Asymptotic Enumeration of Compacted Binary Trees with Height Restrictions

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Michael Wallner

joint work with Antoine Genitrini, Bernhard Gittenberger and Manuel Kauers

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December 11$^{\text{th}}$, 2017

Based on the paper:
Asymptotic Enumeration of Compacted Binary Trees,
submitted to a journal.
ArXiv:1703.10031

Creating a compacted tree

Motivation – Efficiently store redundant information

Example

Consider the labeled tree necessary to store the arithmetic expression

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(* (- (* x x) (* y y)) (+ (* x x) (* y y)))
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which represents $(x^2 - y^2)(x^2 + y^2)$.

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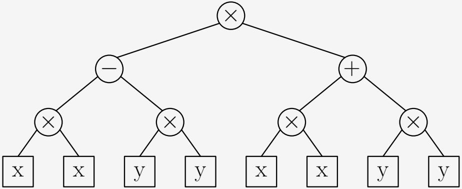

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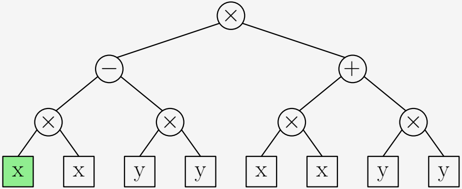

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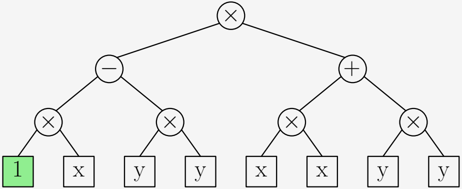

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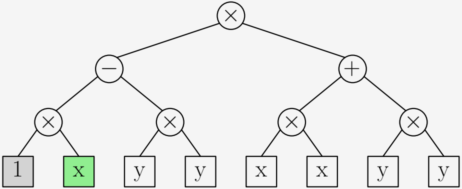

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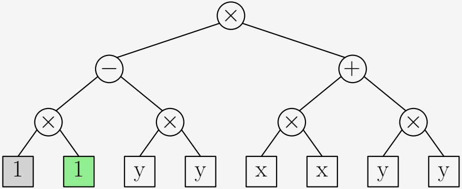

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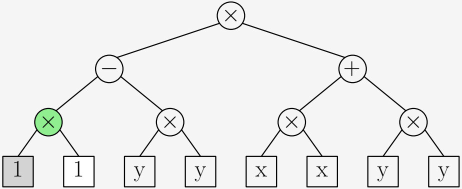

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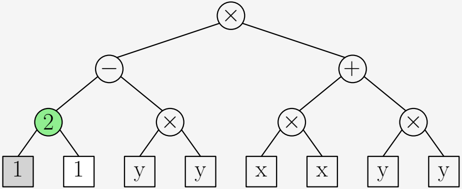

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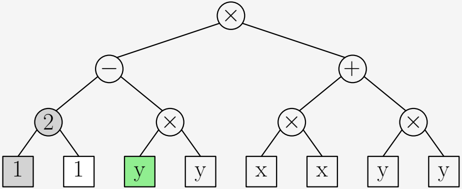

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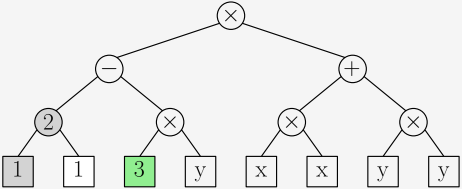

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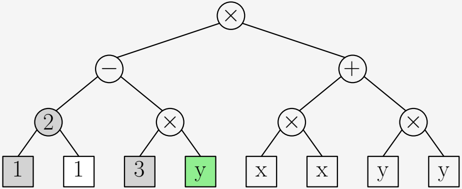

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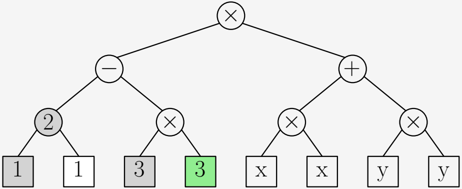

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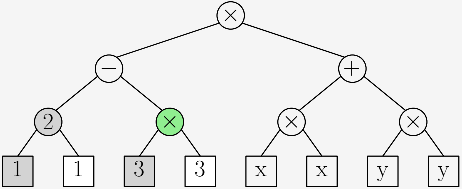

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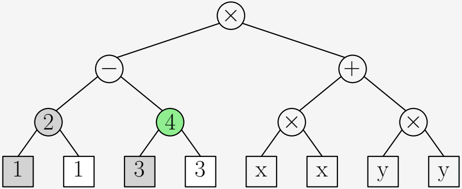

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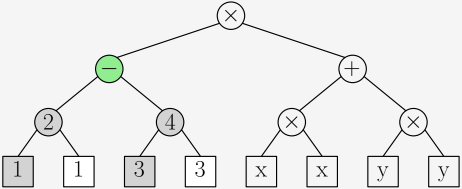

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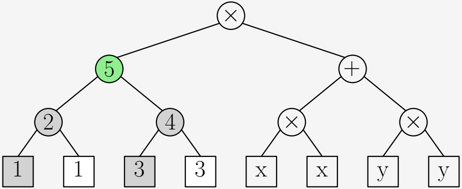

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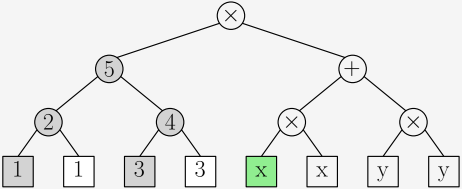

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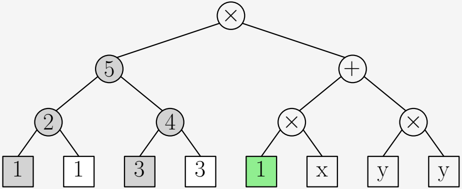

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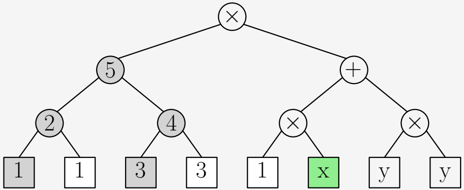

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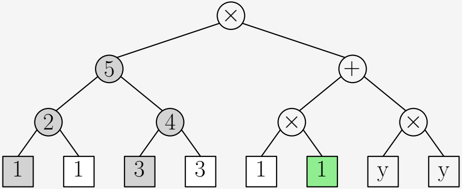

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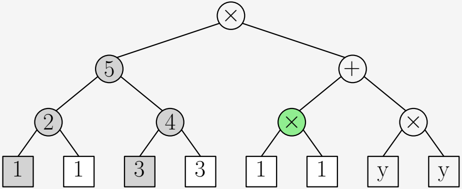

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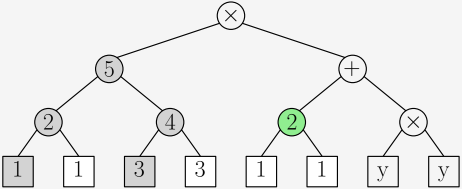

