Analytic Algebraic Combinatorics

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$$g_{\lambda,\mu,\nu}$$

Kronecker Coefficients

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$
$$\mu = (\mu_1, \mu_2, \ldots, \mu_m)$$
$$\nu = (\nu_1, \nu_2, \ldots, \nu_n)$$

Kronecker Coefficients

The Kronecker coefficients $g_{\lambda,\mu,\nu}$ describe the decomposition of the tensor product of two Specht modules (irreducible representations of the symmetric group) into irreducible representations.

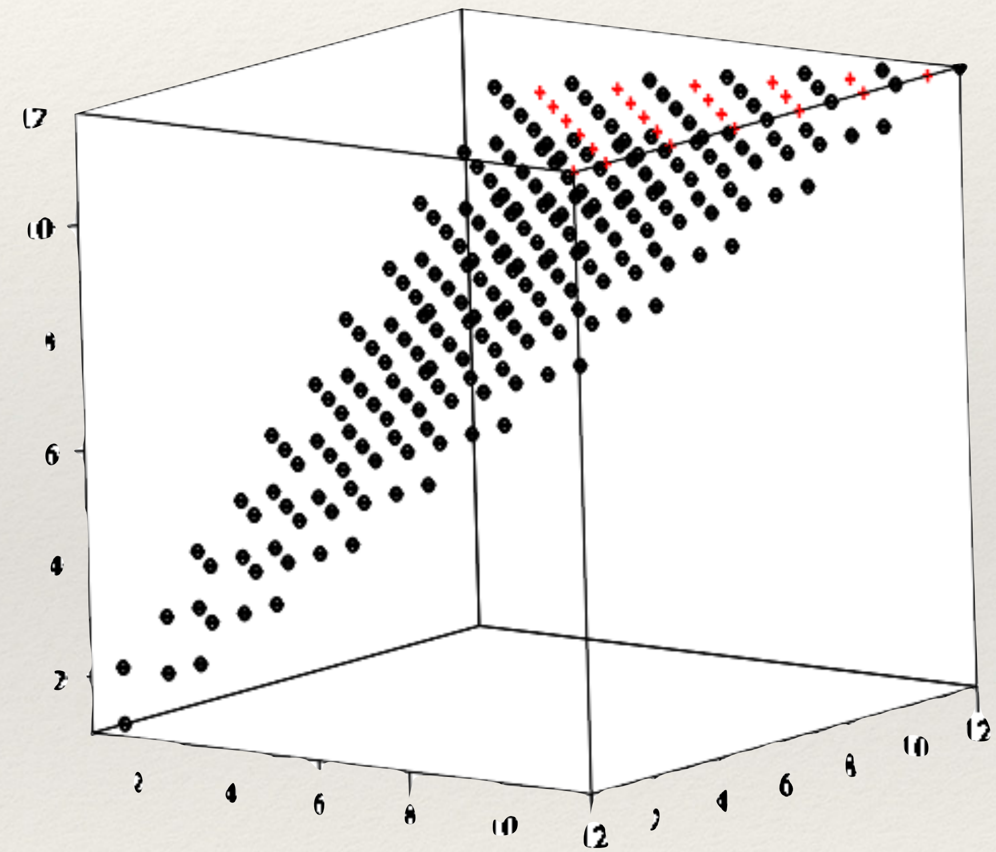
$$V_\mu \otimes V_\nu = \bigoplus g_{\lambda,\mu,\nu} V_\lambda$$

We will not revisit this definition again during the talk.

Mysterious creatures

Many longstanding open problems:

- Combinatorial interpretation
  (like Littlewood-Richardson?)
- Effective computation in the general case.
- Determine when a coefficient is zero.



(24-k, k) (24-i,i) (24-j,j)

The cone of non-zero values. Red = zero

What is known?

Many things!

- For fixed **partition lengths**, the function behaves like a quasipolynomial. Related to **integer points in a polytope**.
- This quasipolynomial is (theoretically) computable but previous results have high complexity in the lengths of the partitions
  - [Baldoni, Vergne, Walter 16] (Maple package); [Christandl, Doran, Walter 12]
- Asymptotics possible using Barvinok algorithm
- **Our contribution:**
  - Minimal dimension of the polytope;
  - **Explicit** expressions for small cases;
  - Applying ACSV to *simplify* the presentation of the asymptotics.

Main result

Simple Arithmetic Formulas

New: An explicit formula

THEOREM 1 (M., Rosas, Sundaram 2018+)

If $|\lambda| \leq 4, |\mu| \leq 2, |\nu| \leq 2, \mu_2 \geq \nu_2$

$$g_{\mu,\nu,\lambda} = [y^{\nu_2}][x^{\mu_2}] \frac{P_\lambda(x,y)}{(1-y/x)(1-xy)(1-x)(1-y)}$$

$$P_\lambda(x,y) = y^{\lambda_3+\lambda_4}(x^{\lambda_2+\lambda_4} - x^{\lambda_2+\lambda_3+1} - x^{\lambda_1+\lambda_4+1} + x^{\lambda_1+\lambda_3+2}) + y^{\lambda_2+\lambda_4+1}(-x^{\lambda_3+\lambda_4-1} + x^{\lambda_2+\lambda_3+1} + x^{\lambda_1+\lambda_4+1})$$

