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INRIA Saclay

On Moment Problems with Holonomic Functions

Florent Bréhard, Mioara Joldes, Jean-Bernard Lasserre

Moments of a mesure

$$m_\alpha = \int_{\mathbb{R}^n} x^\alpha \mathrm{d}\mu \qquad \text{for} \quad \alpha \in \mathbb{N}^n$$ [a]

[a] $\alpha = (\alpha_1, \dots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\mathbb{K}[x]_d$ = polynomials of total degree at most d

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- G n-dim **semi-algebraic** set, with $g \in \mathbb{K}[x]$ vanishing on ∂G
- $f : \mathbb{R}^n \to \mathbb{R}$ **D-finite** = satisfies a "complete" system of PDEs

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- ⇝ **Finite determinancy** of such measures
- ⇝ Solved with **Creative Telescoping**, e.g., [Oaku2013] + Takayama's algorithm

Moment Problems and Applications

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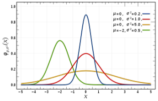
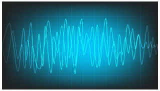
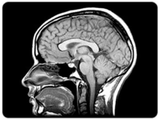
→ **Inverse problem:** reconstruct G and/or f, given finitely many moments m_α



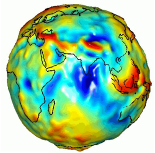
statistics



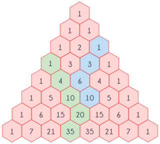
signal processing



medical imaging (MRI)

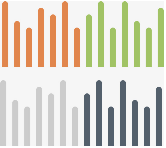


gravimetry



combinatorics

Measures



$(m_\alpha)_{|\alpha| \leqslant N}$

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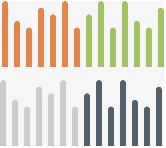

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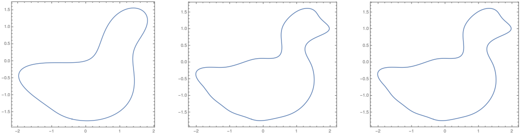
Reconstruction



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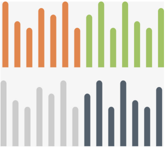

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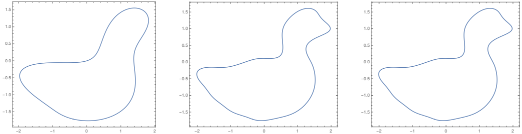
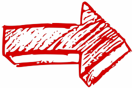


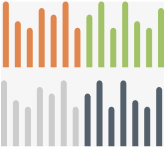


Decision

?

Inverse Problems

Measures



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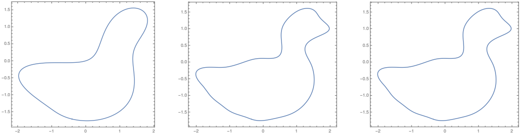

Decision

or

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→ Numerical methods, e.g.:

- Convex polytopes: [GolubMilanfarVarah1999] [GravinLasserrePasechnikRobins2012]
- Planar quadrature domains: [EbenfeltGustafssonKhavinsonPutinar2005]
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Data Recovery From Moments — Exisiting Techniques

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→ Symbolic/algebraic methods:

- A historical starting point: Prony's method
  - reconstructing sparse exponential functions ($\sum_{\alpha \in I} \lambda_\alpha e^{\alpha x}$) from evaluations
  - link with moments of **Dirac** measures
- **Multivariate** extensions of Prony's method, e.g., [Mourrain2018]
- Reconstructing **univariate** piecewise D-finite densities: [Batenkov2009]

Exact Support and/or Density Reconstruction

- Lasserre and Putinar's exact reconstruction algorithm (2015)

Inverse Problem: Exponential-Polynomial Measure, Algebraic Support

Let $G \subset \mathbb{R}^n$, bounded open set, whose algebraic boundary is included in the zero set of a polynomial $g \in \mathbb{K}[x]_d$, and $f(x) = \exp(p(x))$ with $p \in \mathbb{K}[x]_s$. Given p, degree d and moments m_α up to order $|\alpha| = 3d + s$, the coefficients of g can be exactly recovered.