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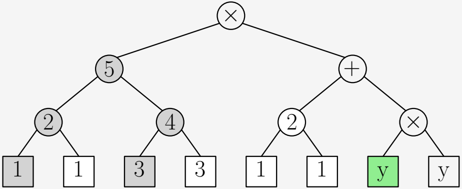

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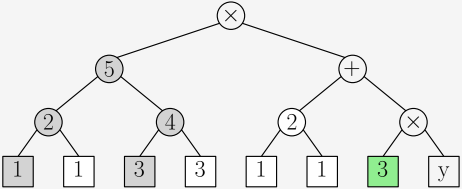

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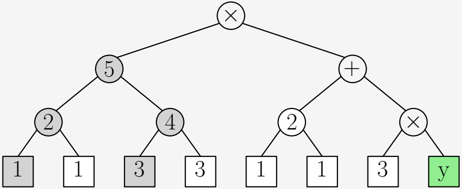

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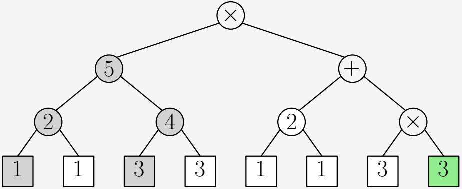

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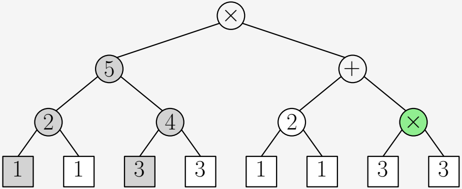

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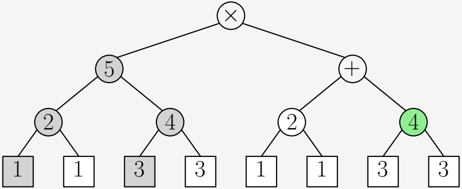

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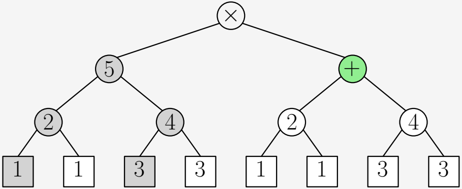

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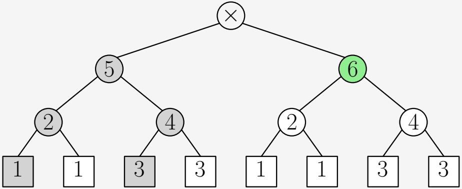

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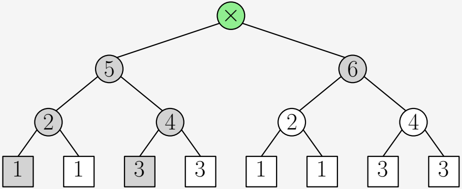

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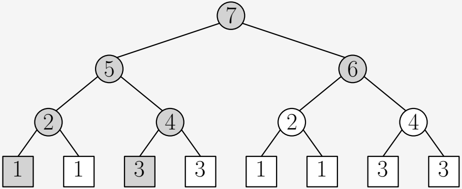

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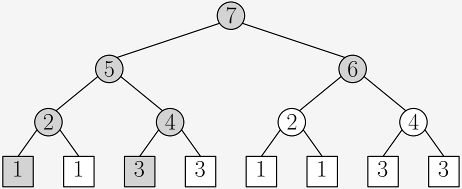

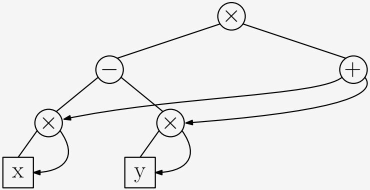

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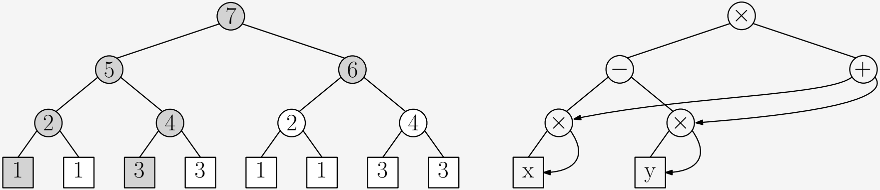

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Definition

Compacted tree is the DAG computed by this procedure.

Compacted trees

- Applications: XML-Compression, Compilers, LISP, Data storage, etc.
- Restrict to unlabeled binary tree
- Important property: Subtrees are unique
- Efficient algorithm to compute compacted tree
  - Bijection
  - Traverse tree post-order
  - If subtree appears twice, delete second one and replace by pointer $\rightarrow$ *directed acyclic graph* (DAG)
- Analyzed by Flajolet, Sipala, Steyaert: A tree of size n has a compacted form of expected size that is asymptotically equal to

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where C is explicit related to the type of trees and the statistical model.

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where C is explicit related to the type of trees and the statistical model.

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Reverse question

How many compacted trees of (compacted) size n exist?

Goals of this talk

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1. Understand compacted trees
2. Find a recurrence relation for compacted trees
3. Use exponential generating functions to count DAGs
4. Solve the (simplified) problem(s)

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Methods

1. Recurrence relations
2. Bijections
3. Generating functions
4. Symbolic method
5. Differential equations
6. Singularity analysis
7. Chebyshev polynomials
8. Guess and prove

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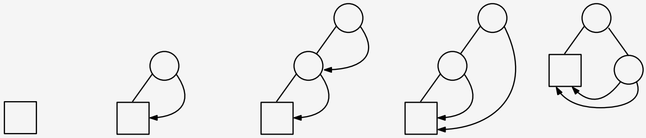

Figure: All compacted binary trees of size $n = 0, 1, 2$.

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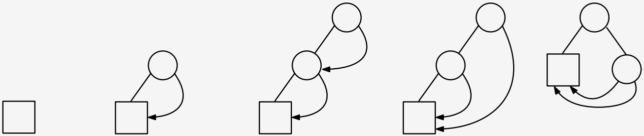

Figure: All compacted binary trees of size $n = 0, 1, 2$.

Example (Compacted binary trees)

size	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
c_n	1	1	3	15	111	1119	14487

$$n! \leq c_n \leq \frac{1}{n+1}\binom{2n}{n} \cdot n!$$

Hence, $c_n = \mathcal{O}(n! 4^n n^{-1/2})$.

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Idea

Every compacted tree of size n can be build from a binary tree of size n by adding pointers.