COROLLARY 2 *(A simple bound)*

$$g_{\mu,\nu,\lambda} \leq [y^{\nu_2-\lambda_3-\lambda_4}][x^{\mu_2-\lambda_2-\lambda_4}] \frac{1}{(1-y/x)(1-xy)(1-x)(1-y)}$$

COROLLARY 3 *(Quasipolynomiality)*

$\tilde{g}_{\mu,\nu,\lambda}$

Partition function representation

THEOREM (M., Rosas, Sundaram 2018+)
If $|\lambda| \leq nm, |\mu| \leq n, |\nu| \leq m,$

$g_{\mu,\nu,\lambda}$ is expressible using a **computable vector partition function** of dimension at most *nm*.

n=2, m=2

$$g_{\mu,\nu,\lambda} = [u^{\nu_2} v^{\mu_2 \nu_2}] \frac{P_\lambda(u, uv)}{(1-uv)(1-uv^2)(1-u)(1-v)} \qquad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

n=2, m=3

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \end{pmatrix}$$

n=m=3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 3 & 4 & 4 & 5 \end{pmatrix}.$$

Proof sketch

Schur function

$$s_\alpha[x_1, \ldots, x_n] = s_\alpha[X] = \frac{\det[x_i^{\alpha_j+j-1}]}{\prod(x_i - x_j)}$$

Jacobi's identity

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_\mu[X] s_\nu[Y]$$

$$\frac{\prod(x_i - x_j)\prod(y_i - y_j)}{\prod(x^i y^j - x^k y^\ell)}\, a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X] a_{\nu+\delta_m}[Y]$$

A small, yet illustrative, example

$$\frac{\prod(x_i - x_j)\prod(y_i - y_j)}{\prod(x^i y^j - x^k y^\ell)} a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X] a_{\nu+\delta_m}[Y]$$

n=m=2

$$\frac{a_{\lambda+\delta_{nm}}[XY]}{(1-y/x)(1-xy)(1-x)(1-y)} = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\,(x^{\mu_1+1} - x^{\mu_2})(y^{\nu_1+1} - y^{\nu_2})$$

Sample Computation

$$g_{\mu,\nu,\lambda} = [u^{\nu_2} v^{\mu_2 \nu_2}] \frac{P_\lambda(u, uv)}{(1-uv)(1-uv^2)(1-u)(1-v)}$$

$$P_\lambda(x,y) = y^{\lambda_3+\lambda_4}(x^{\lambda_2+\lambda_4} - x^{\lambda_2+\lambda_3+1} - x^{\lambda_1+\lambda_4+1} + x^{\lambda_1+\lambda_3+2})$$
$$+y^{\lambda_2+\lambda_4+1}(-x^{\lambda_3+\lambda_4-1} + x^{\lambda_2+\lambda_3+1} + x^{\lambda_1+\lambda_4+1})$$

Remark. If the monomial $x^a y^b$ makes a non-zero contribution from P_λ, then it contributes

$$[u^{\nu_2-b} v^{\mu_2+\nu_2-a-b}] \frac{1}{(1-u)(1-v)(1-uv)(1-uv^2)}$$

$$\lambda = (12,7,4,1), \mu = \nu = (12,12)$$

$$P_\lambda = y^5(x^8 - x^{12} - x^{14} + x^{18}) + y^9(-x^4 + x^{12} + x^{14})$$

$$g_{\mu,\nu,\lambda} = p(7,11) - p(7,7) - p(7,5) - p(7,1) - p(3,11) + p(3,3) + p(3,1)$$
$$= 32 - 20 - 12 + 2 - 10 + 6 + 2 = 0$$

Combinatorial Interpretation

Lattice point enumerators of polytopes

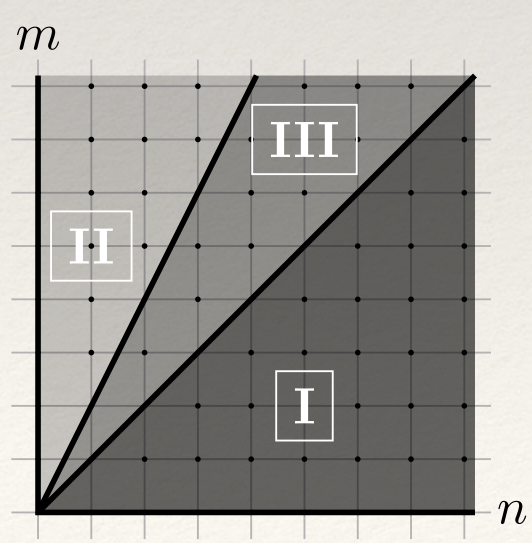
A vector partition function

A vector partition generalizes an integer partition.

$$[u^n v^m]\frac{1}{(1-u)(1-v)(1-uv)(1-uv^2)}$$

$$= \#\left\{ (x_1, x_2, x_3, x_4) \in \mathbb{N}^4 : \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ 2x_4 \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} \right\}$$

$$= \#\{\mathbf{x} \in \mathbb{N}^4 \,|\, A\mathbf{x} = \begin{bmatrix} n \\ m \end{bmatrix}\} \qquad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$