- Key idea: Linear recurrences satisfied by the moments + Stokes' Theorem

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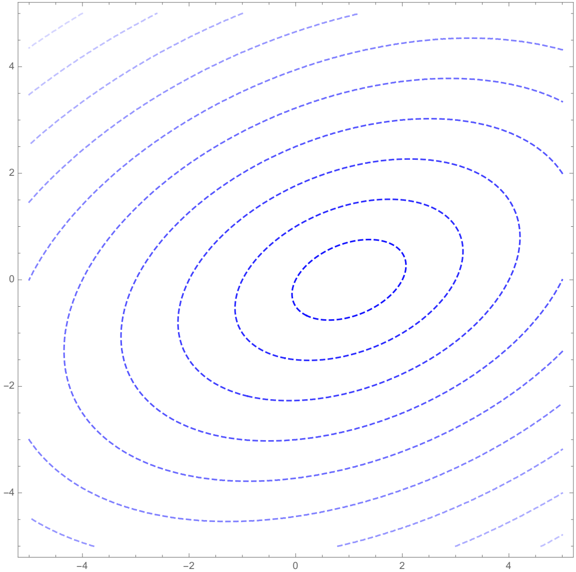

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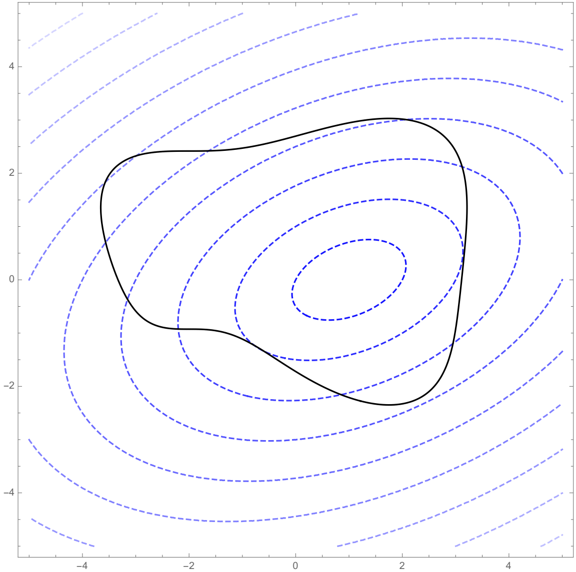

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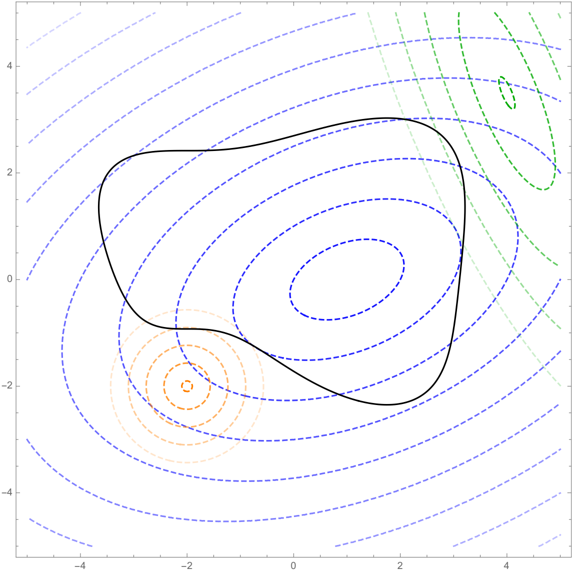