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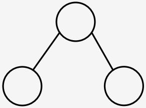

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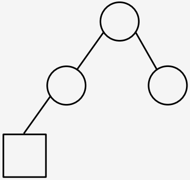

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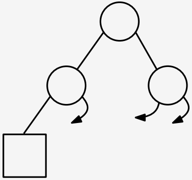

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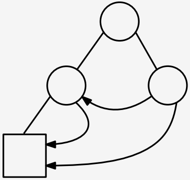

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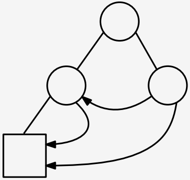

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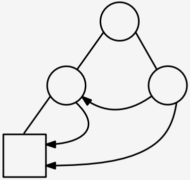

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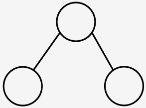

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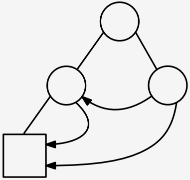

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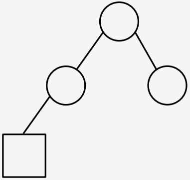

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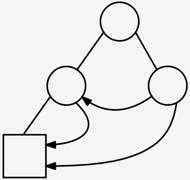

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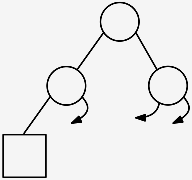

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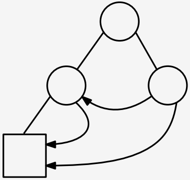

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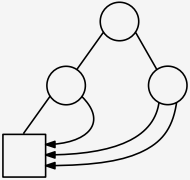

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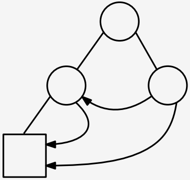

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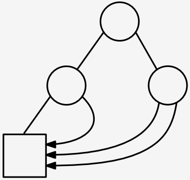

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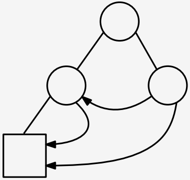

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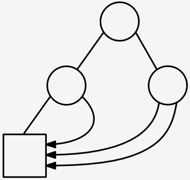
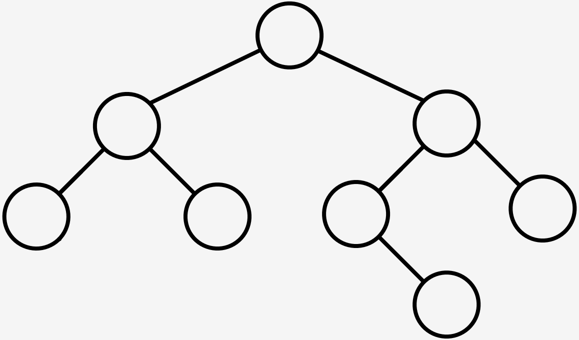


Invalid compacted tree

This spine is associated to 3 valid compacted trees.

A bigger example

We take a binary tree of size 8.



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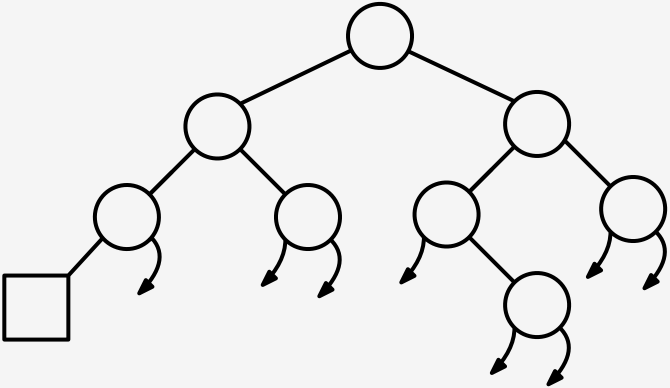

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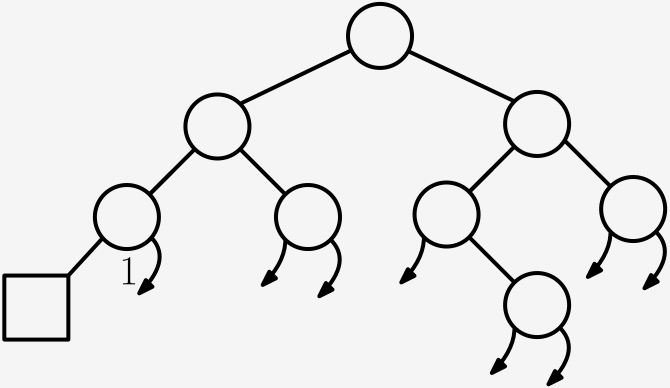

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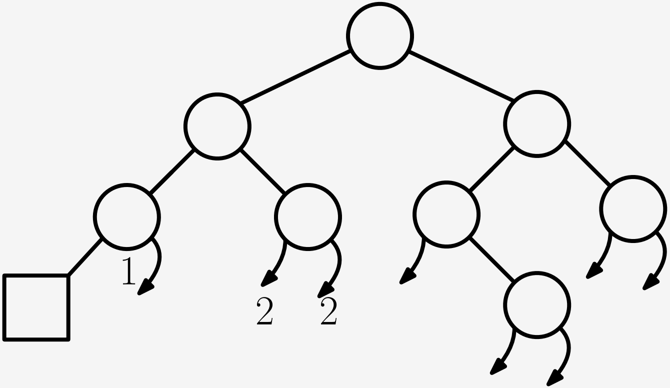

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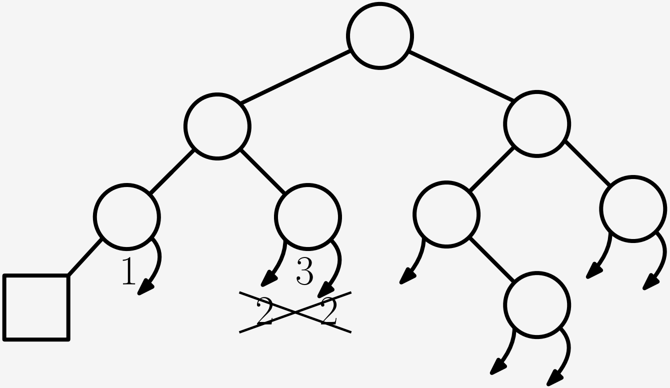

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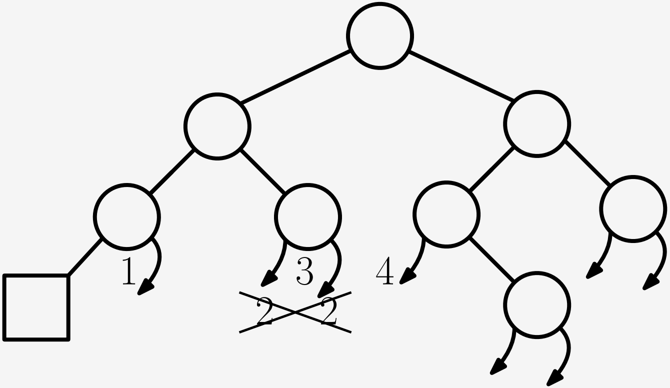