Region		$p_S(n,m)$
I	$m \leq n$	$\frac{m^2}{4} + m + \frac{7}{8} + \frac{(-1)^m}{8}$
II	$2n \leq m$	$\frac{n^2}{2} + \frac{3n}{2} + 1$
III	$n \leq m \leq 2n$	$nm - \frac{n^2}{2} - \frac{m^2}{4} + \frac{n+m}{2} + \frac{7}{8} + \frac{(-1)^m}{8}$

Vector partition function

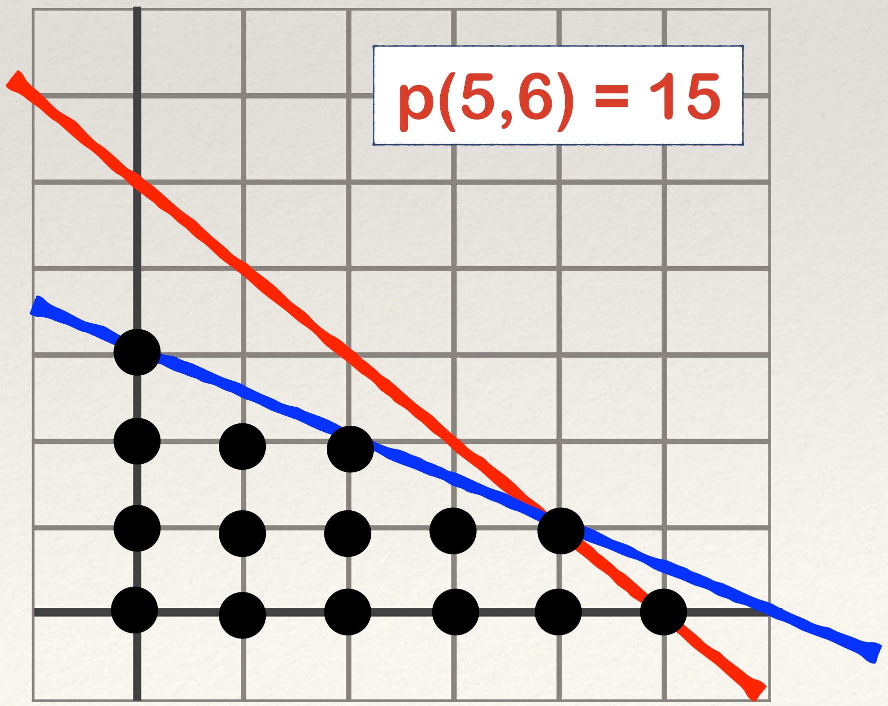
$$[u^n v^m]\,\frac{1}{(1-u)(1-v)(1-uv)(1-uv^2)}$$

$$= \#\left\{ (x_1,x_2,x_3,x_4) \in \mathbb{N}^4 : \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ 2x_4 \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} \right\}$$

$$x_1 + x_3 + x_4 = n \qquad x_2 + x_3 + 2x_4 = m$$

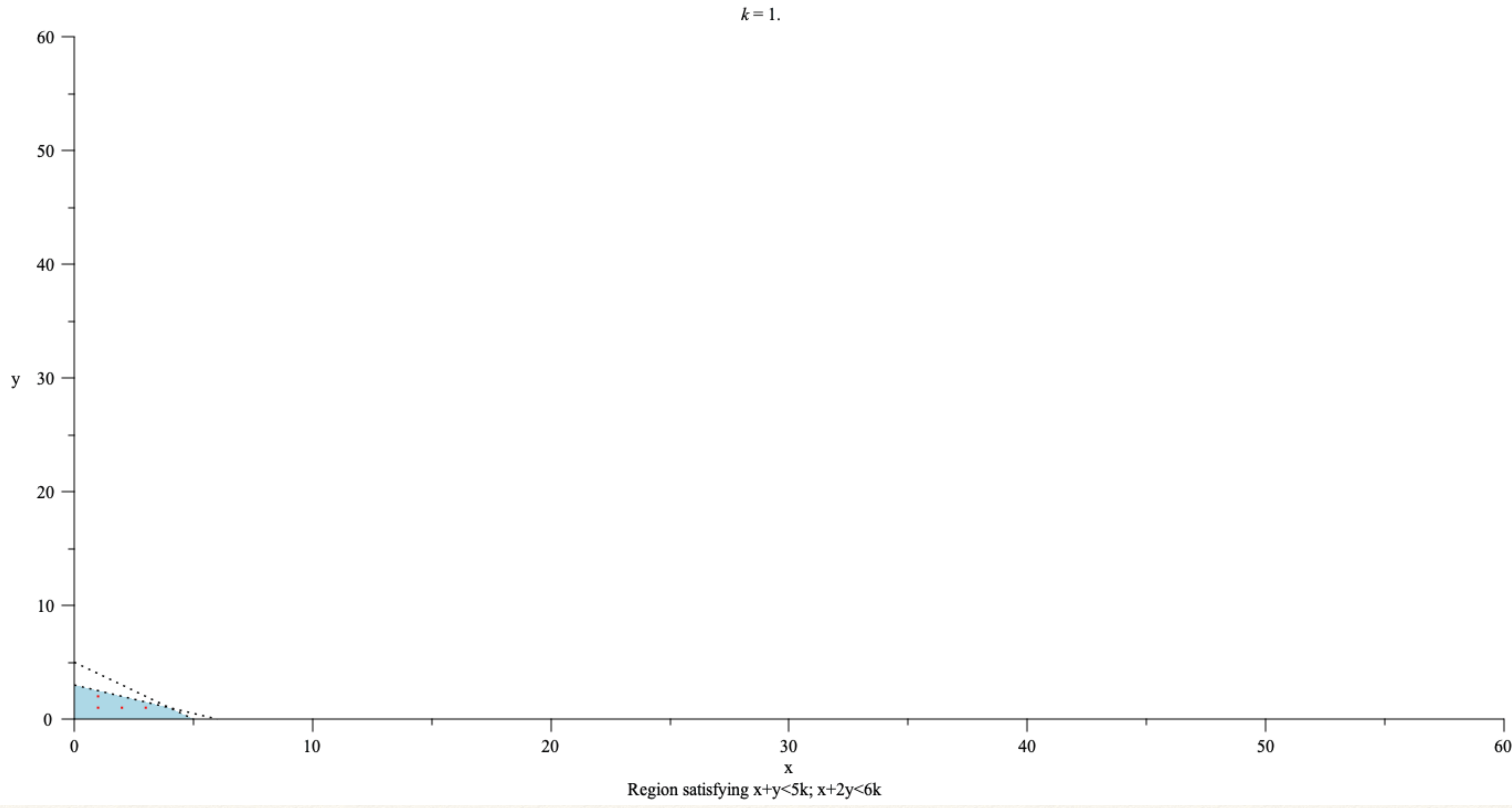
$$x_3 + x_4 \leq n \qquad x_3 + 2x_4 \leq m$$