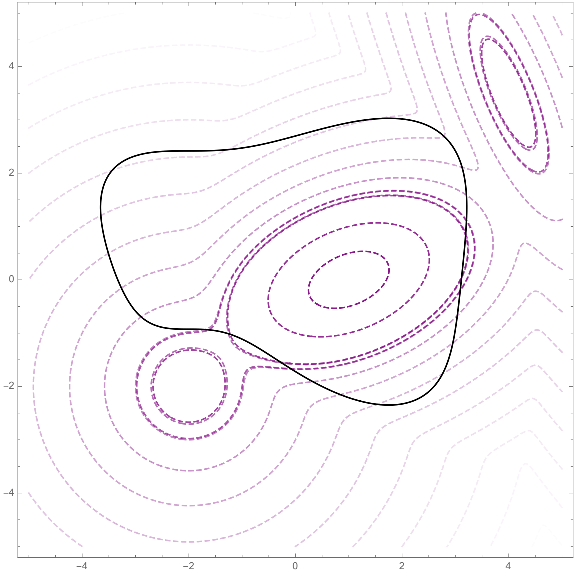
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Our contribution: a computer algebra approach

- generalization in the framework of holonomic distributions
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Our contribution: a computer algebra approach

- generalization in the framework of holonomic distributions
  - ⇒ they satisfy (as a generalized function) a "complete" system of linear PDEs/ODEs with polynomial coefficients
- exact recovery of both support and Exponential-Polynomial density $f = \exp(p)$, with explicit bound on the required number of moments
- similar algorithm for D-finite density, but no a priori bound on the required number of moments

Outline

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1. Differential Ore Algebras

– Differential operators: **non-commutative**, spanned by $x_1, \partial_{x_1}, \ldots, x_n, \partial_{x_n}$

$$\partial_{x_i} f = f'_{x_i} \qquad (x_i f)'_{x_i} = x_i f'_{x_i} + f \qquad \Rightarrow \qquad \partial_{x_i} x_i = x_i \partial_{x_i} + 1$$

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Example: Exponential-Polynomial Density

$f(x) = c\exp(p(x))$ with $p \in \mathbb{K}_s[x]$ (e.g., Gaussian distribution)

$f'_{x_i} - p'_{x_i} f = 0$ $\Rightarrow$ $\mathfrak{Ann}(f)$ generated by the $\partial_{x_i} - p'_{x_i}$ $\Rightarrow$ f is **D-finite**

2. Difference Ore Algebras

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Goals

Recurrences for the moments $m_\alpha = \int_G x^\alpha f(x)\mathrm{d}x$:

- **Direct problem:** $\mathfrak{I} \subseteq \mathfrak{Ann}(f) \overset{?}{\to} \mathfrak{J} \subseteq \mathfrak{Ann}(m_\alpha)$
- **Inverse problem:** Reconstruct G and $\mathfrak{I} \subseteq \mathfrak{Ann}(f)$ from sufficiently many m_α

- Measure $\mu = f\mathbf{1}_G$ as a linear functional:

$$\langle f\mathbf{1}_G, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x) \mathbf{1}_G(x) \mathrm{d}x = \int_G \varphi(x) f(x) \mathrm{d}x$$

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Let $G = [-1, 1]$, $\quad f = 1$, $\quad$ and $\mu = \mathbf{1}_G$

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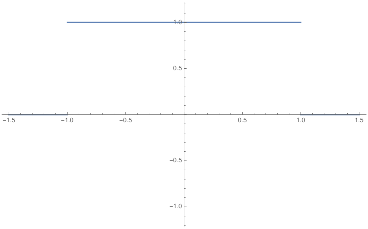

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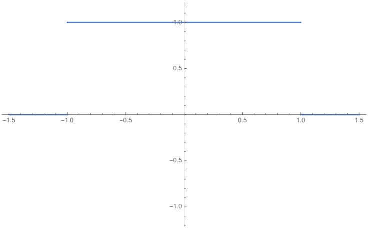

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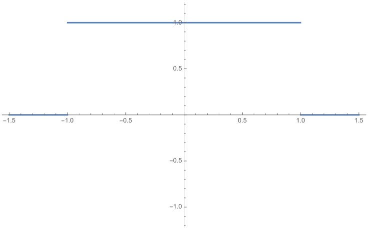

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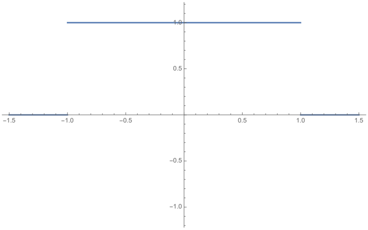