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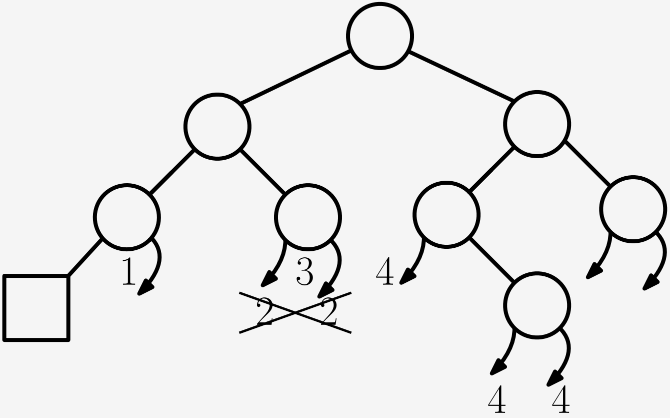

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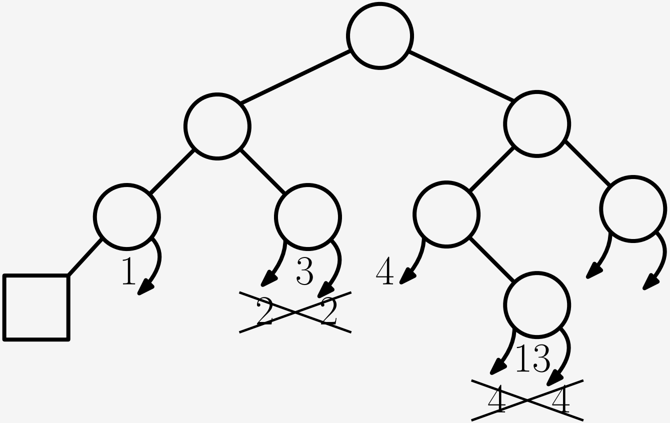

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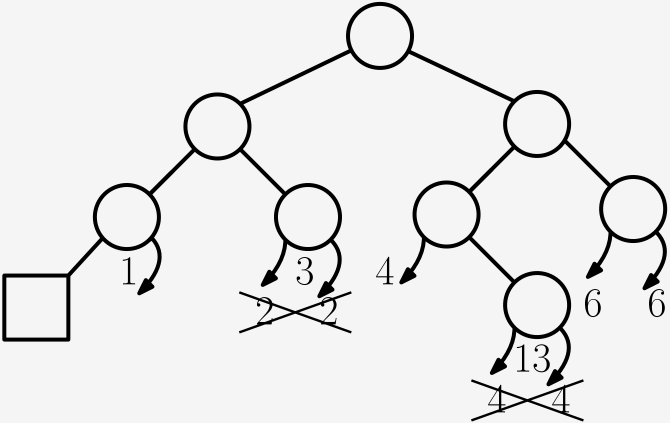

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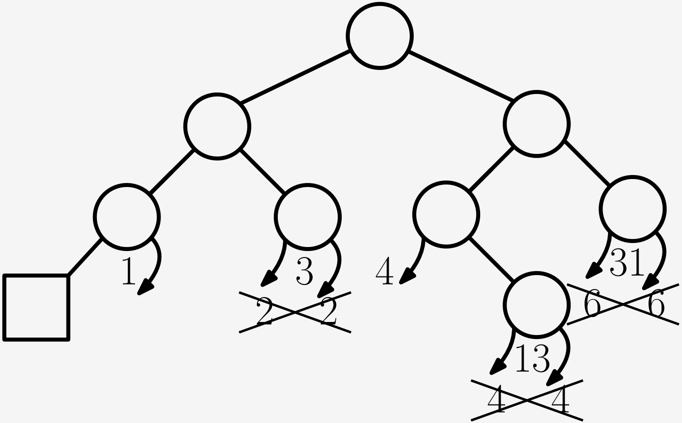

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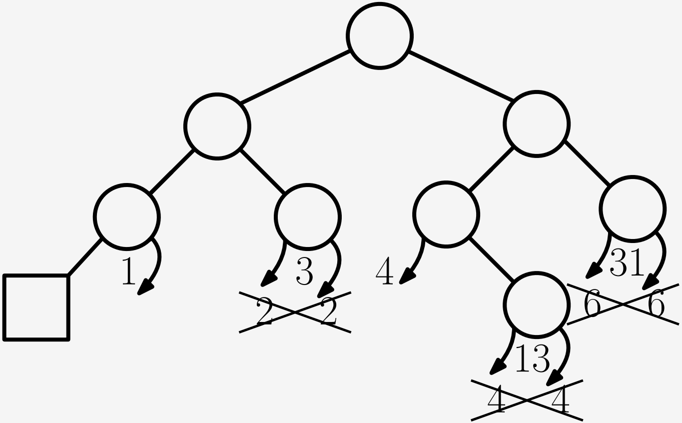

In total, this spine corresponds to $1 \cdot 3 \cdot 4 \cdot 13 \cdot 31 = 4836$ compacted trees.

A recurrence relation

A recurrence for compacted binary trees

Counting formula

Let $n, p \in \mathbb{N}$, then

$$\gamma_{n+1,p} = \sum_{i=0}^{n} \gamma_{i,p}\gamma_{n-i,p+i}, \qquad \text{for } n \geq 1,$$

- Helps us to efficiently compute c_n
- Asymptotic analysis failed (so far)
  One reason: asymptotically every summand matters
- Summands possess 3 (!) dependencies on i

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We are interested in $c_n = \gamma_{n,0}$.

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Counting formula

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Drop the condition of uniqueness of the subtrees, i.e. $c_n \leq r_n$.

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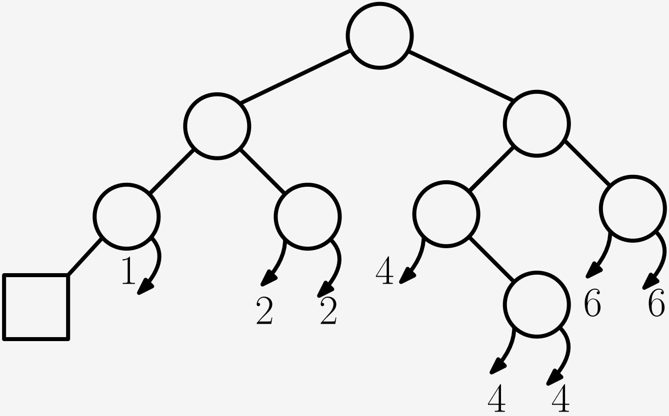

