The following power series F_{19} is Apwenian

$$F_{19}(x) = \Phi(1 - x - x^2 - x^3 + x^4 - x^5 + x^6 - x^7 - x^8 + x^9 + x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + x^{16} - x^{17} - x^{18}).$$

• Apwenian sequences are rather precious! This is the only Apwenian sequence of order 19 among the total of 131072 sequences!

• For the study of $F_{11}(x)$, there are 2274558 types to be considered!

• For proving that $F_{17a}(x)$ is Apwenian, our C program has taken about one week by using 24 CPU cores. No hope for $F_{19}(x)$.

Conjecture/Problems

Problem 2. Find a human proof for Apwenian sequence without computer assistance.

Problem 3. Characterize all the finite ± 1-sequences $\mathbf{v}$ such that $\Phi(\tilde{\mathbf{v}}(x))$ is Apwenian.

Proof by using Jacobi continued fraction (H. 2014)

- Using Jacobi continued fraction
- 1 page

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Jacobi Continued Fraction

$$\mathbf{u} = (u_0, u_1, u_2, \ldots)$$

$$\mathbf{v} = (v_0, v_1, v_2, \ldots)$$

$$\cfrac{1}{1 - v_0 x - \cfrac{u_0 x^2}{1 - v_1 x - \cfrac{u_1 x^2}{1 - v_2 x - \cfrac{u_2 x^2}{\ddots}}}},$$

Notation:

$$\mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{J}[\mathbf{u}/\mathbf{v}] = \mathbf{J}\begin{bmatrix} u_0, u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}.$$

How to find and prove the J-Fraction

Let

$$f = \frac{(1-x)(1+2x) - \sqrt{(1-x)(1-2x)(1+3x)(1+2x-4x^2)}}{4x^2(1-x)}.$$

Find: by computer

Then the J-fraction of f is

$$f = \mathbf{J}\left[\begin{array}{c} (\frac{1}{4}, 2, 2)^* \\ (\frac{1}{2}, \frac{1}{2}, -2)^* \end{array}\right].$$

Proof. Since $\mathbf{u}$ and $\mathbf{v}$ are periodic of same type,

$$f = \cfrac{1}{1 - \frac{1}{2}x - \cfrac{\frac{1}{4}x^2}{1 - \frac{1}{2}x - \cfrac{2x^2}{1 + 2x - 2x^2 f}}}.$$

QED.

Fundamental relation

$$H_n\left(\mathbf{J}\begin{bmatrix} u_0, u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}\right) = u_0^{n-1} u_1^{n-2} \cdots u_{n-3}^2 u_{n-2}.$$

J-Fraction of P2

Thue–Morse sequence

$$P_2(x) = \prod_{k=0}^{\infty} (1 - x^{2^k}) = \mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

$$\mathbf{u} = -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3, 1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15, 1/15, -1/15, -15, 13, -3/13, 3/13, 13/3, -19/3, 3/19, -3/19, \ldots$$

$$\mathbf{v} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

J-Fraction of S2

$$S_2 = S_2(x) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 + x^{2^n}} = \mathbf{J} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

$$\begin{aligned} \mathbf{u} = {} & -3, -1/9, -63, -1/441, -63, -1, -35, -1/11025, -35, -1, \\ & -63, -1/49, -63, -49/81, -1395/49, -1/216225, -1395/49, \\ & -49/81, -63, -1/49, -63, -1, -35, -1/1225, -35, -1, \\ & -63, -1/81, -63, \ldots \end{aligned}$$

$$\begin{aligned} \mathbf{v} = {} & -2, 7/3, 23/3, -167/21, -169/21, 7, 7, -629/105, -631/105, \\ & 7, 7, -57/7, -55/7, 65/9, 391/63, -17663/3255, -17677/3255, \\ & 391/63, 65/9, -55/7, -57/7, 7, 7, -211/35, -209/35, 7, 7, \\ & -73/9, -71/9, \ldots \end{aligned}$$

Too bad

No closed-form expression for u_n, rational numbers.

We cannot prove anything about the Hankel determinants.