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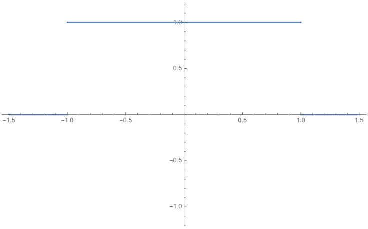

Holonomic Measures

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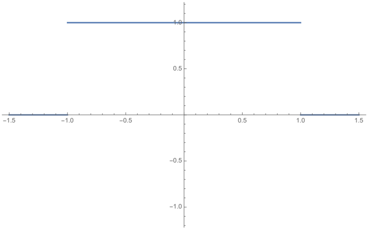

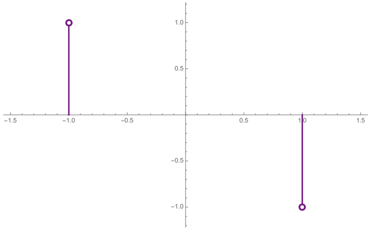

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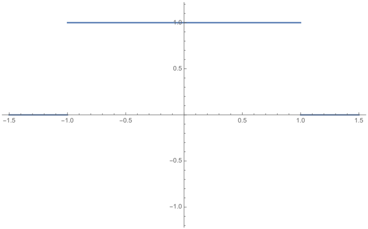

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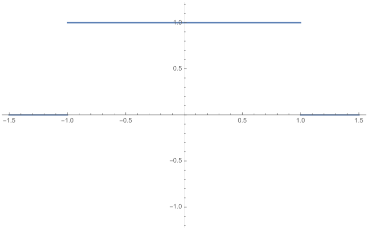

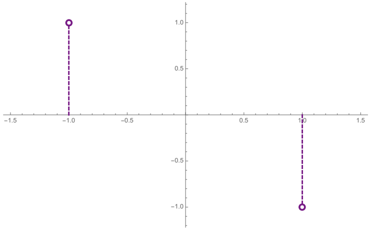

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Example: Lebesgue measure over a segment

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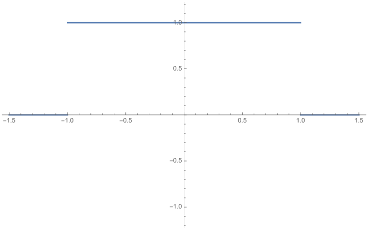

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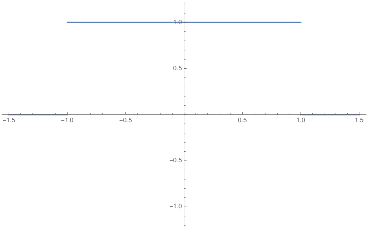

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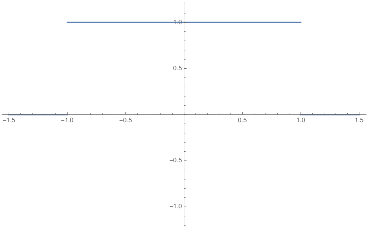

Holonomic Measures

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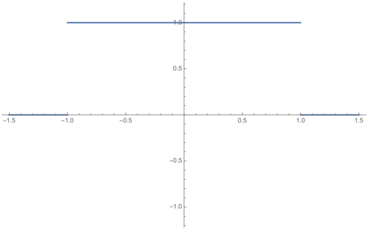

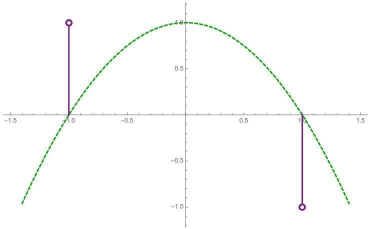

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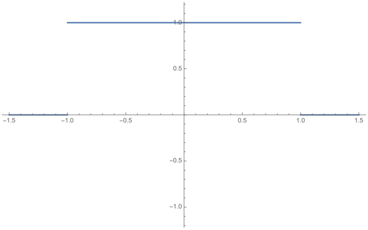