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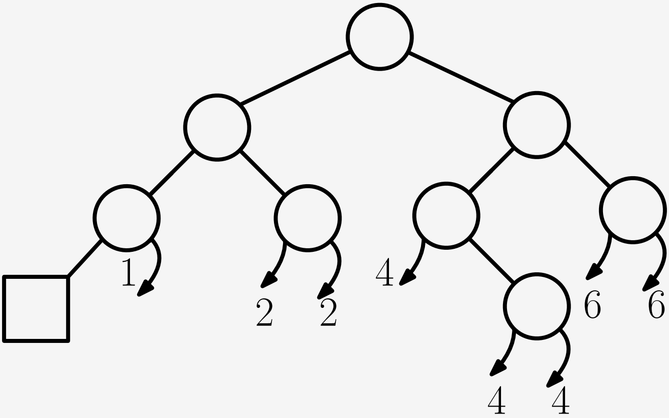
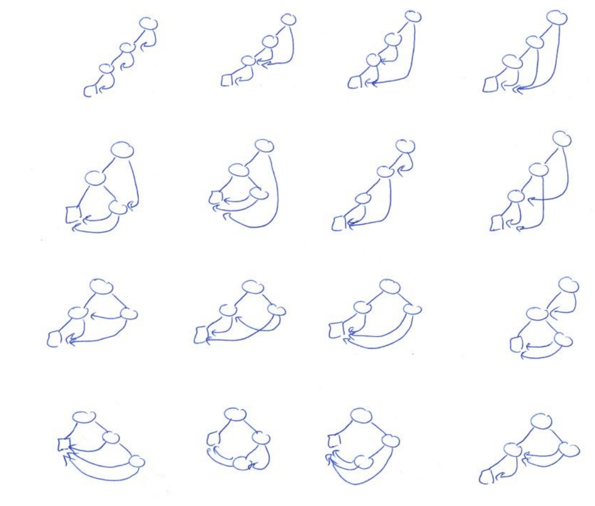


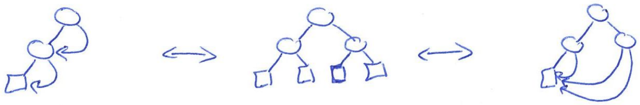
In total, this spine corresponds to $1 \cdot 3 \cdot 4 \cdot 4^2 \cdot 6^2 = 6912$ relaxed trees.
(Recall, that the same spine corresponds to 4836 compacted trees.)

Relaxed compacted binary trees of size 3



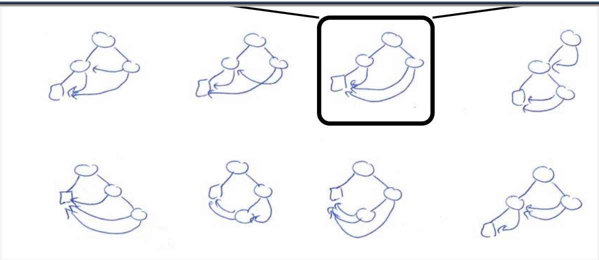
Relaxed compacted binary trees of size 3

The relaxed tree of size 3 which is not a compacted tree



compacted tree binary tree relaxed tree

Reason: subtrees not unique



A recurrence for relaxed compacted binary trees

Counting formula

Let $n, p \in \mathbb{N}$, then

$$\delta_{n+1,p} = \sum_{i=0}^{n} \delta_{i,p}\delta_{n-i,p+i}, \qquad \text{for } n \geq 0,$$

$$\delta_{0,p} = p + 1, \qquad \cancel{\delta_{1,p} = p^2 + p + 1}.$$

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Recursion still too complicated.

Example (Relaxed binary trees)

size	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
c_n	1	1	3	15	111	1119	14487
r_n	1	1	3	16	127	1363	18628

Operations on trees

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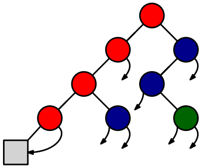

A binary tree with right height 2. Nodes of level 0 are colored in red, nodes of level 1 in blue, and the node of level 3 in green.

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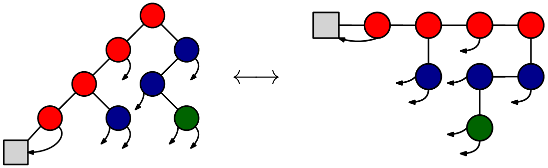

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Relaxed trees of right height $\leq k$

Figure: Right height ≤ 0.

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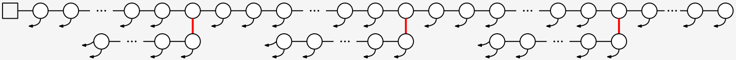

Figure: Right height ≤ 1.

Relaxed trees of right height $\leq k$

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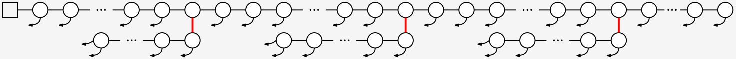

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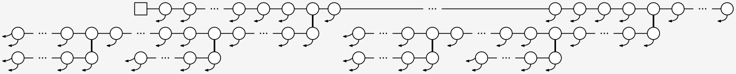

Figure: Right height ≤ 2.

Relaxed trees of right height $\leq k$

Figure: Right height ≤ 0.

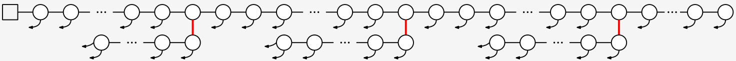

Figure: Right height ≤ 1.

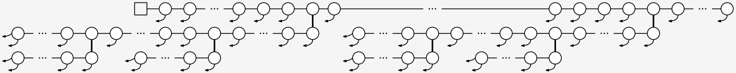

Figure: Right height ≤ 2.

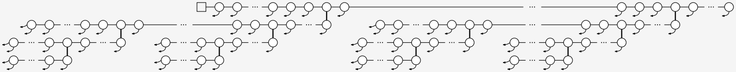

Figure: Right height ≤ 3.

Main idea: Exponential generating functions

- Asymptotic growth: $n!\rho^n \Rightarrow$ exponential generating functions (EGF)
- Upper bound guarantees positive radius of convergence
- Problem: unlabeled structures!

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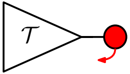
$T(z) \mapsto zT(z)$

Append a new node with a pointer to the class $\mathcal{T}$.



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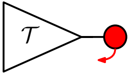

Proof:

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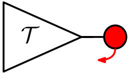

Proof:

$$k![z^k]zT(z) = \underbrace{k}_{k \text{ possible pointers}} \cdot \underbrace{t_{k-1}}_{k-1 \text{ internal nodes}} \qquad \square$$

Construction of $R_0(z)$

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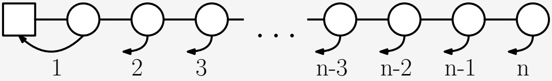

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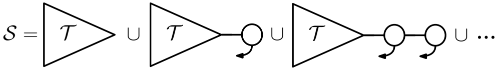
Further constructions

$S : T(z) \mapsto \frac{1}{1-z} T(z)$

Append a (possibly empty) sequence at the root.



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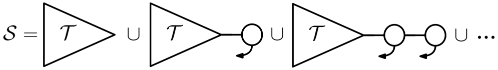
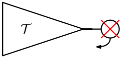


$D : T(z) \mapsto \frac{d}{dz} T(z)$

Delete top node but preserve its pointers.



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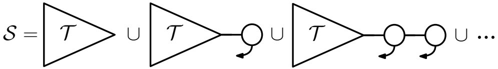

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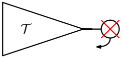

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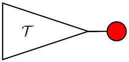

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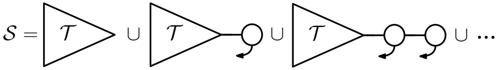

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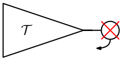

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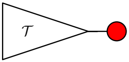
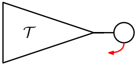


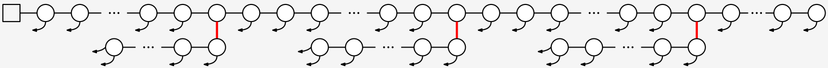
$P : T(z) \mapsto z \frac{d}{dz} T(z)$

Add a new pointer to the top node.