Main idea

let p be a prime number and f a sequence. We want to prove that $H_n(f) \neq 0 \pmod{p}$.

- No closed-form for the coefficients in the J-fraction of f;
- We try to find a "nice" sequence g such that
- (1) $f \equiv g \pmod{p}$
- (2) g has simple J-fraction

By (2) we know $H(g)$.

By (1) we know $H_n(f) \equiv H_n(g) \pmod{p}$

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Question

How to find a *nice* sequence g such that

$$g \equiv f$$

for which each coefficient in the J-fraction of g has a closed-form expression?

J-Fraction of S2

$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1+x^{2^n}} = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

$$\begin{aligned}\mathbf{u} = &-3, -1/9, -63, -1/441, -63, -1, -35, -1/11025, -35, -1,\\ &-63, -1/49, -63, -49/81, -1395/49, -1/216225, -1395/49,\\ &-49/81, -63, -1/49, -63, -1, -35, -1/1225, -35, -1, \ldots\end{aligned}$$

$$\begin{aligned}\mathbf{v} = &-2, 7/3, 23/3, -167/21, -169/21, 7, 7, -629/105, -631/105,\\ &7, 7, -57/7, -55/7, 65/9, 391/63, -17663/3255, -17677/3255,\\ &391/63, 65/9, -55/7, -57/7, 7, 7, -211/35, -209/35, 7, 7\ldots\end{aligned}$$

Only one even number in $\mathbf{u}$ and $\mathbf{v}$. Let

$$g = \mathbf{J}\begin{bmatrix}1, 1, 1, 1, 1, \ldots\\ 0, 1, 1, 1, 1, 1, \ldots\end{bmatrix}$$

J-Fraction of S2

Let

$$g = \mathbf{J}\begin{bmatrix} 1, 1, 1, 1, 1, \ldots \\ 0, 1, 1, 1, 1, 1, \ldots \end{bmatrix}, \quad f = \mathbf{J}\begin{bmatrix} 1, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}$$

$$g = \frac{1}{1 - x^2 f}, \qquad f = \frac{1}{1 - x - x^2 f}$$

$$f = -\frac{1 - x - \sqrt{1 - 2\,x - 3\,x^2}}{2x^2}$$

$$g = \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x}$$

Proof of Coons's Theorem

Let

$$S_2 = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2^n}}{1+x^{2^n}}$$

$$g = \frac{1-\sqrt{\frac{1-3x}{1+x}}}{2x}$$

Since

$$H(g) = (1)^*$$

(?)
$$g \equiv S_2 \pmod 2$$

We have

$$H(S_2) \equiv (1)^* \pmod 2.$$

Crucial Fact

$$(a + x)^p \equiv a^p + x^p \pmod{p}$$

So that

$$f(x^p) \equiv f(x)^p \pmod{p}$$

Proof of Coons's Theorem

$$x^2 S_2(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^n}}{1 + x^{2^n}} = xS_2(x) - \frac{x}{1+x} \pmod 2$$

$$xS_2(x)^2 \equiv S_2(x) - \frac{1}{1+x} \pmod 2$$

We get

$$S_2(x) \equiv \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x} \pmod 2.$$

QED.

J-Fraction of P2

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

$$\mathbf{u} = -2, 1, -1, -1, -1, 1, -1, 1, -3, 1/3, -1/3, -3, 1, -1, 1, 1, -3, 1, -1, -1/3, -5/3, 1/5, -1/5, 15, -17, -1/17, 1/17, -17, 15, \ldots$$

$$\mathbf{v} = 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

Let

$$g = \mathbf{J}\begin{bmatrix} 0, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}$$

J-Fraction of P2

$$P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k}) = \mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

Let

$$g = \mathbf{J}\begin{bmatrix} 0, 1, 1, 1, 1, \ldots \\ 1, 1, 1, 1, 1, 1, \ldots \end{bmatrix}$$

We have

$P_2 \equiv g \pmod 2$, $H_n(g) = 0$, so that $H_n(P_2) \equiv 0 \pmod 2$.