p(n,m) = # integer points inside the lines.



How many integer points in the region?

$$x_3 + x_4 \leq 5k \quad x_3 + 2x_4 \leq 6k$$



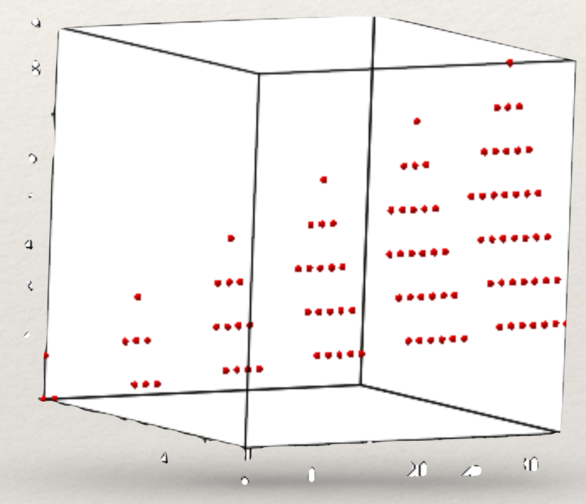
Lattice point enumerator

Ehrhart Theory (1960s)

Dilate the polytope, find the number of integer points inside.

$x_3 + x_4 \le kn \quad x_3 + 2x_4 \le km$

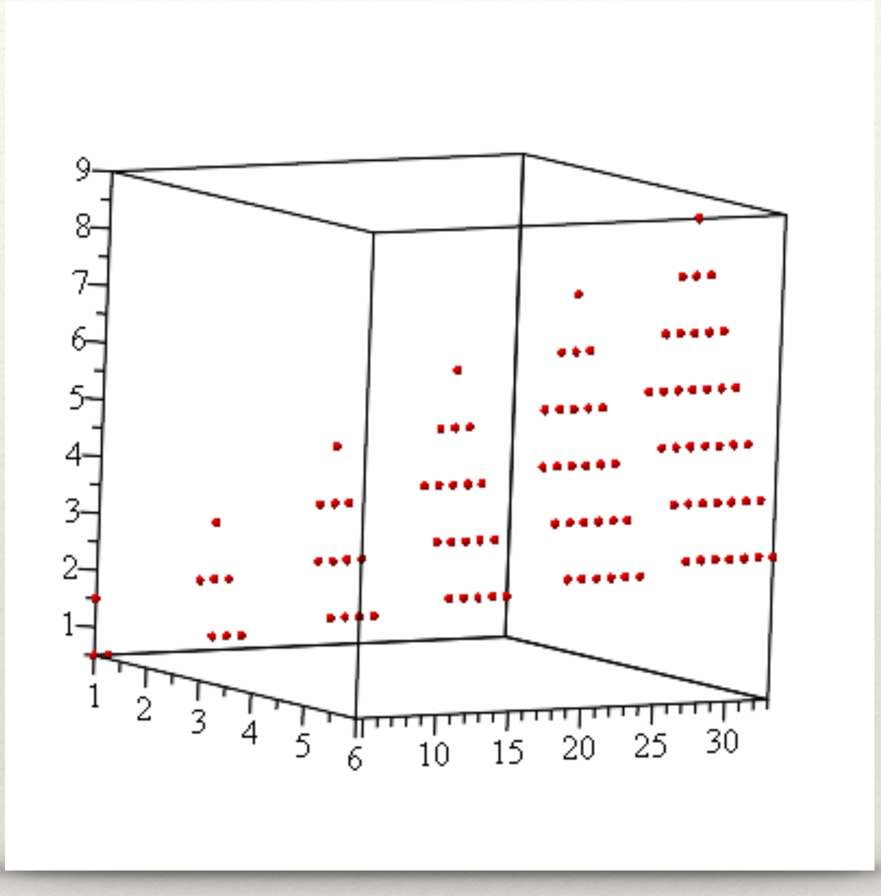
$$= [u^{kn} v^{km}] \frac{1}{(1-uv)(1-uv^2)(1-u)(1-v)}$$