Relaxed binary trees

Construction of $R_1(z)$



Let $R_1(z) = \sum_{\ell \geq 0} r_{1,n} \frac{z^n}{n!}$ be the EGF of relaxed binary trees with bounded left-height ≤ 1.

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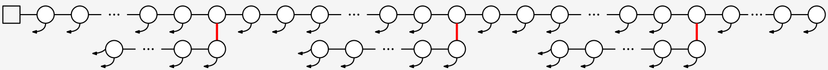

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Decomposition of $R_1(z)$

$$R_1(z) = \sum_{n \geq 0} R_{1,\ell}(z)$$

where $R_{1,\ell}(z)$ is the EGF for relaxed binary trees with exactly ℓ left-subtrees, i.e. ℓ left-edges from level 0 to level 1.

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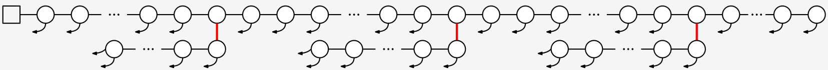

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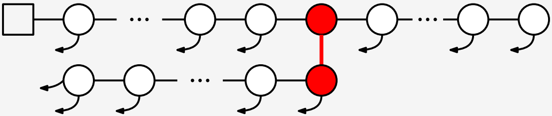
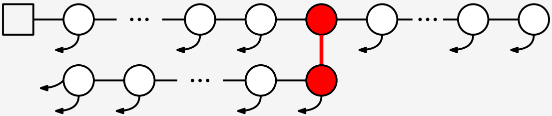
$$R_{1,1}(z) = \ ?$$

Construction of $R_{1,1}(z)$

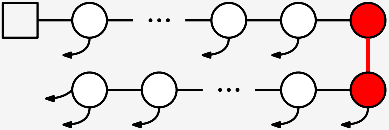


Construction of $R_{1,1}(z)$



Symbolic specification

1 delete initial sequence



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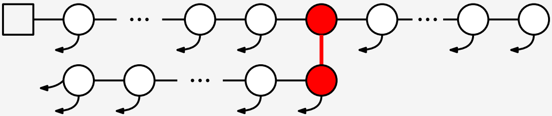
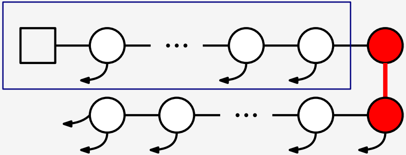


Symbolic specification

1. delete initial sequence
2. decompose



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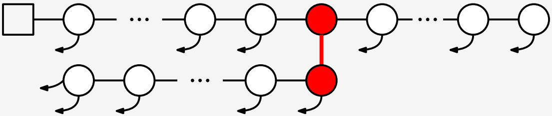

Symbolic specification

1. delete initial sequence
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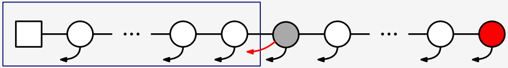
3. append and add pointer



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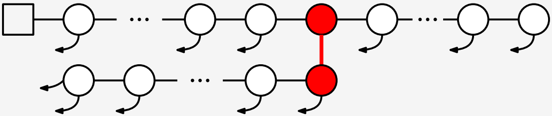

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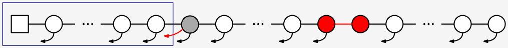
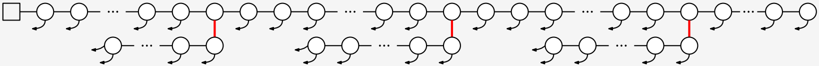


$R_{1,1}(z)$

$$R_{1,1}(z) = \underbrace{S}_{\substack{\text{init.}\\ \text{seq.}}} \circ \underbrace{I}_{\substack{\text{lvl 0}\\ \text{node}}} \circ \underbrace{S \circ P}_{\substack{\text{red pointer}\\ \text{and seq.}}} \left(\underbrace{zR_{1,0}(z)}_{\text{non empty}}\right)$$

$$R_{1,1}(z) = \frac{1}{1-z} \int \frac{1}{1-z} z \left(zR_{1,0}(z)\right)' \, dz$$

Construction of $R_{1,\ell}(z)$



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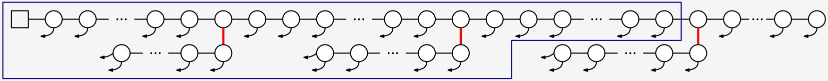
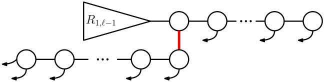


Observation

Same structure as for $R_{1,1}(z)$



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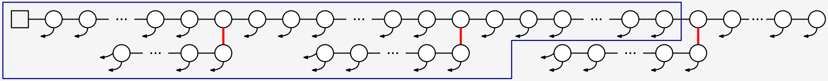

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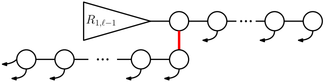

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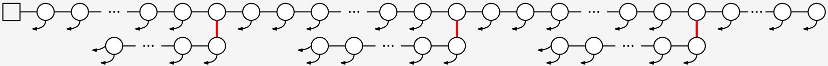
$$R_{1,0}(z) = R_0(z) = \frac{1}{1-z}.$$

Recall that $R_1(z) = \sum_{\ell \geq 0} R_{1,\ell}(z)$. Summing the previous equation (formally) for $\ell \geq 1$ gives

$$\frac{1-2z}{1-z} R_1'(z) - \frac{1}{1-z} R_1(z) - \left((1-z)R_{1,0}(z)\right)' = 0.$$

Closed form of $R_1(z)$

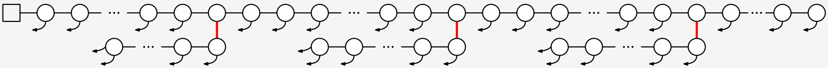


$$\frac{1-2z}{1-z}R_1'(z) - \frac{1}{1-z}R_1(z) - ((1-z)R_{1,0}(z))' = 0.$$

We know that $R_{1,0}(z) = \frac{1}{1-z}$ and get

$$(1-2z)\,R_1'(z) - R_1(z) = 0, \qquad \text{with} \qquad R_1(0) = 1.$$

Closed form of $R_1(z)$



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$$(1-2z)\,R_1'(z) - R_1(z) = 0, \qquad \text{with} \qquad R_1(0) = 1.$$

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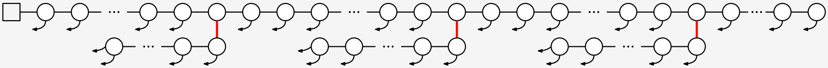
$$R_1(z) = \frac{1}{\sqrt{1-2z}}.$$

Closed form of $R_1(z)$



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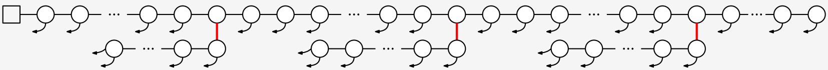

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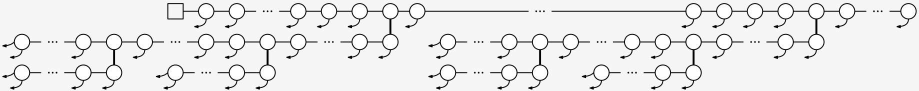
Preprint (ArXiv:1706.07163): [W, 2017, “A bijection of plane increasing trees with relaxed binary trees of right height at most one”].