But we want to prove $H_n(P_2)/2^{n-1} \equiv 1 \pmod 2$!

J-Fraction of P2

$$P_2(x) = \mathbf{J}\begin{bmatrix} -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3 \cdots \\ -1, (1, -1)^* \end{bmatrix}.$$

Delete -2 in $\mathbf{u}$ and -1 in $\mathbf{v}$. Let

$$g = \mathbf{J}\begin{bmatrix} 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, 1, -3, \cdots \\ (1, -1)^* \end{bmatrix},$$

Then

$$H_n(P_2) = (-2)^{n-1} H_{n-1}(g)$$

It suffices to prove that $H_n(g) \equiv 1 \pmod 2$.

Proof of APWW's Theorem

Define g by

$$P_2 = \frac{1}{1 + x + 2x^2 g}, \qquad g = \frac{1}{2x^2}(\frac{1}{P_2} - 1 - x).$$

We have

$$H_n(P_2) = (-2)^{n-1} H_{n-1}(g)$$

$$(?) \qquad \frac{1}{P_2} \equiv \sqrt{(1-x)(1+3x)} \pmod 4,$$

$$g \equiv \frac{1}{2x^2}\left(1 + x - \sqrt{(1-x)(1+3x)}\right) \pmod 2.$$

$$g = \mathbf{J}\left[\begin{matrix} (1)^* \\ (-1)^* \end{matrix}\right],$$

so that $H_n(g) \equiv 1 \pmod 2$. Hence, $H_n(P_2) \neq 0$.

Crucial Lemma

Lemma:

$$\sqrt{1-4x} \equiv 1 + 2\sum_{k=0}^{\infty} x^{2^k} \pmod{4}.$$

Proof of APWW's Theorem

Let

$$f = \sqrt{\frac{1}{(1-x)(1+3x)}}$$

Then

$$(1-x)f(x) = \sqrt{1 - \frac{4x}{1+3x}} \equiv 1+2\sum_{k=0}^{\infty}\left(\frac{x}{1+3x}\right)^{2^k} \pmod 4$$

$$(1-x)f(x) \equiv 1+2\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{2^k} \pmod 4$$

and

$$(1-x^2)f(x^2) \equiv (1-x)f(x) - \frac{2x}{1+x} \pmod 4.$$

Proof of APWW's Theorem

On the other hand,

$$(1-x)P_2(x^2) = P_2(x),$$
$$P_2(x) = \frac{1}{1+x} \pmod 2,$$
$$(1-x^2)P_2(x^2) = (1+x)(1-x)P_2(x^2) = (1+x)P_2(x).$$

$$(1-x^2)P_2(x^2) \equiv (1-x)P_2(x) + \frac{2x}{1+x} \pmod 4.$$

Hence,

$$f \equiv P_2 \pmod 4.$$

QED

New results

Theorem. *Let*

$$P_3 = P_3(x) = \prod_{k \geq 0} (1 - x^{3^k}).$$

Then $H_n(P_3) \equiv (-1)^{n-1} \pmod 3$

New results

Proof. We successively have

$$P_3(x) = (1-x)P_3(x^3) \equiv (1-x)P_3(x)^3 \pmod 3,$$
$$P_3(x)^2 \equiv \frac{1}{1-x} \pmod 3,$$
$$P_3(x) \equiv \sqrt{\frac{1}{1-x}} \pmod 3.$$

$$\sqrt{\frac{1}{1-x}} = \mathbf{J}\left[\begin{matrix} 1/8, (1/16)^* \\ (1/2)^* \end{matrix}\right] \equiv \mathbf{J}\left[\begin{matrix} -1, (1)^* \\ (-1)^* \end{matrix}\right] \pmod 3$$

QED

New results

Theorem.

$$f = f(x) = \prod_{k\geq 0}(1 - x^{3^k} - x^{2\cdot 3^k}). \tag{3.6}$$

Then $H_n(f) \neq 0$.