- For fixed *n*, *m*, this is a **quasipolynomial** in k.
- The leading constant of the polynomial is the area (Pick's theorem)
- Generalized for polytopes in arbitrary dimension.

Dilations of KCs

$$\lambda = (6,3,2), \mu = (8,3), \nu = (7,4)$$



$$g_{(6,3,2),(8,3),(7,4)} = 3$$

$$g_{(12,6,4),(16,6),(14,8)} = 7$$

$$g_{(18,9,6),(24,9),(21,12)} = 12$$

$$g_{k\mu,k\nu,k\lambda} = [u^{2k}v^k]\frac{1}{(1-u)(1-v)(1-uv)(1-uv^2)}$$

Let us make the generating function

One interesting implication of the quasipolynomiality property is that, knowing the Kronecker coefficients asymptotically, in fact we know them completely.

— *Manivel, 2014*

Asymptotic formulas

Analytic **C**ombinatorics in **S**everal **V**ariables

Diagonals of Rational Functions

$$\Delta(1 + x^2 + y + 5xyz + 3xy^2 + 2xy^2z^2 + 3x^2y^2z^2 + \ldots)$$

$$= 1 + 5t + 3t^2 + \ldots$$

Diagonals of series

$$\Delta \sum a_{ijk} x^i y^j z^k := \sum a_{nnn} t^n$$

$$\Delta^{(r,s)} \sum_{i \geq 0 j \geq 0} a_{ij} x^i y^j := \sum_{n \geq 0} a_{rn\, sn} t^n$$

$$\Delta^{(r,s)} \frac{1}{1 - x - y} = \Delta \sum_{i \geq 0, j \geq 0} \binom{i+j}{j} x^i y^j = \sum_{n \geq 0} \binom{(r+s)n}{rn} t^n$$

Generating function as a diagonal

$$\sum z^k [u^{kr} v^{ks}] \frac{1}{(1-uv)(1-uv^2)(1-u)(1-v)}$$

$$= \Delta^{(r,s)} \frac{1}{(1-uv)(1-uv^2)(1-u)(1-v)}$$

- Any lattice point enumerator is a diagonal of a rational function.
- We use ACSV techniques to determine the asymptotic growth of the coefficient

Analytic combinatorics

Problem: Given $F(x, y, z) = \dfrac{G(x, y, z)}{H(x, y, z)}$, write $\Delta^{(r,s)}F(x, y, z) = \sum f_n t^n$

Determine $\Phi(n)$ so that

$$\lim_{n\to\infty} \frac{f_n}{\Phi(n)} = 1$$

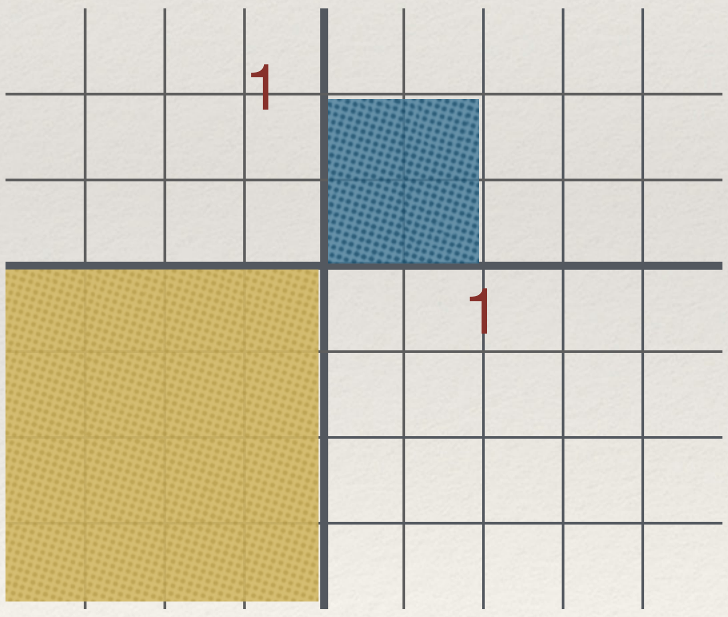
1. Determine "minimal critical point"
2. Decompose the rational G/H into a sum such that denominator of each summand has a transversal intersection at this point
3. Apply formulas to determine contribution at the point