Bounded left-height ≤ 2: $R_2(z)$



Symbolic construction

$$\left(1 - 3z + z^2\right) R_2''(z) + (2z - 3)\, R_2'(z) = 0,$$
$$R_2(0) = 1, \;\; R_2'(0) = 1,$$

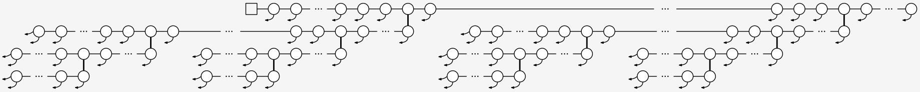
then we get the closed form

$$R_2'(z) = \frac{1}{1 - 3z + z^2},$$

and the coefficients

$$r_{2,n} = n![z^n]R_2(z) = \frac{(n-1)!}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right).$$

Bounded left-height ≤ 3: $R_3(z)$



Symbolic construction

$$\left(1-4z+3z^2\right) R_3'''(z) + (9z-6)\, R_3''(z) + 2R_3'(z) = 0,$$
$$R_3(0)=1,\ R_3'(0)=1,\ R_3''(0)=\frac{3}{2},$$

then we get the closed form

$$R_3(z) = \left(\frac{3z-2+\sqrt{3}\sqrt{1-4z+3z^2}}{\sqrt{3}-2} \right)^{\frac{1}{\sqrt{3}}},$$

and the asymptotics of the coefficients

$$r_{3,n} = n![z^n]R_3(z) = \frac{n!}{\sqrt{6}\,(2-\sqrt{3})^{1/\sqrt{3}}} \frac{3^n}{n^{3/2}\sqrt{\pi}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Differential operators

Theorem

Let $(L_k)_{k\geq 0}$ be a family of differential operators given by

$$\begin{aligned} L_0 &= (1-z), \\ L_1 &= (1-2z)D - 1, \\ L_k &= L_{k-1} \cdot D - L_{k-2} \cdot D^2 \cdot z, \qquad k \geq 2. \end{aligned}$$

Then the exponential generating function $R_k(z)$ for relaxed trees with right height $\leq k$ satisfies

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Proof. Guess and Prove! $\square$

A special class of ODEs

Consider an ordinary generating function of the kind

$$\partial^r Y(z) + a_1(z)\partial^{r-1} Y(z) + \cdots + a_r(z) Y(z) = 0, \tag{1}$$

where the $a_i \equiv a_i(z)$ are meromorphic in a simply connected domain Ω. Let $\omega_\zeta(f)$ be the order of the pole of f at ζ.

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Definition (Regular singularity)

The differential equation (1) is said to have a singularity at ζ if at least one of the $\omega_\zeta(f)$ is positive. The point ζ is said to be a *regular singularity* if

$$\omega_\zeta(a_1) \leq 1, \qquad \omega_\zeta(a_2) \leq 2, \qquad \dots, \qquad \omega_\zeta(a_r) \leq r,$$

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Relaxed trees

$$\ell_{k,k}(z)\partial^k R_k(z) + \ell_{k,k-1}(z)\partial^{k-1} R_k(z) + \ldots + \ell_{k,0}(z) R_k(z) = 0$$

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Given an equation of the form (1) and a regular singular point ζ, the *indicial polynomial* $I(\alpha)$ at ζ is defined as

$$I(\alpha) = \alpha^{\underline{r}} + \delta_1 \alpha^{\underline{r-1}} + \cdots + \delta_r, \qquad \alpha^{\underline{\ell}} := \alpha(\alpha-1)\cdots(\alpha-\ell+1),$$

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All the solutions of the differential equations behave for $z \to \zeta$ like

$$(z-\zeta)^\alpha \log(z-\zeta)^\beta$$

for some $\alpha \in \mathbb{C}, \beta \in \mathbb{N}$.

- α is a root of the indicial polynomial
- β is related to multiple roots of the indicial polynomial and roots at integer distances

A basis for our class of ODEs

Theorem

Consider a differential equation (1) *and a regular singular point* ζ *such that* $\boldsymbol{\omega_\zeta(a_i) \leq 1}$ *for all* $i = 1, \ldots, r$, *and* $\delta_1 \geq 0$.

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where H_m *is analytic at* 0 *(*$H_m(0) \neq 0$*).*

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$$(z-\zeta)^{r-1-\delta_1} H(z-\zeta) + H_0(z-\zeta) \left(\log(z-\zeta)\right)^k, \quad \textit{with} \quad k \in \{0,1\};$$

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3. *For* $\delta_1 \notin \mathbb{Z}$ *it is of the form*
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Chebyshev polynomials

The **Chebyshev polynomials of the first kind** $T_n(z)$ are defined by the recurrence relation

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \\ T_{n+2}(z) &= 2zT_{n+1}(z) - T_n(z). \end{aligned}$$

The **Chebyshev polynomials of the second kind** $U_n(z)$ are defined by the recurrence relation

$$\begin{aligned} U_0(z) &= 1, \\ U_1(z) &= 2z, \\ U_{n+2}(z) &= 2zU_{n+1}(z) - U_n(z). \end{aligned}$$

Properties of $\ell_{k,k}(z)$

Lemma (Transformed leading coefficient)

For the leading coefficient we get

$$\ell_{k,k}(z) = z^{\frac{k+2}{2}} U_{k+2}\left(\frac{1}{2\sqrt{z}}\right) = \sum_{n=0}^{\lfloor \frac{k+2}{2} \rfloor} (-1)^n \binom{k+2-n}{n} z^n,$$

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Lemma

The roots of $\ell_{k,k}(z)$ are real, positive, and distinct. Let ρ_k be the smallest real root of $\ell_{k,k}(z)$. Then, we have

$$\rho_k = \frac{1}{4\cos^2\left(\frac{\pi}{k+3}\right)}.$$

Furthermore, ρ_k is not a root of $\ell_{k,k-1}(z)$.

