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Structure of G-operators in the broad sense

I. G-functions

Def: A G-function in the strict sense (ITSS) is $f(z) = \sum a_n z^n \in \overline{\mathbb{Q}}[[z]]$

broad sense (ITBS)

a) f is solution of a linear differential equation with coeffs in $\overline{\mathbb{Q}}(z)$

b) $\exists C > 0 : \forall n, \boxed{a_n} \leq C^{n+1}$ where $\boxed{\alpha} = \max\limits_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(\alpha)|$

b') $\forall \varepsilon > 0 : \exists n_0(\varepsilon) : \forall n \geq n_0(\varepsilon), \boxed{a_n} \leq (n!)^{\varepsilon}$

c) $\exists C > 0 : \mathrm{den}(a_0, \dots, a_n) \leq C^{n+1}$ $d_n a_0, \dots, d_n a_n \in \mathcal{O}_{\overline{\mathbb{Q}}}$

c') $\forall \varepsilon > 0 : \exists n_0(\varepsilon) \in \mathbb{N} \quad \forall n \geq n_0(\varepsilon) \ \mathrm{den}(a_0, \dots, a_n) \leq (n!)^{\varepsilon}$

Rk. b') $\Rightarrow$ b)

c') $\overset{?}{\Rightarrow}$ c) : conjecture

Ex. $\cdot \sum_n z^n = \dfrac{1}{1-z}$

$\cdot \sum \dfrac{z^n}{n} = -\log(1-z) \qquad \mathrm{lcm}(1, 2, \dots, n) \leq e^{n(1+o(1))}$

$\cdot \mathrm{Li}_s(z) = \sum \dfrac{z^n}{n^s} \qquad \mathrm{Li}_s(1) = \zeta(s)$

- $$\frac{2}{1+\sqrt{1-4z}} = \sum \frac{\binom{2n}{n}}{n+1} z^n$$

- ${}_nF_{n-1}(\alpha, \beta, z)$

- exp is not a G-function.

The E-functions are the $\sum \frac{a_n}{n!} z^n \xleftrightarrow{\text{Laplace transform}}$ such that $\sum a_n z^n$ is G-func°

ex: exp

II. G-operators IT 8 S

$D = \frac{d}{dz}$

1. Fuchsian operators

$$L = B_0(z) D^n + B_1(z) D^{n-1} + \dots + B_n(z) \in \overline{\mathbb{Q}}(z)[D]$$

The singularities of L are the poles of the B_k/B_0.

ordinary pts ——— non ———

↳ basis of solutions $(f_1(z-\alpha), \dots, f_n(z-\alpha))$ $f_i(u) \in \overline{\mathbb{Q}}[[u]]$

<u>Def.</u> 0 is reg. singular if $\forall k$, B_k/B_0 has a pole of order at most k in 0.

$\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is reg sing $\Longleftrightarrow$ 0 is a reg sing point of L_α obtained by $u = z - \alpha$ $\left(u = 1/z\right)$

L is fuchsian if all singularities are regular.

<u>Ex</u>:
- $(1-z)D^2 - D$ is fuchsian
- $D - 1$ is not fuchsian

Frobenius: around α reg sing, there is a basis of solutions

$$\left(f_1(z-\alpha), \ldots, f_n(z-\alpha)\right)(z-\alpha)^{C_\alpha}$$

$f_i \in \overline{\mathbb{Q}}[[u]]$, $C_\alpha \in M_n(\overline{\mathbb{Q}})$

$$\sum_{k,\ell} (z-\alpha)^{\alpha_\ell} \log^k(z-\alpha)\, g_{k,\ell}(z-\alpha)$$

$g_{k,\ell} \in \overline{\mathbb{Q}}[[u]]$

2. Galochkin condition

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad f' = G f \qquad G \in \mathcal{M}_n(\overline{\mathbb{Q}}(z))$$

$$T(z)\, G(z) \in \mathcal{M}_n(\overline{\mathbb{Q}}[z])$$

$$f^{(s)} = G_s f \qquad G_{s+1} = G_s G + G_s'$$

$$T^s \underbrace{G_s}_{s!} \in \mathcal{M}_n(\overline{\mathbb{Q}}[z])$$

Def: Galochkin condition ITSS.

$$q_s = \operatorname{den}\left(TG, T^2\frac{G_2}{2}, \dots, T^s\frac{G_s}{s!}\right) \geq 1$$

$$(\mathcal{G}) \quad \exists\, C > 0 : \quad q_s \leq C^{s+1} \quad \forall s \in \mathbb{N}.$$

ITBS (André): $(\mathcal{G}') \quad \forall \varepsilon > 0,\ \exists\, s_0(\varepsilon) \in \mathbb{N} : \forall s \geq s_0(\varepsilon) \quad q_s \leq (s!)^{\varepsilon}$

Ex: $L = (1-z)D^2 - D \qquad A_L = \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{1-z} \end{pmatrix} \qquad \frac{(1-z)^s}{s!}(A_L)_s = \begin{pmatrix} 0 & \frac{1-z}{s} \\ 0 & 1 \end{pmatrix}$

· $D-1$ $\quad A_L = 1 \rightsquigarrow \frac{(A_L)_s}{s!} = \frac{1}{s!}$ $\quad q_s = s!$

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<u>Def</u>. $L \in \overline{\mathbb{Q}}(z)[D]$ is a G-operator ITSS (ITBS)
if A_L satisfies $(\mathcal{G})$ (resp. $(\mathcal{G}')$)

3. <u>Chudnovsky's Theorem</u>

<u>Th</u>: If $^{\forall i}$ f_i is a G-funct° and $(f_1, \dots, f_r)$ is free over $\overline{\mathbb{Q}}(z)$,
then G satisfies $(\mathcal{G})$

<u>Particular case</u> : $G = A_L$, L is an operator such that

$L(f_1(z)) = 0$ with <u>minimal order</u> μ $\quad (*)$

$(f_1, f_1', \dots, f_1^{(\mu-1)})$ is free over $\overline{\mathbb{Q}}(z)$

$\Longrightarrow$ L is a G-operator !!!