New results

Proof. We successively have

$$f = f(x) = \prod_{k \geq 0} (1 - x^{3^k} - x^{2 \cdot 3^k}) = (1 - x - x^2) f(x^3);$$

$$f \equiv \sqrt{\frac{1}{1 - x - x^2}} \pmod 3.$$

$$\sqrt{\frac{1}{1 - x - x^2}} = \mathbf{J}\begin{bmatrix} 5/8, (5/16)^* \\ (1/2)^* \end{bmatrix} \equiv \mathbf{J}\begin{bmatrix} 1, (-1)^* \\ (-1)^* \end{bmatrix} \pmod 3.$$

So that

$$H(f) \equiv (1, (1, 1, 2, 2)^*) \pmod 3.$$

QED

Automatic proof 2 (H. 2015)

Results from automatic proof 2.

Theorem. *For each pair of positive intergers a, b, let*

$$G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}}.$$

Then $H(G_{a,b})$ is periodic modulo 2.

New results (H, 2015)

The following relations are calculated and proved by a computer program automatically.

$$H(G_{0,0}) \equiv (1)^* \pmod 2;$$

Michael Coons, 2013

$$H(G_{0,1}) \equiv 1, 1, (0)^* \pmod 2;$$
$$H(G_{1,0}) \equiv (1)^* \pmod 2;$$

$$H(G_{0,2}) \equiv (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^* \pmod 2;$$

Guo, Wu, Wen, 2013

$$H(G_{1,1}) \equiv (1, 1, 0, 0, 1, 1)^* \pmod 2;$$
$$H(G_{2,0}) \equiv (1, 1, 0, 0)^* \pmod 2;$$

New results (H, 2015)

$$H(G_{0,3}) \equiv \big(1^5 0^2 1^1 0^6 1^3 0^2 1^2 0^2 1^2 0^4 1^1 0^4 1^1 0^2 1^1 0^2 1^1$$
$$0^4 1^1 0^4 1^2 0^2 1^2 0^2 1^3 0^6 1^1 0^2 1^4\big)^* \pmod 2;$$
$$\text{[period is 74]}$$

$$H(G_{1,2}) \equiv 1, 1, 1, (0)^* \pmod 2;$$
$$H(G_{2,1}) \equiv (1, 1, 1, 1, 1, 1, 0, 0)^* \pmod 2;$$
$$H(G_{3,0}) \equiv (1, 1, 0, 0, 0, 0, 0, 0)^* \pmod 2;$$

New results (H, 2015)

$H(G_{0,4}) \equiv (1^9 0^2 1^1 0^2 \cdots 1^1 0^2 1^8)^*$ [period is 1078];

Oh là là

$H(G_{1,3}) \equiv (1,1,1,1,1,0,0,0,0,1,1,0,0,0,0,1,1,1,1,1)^* \pmod 2$;

$H(G_{2,2}) \equiv (1,1,0,0,0,0,0,0,1,1,0,0)^* \pmod 2$;

$H(G_{3,1}) \equiv (1,1,1,0,0,0,0,1,1,1,0,0,0,0,0,0)^* \pmod 2$;

$H(G_{4,0}) \equiv (1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^* \pmod 2$.

Real number and continued fraction

Real numbers	$\longleftrightarrow$	Continued fractions
$\sqrt{2}$		$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$

Real number and continued fraction

Real numbers $\longleftrightarrow$ Continued fractions

$$\sqrt{2} \qquad 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$

Quadratic numbers $\longleftrightarrow$ Periodic continued fractions

(Euler, Lagrange, Galois)

Similar result for J-fraction ?

Real numbers	$\longleftrightarrow$	Continued fractions
$\sqrt{2}$		$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$
Quadratic numbers	$\longleftrightarrow$	Periodic continued fractions
Formal power series	$\longleftarrow$	Jacobi continued fractions

Similar result for J-fraction ?

Real numbers	$\longleftrightarrow$	Continued fractions
Quadratic numbers	$\longleftrightarrow$	Periodic continued fractions
Formal power series	$\longleftarrow$	Jacobi continued fractions

Remark: The $\longrightarrow$ *in the third relation is missing.*

Condition: The Jacobi continued fraction of a power series $F(x)$ exists if and only if all the Hankel determinants of $F(x)$ are nonzero.