Analyzing the other polynomials

- Using the recurrence we get $\ell_{k,k-1}(z) = \frac{k}{2}\ell'_{k,k}(z)$;
- For $k \geq 2$ and $0 \leq i \leq \lfloor \frac{k-2}{2} \rfloor$ it holds that $\ell_{k,i}(z) \equiv 0$;
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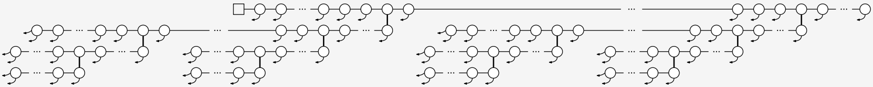
Proposition

The indicial polynomial $I_k(\alpha)$ of the k-th differential equation is given by $I_k(\alpha) = \alpha^{\underline{k-1}}(\alpha - (\frac{k}{2} - 1))$.

Asymptotics of relaxed trees with bounded right height



Theorem

The number $r_{k,n}$ of relaxed trees with right height at most k is for $n \to \infty$ asymptotically equivalent to

$$r_{k,n} \sim \gamma_k n! \left(4 \cos\left(\frac{\pi}{k+3}\right)\right)^n n^{-k/2},$$

where $\gamma_k \in \mathbb{R}$ is independent of n.

Compacted binary trees

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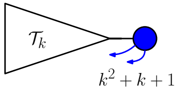
Uniqueness of subtrees



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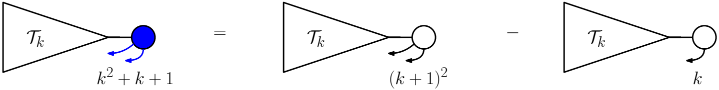

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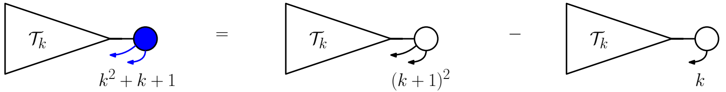

Let $(M_k)_{k\geq 0}$ be a family of differential operators such that the EGF $C_k(z)$ for compacted binary trees with right height $\leq k$ satisfies

$$M_k \cdot C_k = 0.$$

Compacted binary trees

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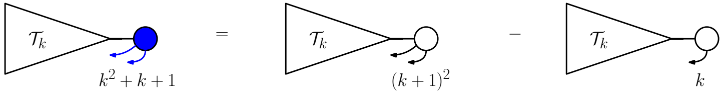

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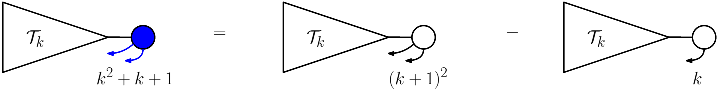

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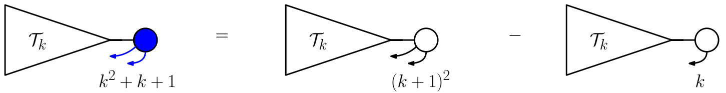

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$$(3z^2-4z+1)\frac{d^4}{dz^4}C_3(z) - (4z^2-18z+10)\frac{d^3}{dz^3}C_3(z) + \cdots$$

$$\cdots + (z^2-12z+14)\frac{d^2}{dz^2}C_3(z) + (z-3)\frac{d}{dz}C_3(z) = 0.$$

Properties of M_k

Theorem

The operator $M_k(\cdot)$ decomposes into

$$M_k = m_{k,k}(z)D^{k+1} + m_{k,k-1}(z)D^k + \ldots + m_{k,0}(z)D + m_{k,-1}(z),$$

where the $m_{k,i}(z)$ are polynomials.

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$$\begin{aligned}
m_{k,-1}(z) &= 0, \\
m_{k,0}(z) &= \begin{cases} -2z+3, & \text{for } k \text{ even}, \\ z-3, & \text{for } k \text{ odd}, \end{cases} \\
m_{k,i}(z) &= m_{k-1,i-1}(z) + (i+1)m_{k-2,i}(z) + (z-i-2)m_{k-2,i-1}(z) \\
&\quad - zm_{k-2,i-2}(z), \qquad 1 \leq i \leq k-1, \\
m_{k,k}(z) &= m_{k-1,k-1}(z) - zm_{k-2,k-2}(z), \\
m_{k,i}(z) &= 0, \qquad i > k.
\end{aligned}$$

The initial polynomials are $m_{0,-1}(z) = -1$, $m_{0,0} = 1-z$, $m_{1,-1} = 0$, $m_{1,0} = z-3$, and $m_{1,1}(z) = 1-2z$.

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$$\boldsymbol{m_{k,k}(z) = \ell_{k,k}(z)}$$

Analysis of the polynomials $m_{k,i}(z)$

- As $m_{k,k}(z) = \ell_{k,k}(z)$ we have the same dominant singularity ρ_k;
- Using the recurrence relation we can express $m_{k,k-1}(z)$ by the Chebyshev polynomials of first and second kind.

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The indicial polynomial is given by

$$I_k(\alpha) = \alpha^k(\alpha - (k - \delta_1)).$$

Asymptotics of compacted trees with bounded right height

Theorem (Main result)

The number $c_{k,n}$ of compacted trees with right height at most k is asymptotically equal to

$$c_{k,n} \sim \kappa_k n! \left(4 \cos \left(\frac{\pi}{k+3} \right)^2 \right)^n n^{-\frac{k}{2} - \frac{1}{k+3} - \left(\frac{1}{4} - \frac{1}{k+3}\right) \cos\left(\frac{\pi}{k+3}\right)^{-2}},$$

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Comparing compacted and relaxed trees

Asymptotics of compacted and relaxed trees

$$c_{k,n} \sim \kappa_k n! \rho_k^{-n} n^{\alpha} \qquad \text{and} \qquad r_{k,n} \sim \gamma_k n! \rho_k^{-n} n^{\beta}.$$

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1	$\frac{1}{2}$	0.500	$-\frac{3}{4}$	-0.750	$-\frac{1}{2}$	-0.5
2	$\frac{3}{2} - \frac{\sqrt{5}}{2}$	0.382	$-\frac{3}{2} + \frac{\sqrt{5}}{10}$	-1.276	-1	-1.0
3	$\frac{1}{3}$	0.333	$-\frac{16}{9}$	-1.778	$-\frac{3}{2}$	-1.5
4	$(2\cos(\frac{\pi}{7}))^{-2}$	0.308	$-\frac{15}{7} - \frac{3}{28\cos(\pi/7)^2}$	-2.275	-2	-2.0
5	$1 - \frac{\sqrt{2}}{2}$	0.293	$-\frac{25}{8} + \frac{\sqrt{2}}{4}$	-2.772	$-\frac{5}{2}$	-2.5
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Corollary (Proportion of compacted among relaxed trees)

$$\frac{c_{k,n}}{r_{k,n}} \sim \kappa n^{\delta_1 - \frac{k}{2} - 1}$$

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$$\frac{c_{k,n}}{r_{k,n}} \sim \kappa n^{\delta_1 - \frac{k}{2} - 1} = \kappa n^{-\frac{1}{k+3} - \left(\frac{1}{4} - \frac{1}{k+3}\right)\frac{1}{\cos^2\left(\frac{\pi}{k+3}\right)}}$$

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Thanks