<u>Th</u> (André - Chudnovsky - Katz)

If $(*)$, if $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$, there is a basis of solutions over $\overline{\mathbb{Q}}(z)$ of $L(y(z)) = 0$

of the form $\left(g_1(z-\alpha), \dots, g_n(z-\alpha)\right)(z-\alpha)^{C_\alpha}$

$\to$ G-functions. $(\ast 1)$ $\in M_n(\mathbb{Q})$

- Katz: globally nilpotent $\Longrightarrow$ $(\ast 1)$ (except g_i G-funct$^{\circ s}$)
- André–Bombieri: $(\mathcal{G})$ $\Longrightarrow$ L globally nilpotent
- Chudnovsky: L satisfies $(\mathcal{G})$

III. G-op ITBS

Th (L.): $f(z)$ G-funct° ITBS, L its minimal operator

Then L is fuchsian, satisfies $(\mathcal{G}')$, and if α is an ordinary point, $\exists$ a basis $(f_1(z-\alpha), \dots, f_n(z-\alpha))$ of solutions

f_i G-funct°s ITBS

Dem:

1. L satisfies $(\mathcal{G}')$ (adapting Chudnovsky)

(2) $(\mathcal{C}_y')$ $\Longrightarrow$ 0 reg singularity.

$$f(z) \in \mathbb{Q}[[z]], \quad L = z^n D^n - z^{n-1} B_{n-1}(z) D^{n-1} - \cdots - B_0(z)$$

$$B_i(z) \in \mathbb{Q}(z)$$

$$L(y(z)) = 0 \Longrightarrow z^m y^{(m)}(z) = \sum_{r=0}^{m-1} z^r \underbrace{A_{m,r}(z)}_{\in \mathbb{Q}(z)} y^{(r)}(z)$$

0 sing reg $\Longleftrightarrow$ $\forall k$, B_k has no pole at 0

$$\Longleftrightarrow \lambda := \max\left(v(B_{n-1}), \frac{v(B_{n-2})}{2}, \dots, \frac{v(B_0)}{n} \right) \leq 0$$

$\hookrightarrow$ order of the pole

Assume $\lambda > 0$

$$\tilde{A}_{m,k} = z^{(m-k)\lambda} A_{m,k} \xrightarrow[z \to 0]{} \alpha_{m,k}$$

$$\tilde{B}_k = z^{(n-k)\lambda} B_k \xrightarrow[z \to 0]{} \beta_k$$

$$(\beta_0, \dots, \beta_{n-2}) \neq 0$$

$$U_m = \begin{pmatrix} \alpha_{m,n-1} \\ \vdots \\ \alpha_{m,0} \end{pmatrix}, \qquad U_{m+1} = B\,U_m$$

$$B = \begin{pmatrix} \beta_{n-2} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ \beta_0 & 0 & \cdots & 0 \end{pmatrix} \implies U_m \neq 0 \ \forall m$$

$$\boxed{U_m = B^m \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}$$

$N \in \mathbb{N}$, $N \geq n$.

$$q_N = \mathrm{den}\left(T(z)^{m-n+1} \frac{A_{m,r}(z)}{m!},\ \begin{matrix} 0 \leq r \leq n-1 \\ m \leq N \end{matrix} \right)$$

Assume $(\mathcal{G}')$: $\forall N \geq n_0(\varepsilon)$ $q_N \leq (N!)^{\varepsilon}$

$$\Pi(z) = z^{u}\, T_1(z) \qquad \underbrace{T_1(0)}_{\gamma} \neq 0, \quad T_1(z) \in \mathbb{Z}[z]$$

$$q_N \, \gamma^{N-n+1} \frac{\alpha_{N,r}}{N!} \in \mathbb{Z}^* \longrightarrow$$

$\forall \varepsilon > 0$, $\varepsilon < 1$, $\exists N_0(\varepsilon)$:

$$\left| q_N \, \gamma^{N-n+1} \frac{\alpha_{N,r}}{N!} \right| \leq \frac{(N!)^{\varepsilon}}{N!} \xrightarrow[N\to\infty]{} 0$$

So $\Lambda \mathfrak{S} \partial$: 0 is reg singular.

IV. Applications

(Beukers)
Th 1: $f = (f_1, \dots, f_n)$ family of E-f^{ons} ITSS

$$ {}^t f' = A\, {}^t f, \quad A \in \mathcal{M}_n(\overline{\mathbb{Q}}(z)) $$

$T(z)$ s.t. $T(z)A(z) \in \mathcal{M}_n(\overline{\mathbb{Q}}[z])$, $\xi \in \overline{\mathbb{Q}}$ $\xi T(\xi) \neq 0$

Then if $P(f_1(\xi), \dots, f_n(\xi)) = 0$,

$\hookrightarrow \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$

$\exists\, Q \in \mathbb{Q}[z, X_1, \ldots, X_n]$ such that

$$Q(z, f_1(z), \ldots\ldots, f_n(z)) = 0$$

and $Q(\xi, X_1, \ldots, X_n) = P$.

$\rightsquigarrow$ Siegel–Shidlovskii

André 2014: Th 1 is true ITBS

Using Laplace transform, we give a new proof of André's Thm.