Hankel Continued fraction

A Hankel continued fraction (H-fraction) is a continued fraction of the following form

$$F(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+2}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+2}}{1 + u_3(x)x - \ddots}}}$$

where

- $v_j \neq 0$ are contants,
- k_j are nonnegative integers
- $u_j(x)$ are polynomials of degree less than or equal to k_{j-1}.

By convention, 0 is of degree -1.

Hankel Continued fraction

P-Fractions : Arne Magnus (1962)

P-Paths: Emmanuel Roblet (1994), PhD thesis

Fundamental Theorem

(i) Each H-fraction defines a power series, and conversely, for each power series $F(x)$, the H-fraction expansion of $F(x)$ exists and is unique.

$$\text{power series} \longleftrightarrow \ H\text{-fraction}$$

Fundamental Theorem

(i) Each H-fraction defines a power series, and conversely, for each power series $F(x)$, the H-fraction expansion of $F(x)$ exists and is unique.

$$\text{power series} \longleftrightarrow \quad H\text{-fraction}$$

(ii) All non-vanishing Hankel determinants of $F(x)$ are given by

$$H_{s_j}(F(x)) = (-1)^{\epsilon} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}},$$

where $\epsilon = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 0$.

Example

Let

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1+x}}}{2x^4} \in \mathbb{Q}[[x]].$$

Then

$$f(x) = \cfrac{1}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{1 + x - \cfrac{x^4}{\ddots}}}}}}.$$

Hence

$$H(f) = (1, 1, 0, 0, -1, -1, 0, 0)^*.$$

Theorem (H., 2014)

Let p be a prime number and $F(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[x]$ are three polynomials. Then, the Hankel continued fraction expansion of $F(x)$ exists and is ultimately periodic. Also, the Hankel determinant sequence $H(F)$ is ultimately periodic.

Power series analog of Euler-Lagrange Theorem for real numbers.

Algorithm NextABC

Prototype: $(A^*, B^*, C^*; k, A_k, D) = \texttt{NextABC}(A, B, C)$

Input: $A(x), B(x), C(x) \in \mathbb{F}[x]$ three polynomials such that $B(0) = 1$, $C(0) = 0, C(x) \neq 0, A(x) \neq 0$;

Output: $A^*(x), B^*(x), C^*(x) \in \mathbb{F}[x]$, $k \in \mathbb{N}^+$, $A_k \neq 0 \in \mathbb{F}$, $D(x) \in \mathbb{F}[x]$ a polynomial of degree less than or equal to $k+1$ such that $D(0) = 1$.

Lemma

If $F(x)$ is the power series defined by

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

Then, $F(x)$ can be written as

$$F(x) = \frac{-A_k x^k}{D(x) - x^{k+2}G(x)}$$

where $G(x)$ is a power series satisfying

$$A^*(x) + B^*(x)G(x) + C^*(x)G(x)^2 = 0.$$

./..

Lemma (continued)

Furthermore, $A^*(x), B^*(x), C^*(x)$ are three polynomials in $\mathbb{F}[x]$ such that $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$ and

$$\deg(A^*) \leq d; \;\deg(B^*) \leq d + 1; \;\deg(C^*) \leq d + 2,$$

where

$$d = d(A, B, C) = \max(\deg(A), \deg(B) - 1, \deg(C) - 2).$$

Notation

$$\frac{v_0}{u_1+}\,\frac{v_1}{u_2+}\,\frac{v_2}{u_3+}\,\frac{v_3}{u_4+}\cdots = \cfrac{v_0}{u_1+\cfrac{v_1}{u_2+\cfrac{v_2}{u_3+\cfrac{v_3}{u_4+\ddots}}}}.$$