Visualizing the domain of convergence

$$\{(|x|,|y|) : (x,y) \in \mathbb{C}^2, F(x,y) \text{ convergent } \}$$

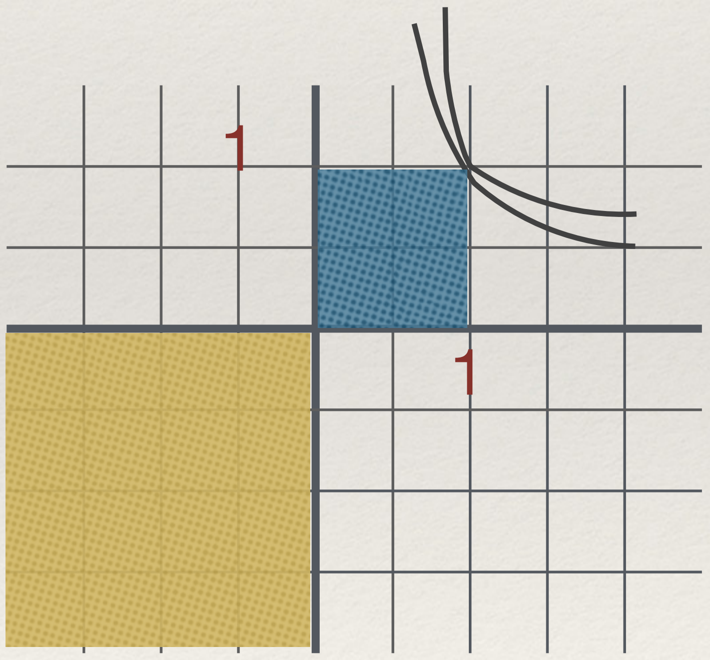
$$\{(\log|x|,\log|y|) : (x,y) \in \mathbb{C}^2, F(x,y) \text{ convergent } \}$$

$$\frac{1}{(1-x)(1-y)}$$



$|x| < 1 \quad |y| < 1$

$$\frac{1}{(1-xy)(1-xy^2)(1-x)(1-y)}$$



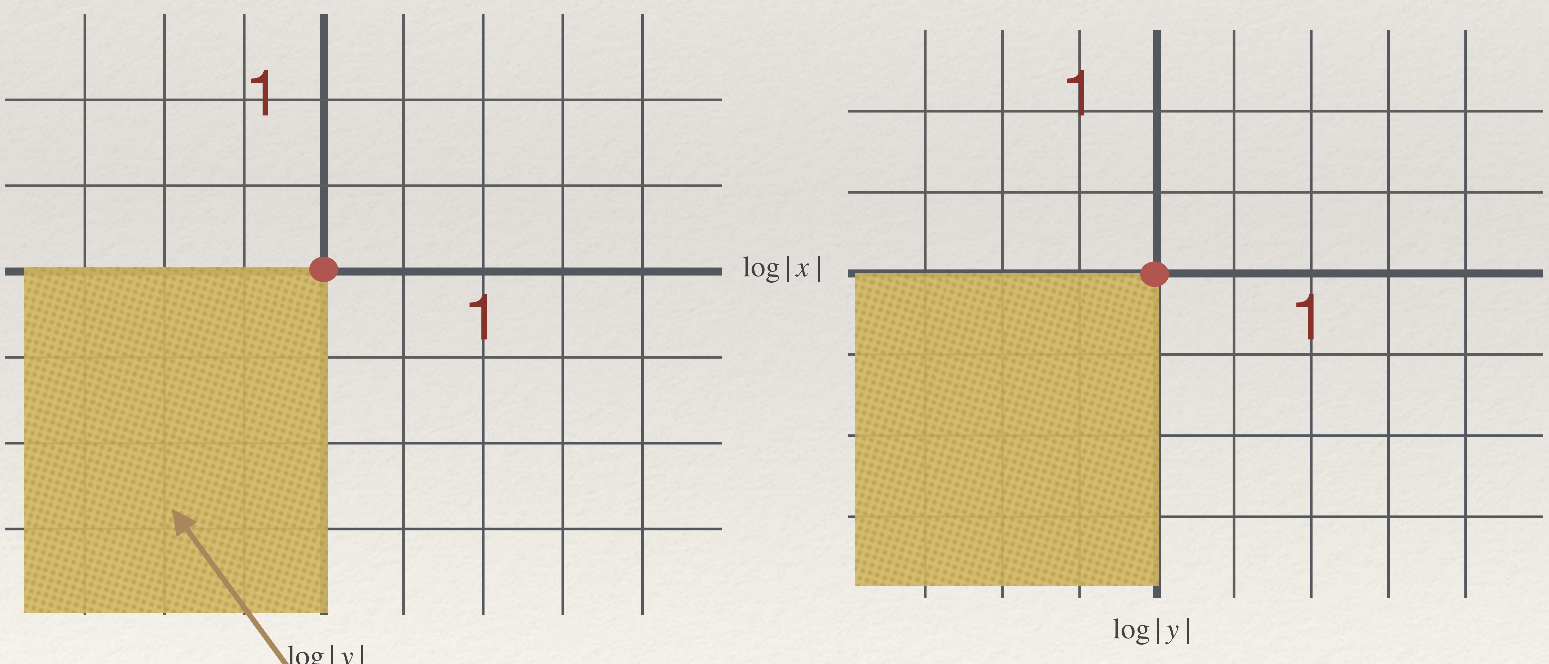
$|x| < 1 \quad |y| < 1$

TEST:
What will be the domain of given partition function?

