Périodes en géométrie algébrique effective

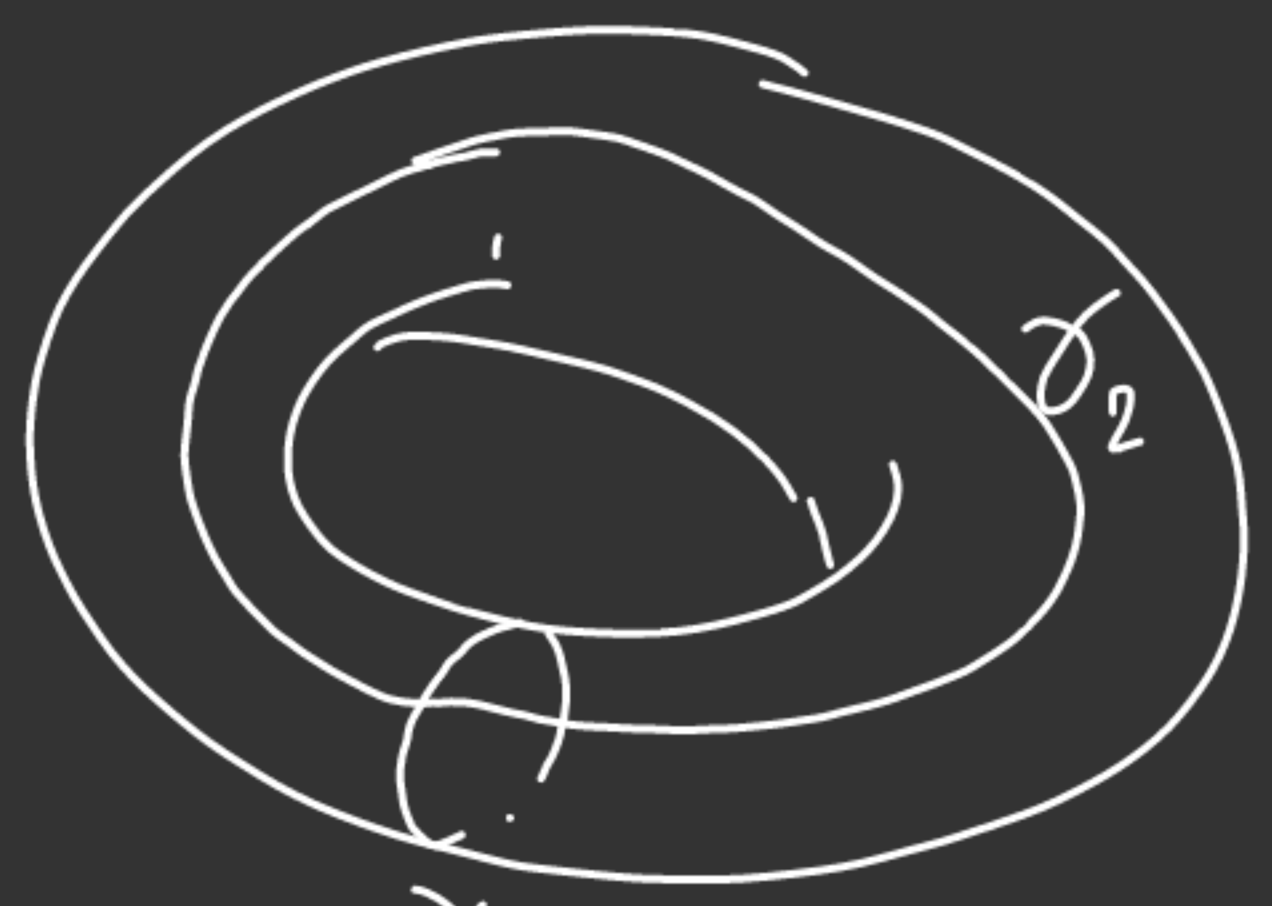
(avec E. Sertöz)

1. What is a period?

$$X = \{ y^2 = P(x) \}$$

deg 3

$$-2y\,dy + P'(x)\,dx = 0$$



$$H_1(X) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$$

$$\omega_i = \int_{\gamma_i} \frac{dx}{y}$$

polynomials

without poles at ∞

$$\Omega^1(X) = \{ A(x,y)\,dx + B(x,y)\,dy \}$$

$$= \mathbb{C} \cdot \frac{dx}{y}$$

$$X \simeq \frac{\mathbb{C}}{\omega_1\mathbb{Z}+\omega_2\mathbb{Z}}$$

$\Omega(X)$: matrix of periods

$H_2(X)$

For surfaces.

$X \subseteq \mathbb{P}^3$ $\dim X = 2$

$\gamma_1, \dots, \gamma_r$

$\alpha_1, \dots, \alpha_m \rightsquigarrow \int_{\gamma_i} \alpha_j$

$\Omega^2(X)$

Ex. $\{x^4+y^3z+xyzw+z^3w+yw^3=0\}$.

deg	1	2	3	4
#s rat. c. $\subseteq X$	0	0	0	133056

2. Numerical computation of periods.

$F(x,y,z,w)$ hom. polynomial of degree 4.

$X = \{F=0\} \subseteq \mathbb{P}^3$.

$\Omega^2(X) = \mathbb{C}\,\omega_X$

$\gamma_1, \dots, \gamma_{22} \in H_2(X)$

$\Omega(X) \in \mathbb{C}^{22}$

$G_t = (1-t)\cdot(x^4+y^4+z^4+w^4) + t\cdot F$

$\{G_0 = 0\}$: Fermat hypersurface

$$\omega_i(t) = \oint \frac{dx\,dy\,dz}{G_t(x,y,z,1)} = \int_{\gamma_i} \omega_X$$

$\rightsquigarrow$ P.F diff eq

$$\mathcal{L}(\omega_i) = 0.$$

$$\sum_i a_i \frac{d^i}{dt^i}\omega = 0$$

3. Picard group.

$X \subseteq \mathbb{P}^3(\mathbb{C})$ $\quad \mathbb{Z}^{22}$

$$\mathrm{Pic}(X) = \left\langle \left\{ \gamma \in H_2(X) \;\middle|\; \begin{array}{l} \gamma \text{ is the homology} \\ \text{class of an alg.} \\ \text{curve} \end{array} \right\} \right\rangle \simeq \mathbb{Z}^{\rho}$$

- $X = \{x^4 + y^4 + w^4 + z^4\}$
  $\rho = \mathrm{rk}(\mathrm{Pic}\,X) = 20$

- X is very general
  $\rho(X) = 1$

Thm (Lefshetz)

$$\mathrm{Pic}(X) = \left\{ \gamma \in H_2(X) \;\middle|\; \int_\gamma \omega_X = 0 \right\}$$

$$= \left\{ (a_i) \in \mathbb{Z}^{22} \;\middle|\; 0 = \sum_i a_i \int_{\gamma_i} \omega_X \right\}$$