Example 1

Let $p = 5$ and

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_5[[x]]$$

or

$$-1 + (1 - x^4)F + (-x + x^5)F^2 = 0.$$

$$A := -1; \quad B := 1 - x^4; \quad C := -x + x^5;$$

$$B(0) = 1, \quad C(0) = 0, \quad C(x) \neq 0$$

By Algorithm HFrac, F has the following H-fraction expansion

$$\frac{1}{1+4x} + \Big(\frac{4x^2}{1+3x} + \frac{3x^2}{1+x} + \frac{4x^3}{1+3x+2x^2} + \frac{4x^3}{1+x} + \frac{3x^2}{1+3x} + \frac{4x^2}{1+3x} + \frac{4x^2}{1+3x} + \Big)^*.$$

$$H(g) = (1,1,1,2,0,2,4,1,4,1,4,2,0,2,1,1)^*.$$

Example 2

Same F as Example 1, but with $p = 2$

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_2[[x]]$$

$$F = \frac{1}{1+x} + \left(\frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \right)^*.$$

$H(F) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*.$

Theorem [H, 2014].

For each pair of positive intergers a, b, let

$$G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}}.$$

Then $H(G_{a,b}) \pmod 2$ is periodic.

Proof

Let $f(x) = G_{a,b}(x) \in \mathbb{F}_2[[x]]$. Then

$$x^{2^a} f(x) = \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^{a+1}} f(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^a} f(x^2) = f(x) - \frac{1}{1 - x^{2^b}};$$

$$1 + (1 + x^{2^b}) f(x) + x(1 + x^{2^b}) x^{2^a - 1} f(x)^2 = 0.$$

By the Main Theorem, the Hankel determinant sequence $H(f)$ is ultimately periodic.
QED.

Stern sequence (1858)

$(a_n)_{n=0,1,\dots}$ is defined by $a_0 = 0, a_1 = 1$ and for $n \geq 1$

$$a_{2n} = a_n, \qquad a_{2n+1} = a_n + a_{n+1}.$$

The generating function for Stern's sequence is denoted by

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Theorem

$$H_n(S)/2^{n-2} \equiv (0,0,1,1)^* \pmod 2.$$

Proof

Define $G(x)$ by

$$S(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 + 2x + 2x^2 G(x)}}.$$

The power series $G(x)$ satisfies the following relation

$$(1 + x + x^2) + (1 + x + x^2)G(x) + x^4 G(x^2) \equiv 0 \pmod{2}.$$

By Algorithm HFrac, we get $H(G) \equiv (1,1,0,0)^* \pmod{2}$. Hence

$$H_n(S)/2^{n-2} \equiv (0,0,1,1)^* \pmod{2}.$$

QED

Irrationality exponents

Theorem (Bugeaud, H., Wen, Yao; 2016)

Let $f(z) \in \mathbb{Z}[[z]]$ be the power series defined by

$$f(z) = \prod_{n \geq 0} \Big(1 + uz^{2^n} + 2z^{2^{n+1}} \frac{C(z^{2^n})}{D(z^{2^n})} \Big),$$

where $u \in \mathbb{Z}$, and $C(z), D(z) \in \mathbb{Z}[z]$ with $D(0) = 1$. Let $b \geq 2$ be an integer such that $D\big(\frac{1}{b^{2^m}}\big) f\big(\frac{1}{b^{2^m}}\big) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 4$ is not a rational function, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

For all integers $\alpha, \beta \geq 0$, define

$$F_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 + z^{2^{n+\beta}}}, \qquad G_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 - z^{2^{n+\beta}}}$$

Theorem (Bugeaud, H., Wen, Yao; 2016)

Let $\alpha, \beta \geq 0$ be integers such that $\beta \neq \alpha + 1$. Let $b \geq 2$ be an integer. Then both $F_{\alpha,\beta}(1/b)$ and $G_{\alpha,\beta}(1/b)$ are transcendental, and their irrationality exponent are equal to 2.