The minimal critical point

Find (log|x|,log|y|) which maximizes **r log |x|+s log |y|**
(with (x,y) in closure of the domain of convergence)

$$\frac{1}{(1-x)(1-y)} \qquad \frac{1}{(1-xy)(1-xy^2)(1-x)(1-y)}$$



$(\log|x|, \log|y|) : (x,y) \in \overline{\mathscr{D}}$

Exponential growth:

$$\left(|x|^{-r} |y|^{-s} \right)^n$$

The minimal critical point for any partition functions is (1,1,…1)

The exponential growth is always 1.

(We knew this, but here we deduced it on our own.)

Subexponential growth

1. Separate the rational into terms such that each has a transversal intersection at its critical point.
2. Apply a formula to each term.
3. Given *(r,s)*, sum over contributing terms.

$$\frac{1}{(1-uv)(1-uv^2)(1-u)(1-v)}$$

$$= \frac{(1+v)(uv+1)}{\left(-uv^2+1\right)^2(1-u)^2} - \frac{(1+v)\,u\,(uv+1)}{(1-u)^3\left(-uv^2+1\right)} + \frac{(1+v)\,u^2}{(1-u)^3(-uv+1)} - \frac{v^3}{(1-v)^2\left(-uv^2+1\right)^2} - \frac{v^3}{(1-v)^3\left(-uv^2+1\right)} + \frac{v^2}{(1-v)^3(-uv+1)}$$

Splitting the rational into parts

$$\begin{array}{cccc} \multicolumn{4}{c}{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}} \\ 1 & 2 & 3 & 4 \end{array} \qquad H1 = (1-u) \quad H2 = (1-v) \quad H3 = (1-uv) \quad H4 = (1-uv^2)$$

$$(1-uv) = (1-u) + u(1-v) \qquad H3 = H1 + uH2$$

$$\frac{1}{H1\,H2\,H3\,H4} = \frac{H3}{H1\,H2\,H3^2\,H4} = \frac{H1 + uH2}{H1\,H2\,H3^2\,H4} = \frac{1}{H2\,H3^2\,H4} + \frac{u}{H1\,H3^2\,H4}$$

$$= \frac{(1+v)(uv+1)}{\left(-uv^2+1\right)^2(1-u)^2} - \frac{(1+v)\,u\,(uv+1)}{(1-u)^3\left(-uv^2+1\right)} + \frac{(1+v)\,u^2}{(1-u)^3(-uv+1)} - \frac{v^3}{(1-v)^2\left(-uv^2+1\right)^2} - \frac{v^3}{(1-v)^3\left(-uv^2+1\right)} + \frac{v^2}{(1-v)^3(-uv+1)}$$

Formula

Theorem 10.3.1 (Pemantle Wilson 2013)

$$\frac{G(x,y)}{H_1(x,y)^k H_2(x,y)^\ell} = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} x^i y^j \qquad M = \left.\begin{bmatrix} \frac{\partial H_1(x,y)}{\partial x} & \frac{\partial H_1(x,y)}{\partial y} \\ \frac{\partial H_2(x,y)}{\partial x} & \frac{\partial H_2(x,y)}{\partial y} \end{bmatrix}\right|_{(x,y)=(1,1)}$$

$$a_{r,s} \sim \frac{1}{(k-1)!(\ell-1)!} \frac{G(x_0,y_0)}{\det(M)} \big((r,s)\times M\big)^{(k-1,\ell-1)}$$