Special cases:

$\alpha = 0$ and $\beta = 0$: Coons (2013)

$\alpha = 0$ and $\beta = 2$: Guo, Wu, Wen (2014)

Recall that Stern's sequence $(a_n)_{n\geq 0}$ and its twisted version $(b_n)_{n\geq 0}$ are defined, respectively, by

$$\begin{cases} a_0 = 0,\ a_1 = 1, \\ a_{2n} = a_n,\ a_{2n+1} = a_n + a_{n+1},\ (n \geq 1), \end{cases}$$

and

$$\begin{cases} b_0 = 0,\ b_1 = 1, \\ b_{2n} = -b_n,\ b_{2n+1} = -(b_n + b_{n+1}),\ (n \geq 1). \end{cases}$$

Put $S(z) = \sum_{n=0}^{\infty} a_{n+1} z^n$ and $T(z) = \sum_{n=0}^{\infty} b_{n+1} z^n$.

Theorem (Bundschuh and Väänänen; 2013)

$$\mu(S(1/b)) \leq 2.929$$

and

$$\mu(T(1/b)) \leq 3.555$$

for all integers $b \geq 2$.

Theorem (Bugeaud, H., Wen, Yao; 2016)

For all integers $b \geq 2$, both $S(1/b)$ and $T(1/b)$ are transcendental and their irrationality exponents are equal to 2.

Theorem (Bugeaud, H., Wen, Yao; 2016)

Let $f(z) \in \mathbb{Z}[[z]]$ be a power series defined by

$$f(z) = \prod_{n=0}^{\infty} \frac{C(z^{3^n})}{D(z^{3^n})},$$

with $D(z), C(z) \in \mathbb{Z}[z]$ such that $C(0) = D(0) = 1$. Let $b \geq 2$ be an integer such that $C(\frac{1}{b^{3^m}})D(\frac{1}{b^{3^m}}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 3$ is not a rational function, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

Theorem (Bugeaud, H., Wen, Yao; 2016)

Let

$$F_5(z) := (1 - z - z^2 - z^3 + z^4)\, F_5(z^5).$$

For all integers $b \geq 2$, all the $F_5(1/b)$ are transcendental and their irrationality exponents are equal to 2.

Conjectures

Conjecture. Let

$$f(x) := \Phi(1 - x - x^2 - x^3 + x^4 - x^5 + x^6 - x^7 - x^8 + x^9 + x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + x^{16} - x^{17} - x^{18})$$

Then,

$$H_n(f)/2^{n-1} \equiv 1 \pmod 2.$$

$$P_1 = 3\prod_{n=1}^{\infty}(1 - x^{3^n}) - \frac{2}{1-x}.$$

Proposition. *$H_k(P_1) \neq 0$ for all k.*

$$C_2 = 3\prod_{n=1}^{\infty}(1 + x^{3^n}) - \frac{2}{1-x}.$$

Conjecture. *$H_k(C_2) \neq 0$ for all k.*

$$P_3 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}.$$

Proposition. *$H_k(P_3) \neq 0$ for all k.*

$$C_4 = 1 + \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}.$$

Conjecture. *$H_k(C_4) \neq 0$ for all k.*

$$F_5 = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+2}}}$$

Proposition. $H(F_5) \pmod{2}$ *is periodic.*

Conjecture. $H(F_5) \pmod{p}$ *is not periodic for prime integer* $p \geq 3$.

$$P_7 = \prod_{n \geq 0} (1 - x^{5^n} - x^{2 \cdot 5^n} - x^{3 \cdot 5^n} + x^{4 \cdot 5^n})$$

Proposition. *$H_k(P_7) \neq 0$ for all k.*

$$C_8 = \prod_{n \geq 0} (1 - x^{6^n} - x^{2 \cdot 6^n} - x^{3 \cdot 6^n} + x^{4 \cdot 6^n} - x^{5 \cdot 6^n})$$

Conjecture. *$H_k(C_8) \neq 0$ for all k.*

$$C_9 = \prod_{n \geq 0} (1 - x^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{n(n+1)/2}.$$

Conjecture. *$H_k(C_9) \neq 0$ for all k.*

$$C_{10} = \sum_{n=0}^{\infty} (-1)^n x^{n(n+1)/2}.$$

Conjecture. *$H_k(C_{10}) \neq 0$ for all k.*

Thank you all for your attention!