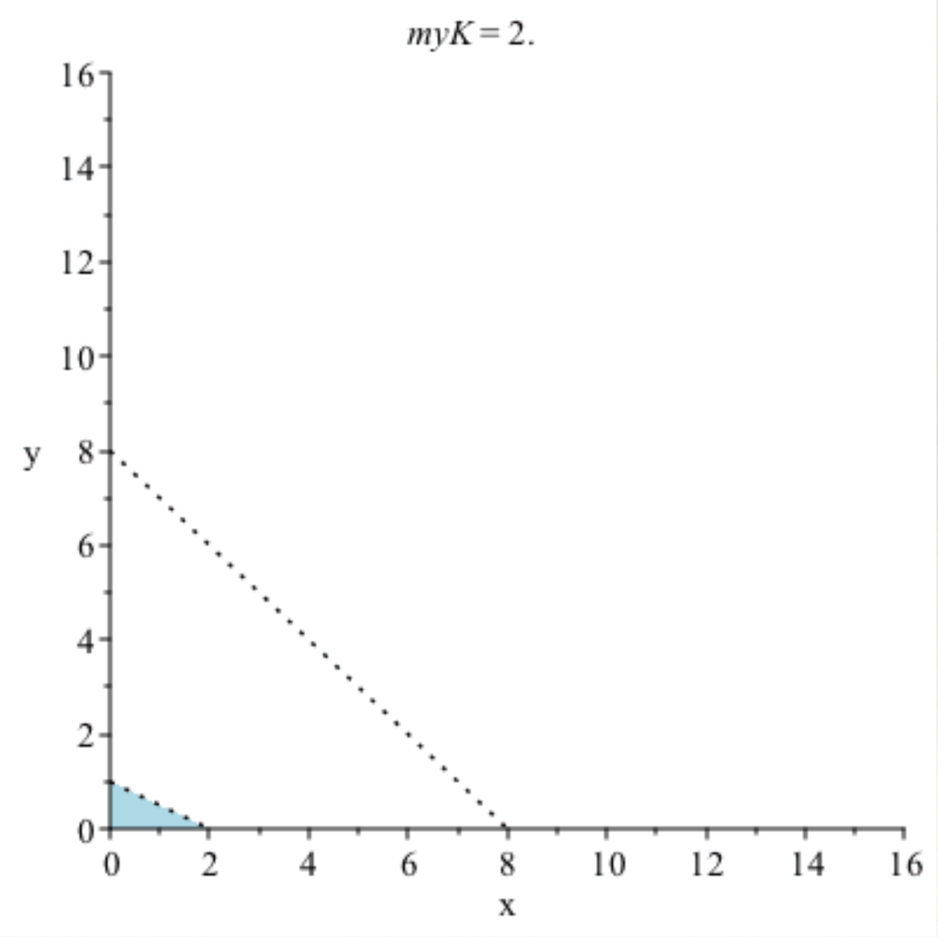
Notation: $(x,y)^{(a,b)} := x^a y^b$

Asymptotic formulas for atomic Kronecker coefficients

$$\sum f_n z^n = \Delta^{(r,s)} \frac{1}{(1-u)(1-uv)(1-uv^2)(1-v)}$$

$$f_n \sim$$

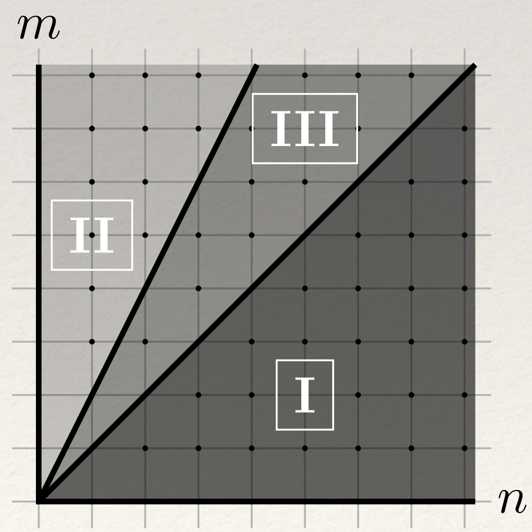
$r < s$	$\frac{(rn)^2}{4}$
$r = s$	$\left(\frac{rn}{2}\right)^2$
$r/2 < s < r$	$rs\,n^2 - \frac{(sn)^2}{2} - \frac{(rn)^2}{4}$
$s \leq r/2$	$\frac{sn^2}{2}$



$$(r,s) = (2,1) \implies f_n \sim n^2/2$$

Easy to confirm

$r < s$	$\frac{(rn)^2}{4}$
$r = s$	$\left(\frac{rn}{2}\right)^2$
$r/2 < s < r$	$rs\,n^2 - \frac{(sn)^2}{2} - \frac{(rn)^2}{4}$
$s \le r/2$	$\frac{sn^2}{2}$



Region		$p_S(n,m)$
I	$m \le n$	$\frac{m^2}{4} + m + \frac{7}{8} + \frac{(-1)^m}{8}$
II	$2n \le m$	$\frac{n^2}{2} + \frac{3n}{2} + 1$
III	$n \le m \le 2n$	$nm - \frac{n^2}{2} - \frac{m^2}{4} + \frac{n+m}{2} + \frac{7}{8} + \frac{(-1)^m}{8}$

Asymptotic formulas for general dilated Kronecker coefficients?

$$g_{\mu,\nu,\lambda} = [u^{\nu_2} v^{\mu_2 \nu_2}] \frac{P_\lambda(u, uv)}{(1-uv)(1-uv^2)(1-u)(1-v)}$$

- In general, we will have diagonals with a more complicated numerator
- This affects the constant term, primarily ….
- … but when it is zero, the degree of the sub-exponential growth can drop.

Example

$$g_{(k,k),(k,k),(k,k)} = [x^k y^k](x^k - 2x^{k+1} + x^{k+2})\bar{F}_{2,2}(x,y)$$

$$= [u^k v^k]\frac{(1-2v+v^2)}{(1-u)(1-v)(1-uv)(1-uv^2)}$$

$$= \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd}\,. \end{cases}$$

- The numerator $(1\text{-}2v+v^2)$ is *0* at *(1,1)* as its derivative, *(-2+2v).* As a consequence, the degree drops by 2. It is “quasi-constant”.
- This example illustrates that KCs are **not** pure polytope enumerators.

Perspectives

- Determine exact and asymptotic formulas for Kronecker coefficient dilations of “higher dimension”
- Combinatorial interpretation of Kronecker Coefficients
- Develop inequalities to determine when it is zero.
- Develop more automated/ computational methods for ACSV
- Many problems from representation theory are diagonal problems. Apply similar tools.

For more details…

An elementary approach to the quasipolynomiality of the Kronecker coefficients

Marni Mishna, Mercedes Rosas, Sheila Sundaram

https://arxiv.org/abs/1811.10015

Merci
beaucoup