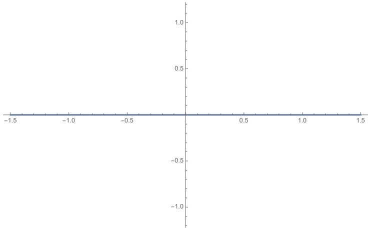

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- Ore polynomials acting on **distributions**: $\langle L\,T, \varphi\rangle = \langle T, L^*\,\varphi\rangle$

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- $\mathfrak{Ann}(T)$ in $\mathbb{K}[x]\langle\partial_x\rangle$ $\quad \Rightarrow \quad$ **holonomic** instead of D-finite

– Again, with $G = [-1, 1]$, and using $\varphi = x^k$:

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⇒ Recurrence satisfied by the moments (m_k):

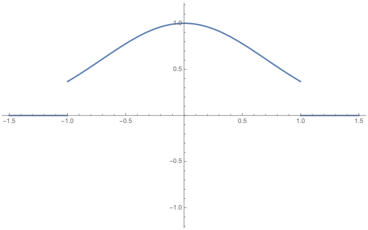
$$(k+2)\boldsymbol{m}_{\boldsymbol{k+1}} - k\boldsymbol{m}_{\boldsymbol{k-1}} = 0$$

This is indeed true...

$$m_k = \int_{-1}^{1} x^k \mathrm{d}x = \begin{cases} \frac{2}{k+1} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

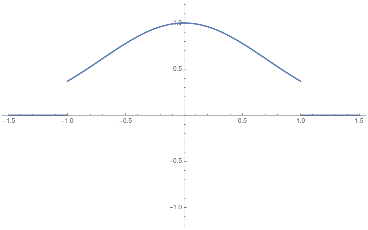
– $\mu = f\mathbf{1}_G$ with $G = [-1, 1]$ and $f(x) = \exp(-x^2)$:

$$\langle \mu, \varphi \rangle = \int_{-1}^{1} \varphi f \mathrm{d}x$$



- $\mu = f\mathbf{1}_G$ with $G = [-1, 1]$ and $f(x) = \exp(-x^2)$:

$$\int_{-1}^{1} \varphi \, (\partial_x - 2x) f \, \mathrm{d}x$$



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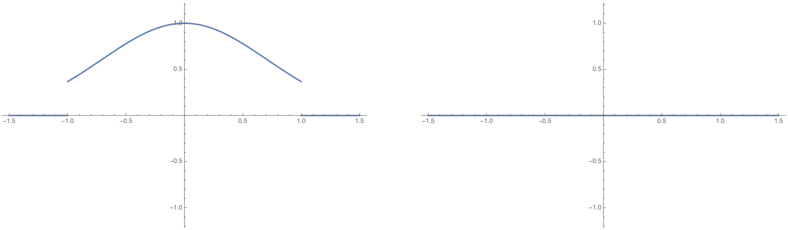

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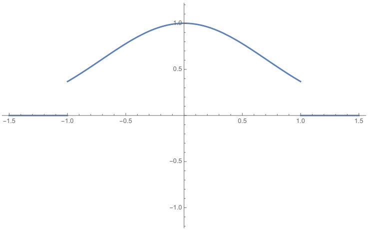
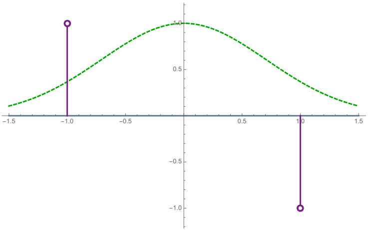
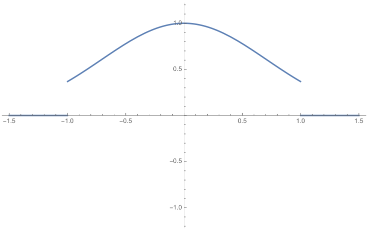




Using Integration by Parts

– $\mu = f\mathbf{1}_G$ with $G = [-1, 1]$ and $f(x) = \exp(-x^2)$:

$$0 = \int_{-1}^{1} \varphi \underbrace{(1 - x^2)(\partial_x - 2x) f}_{=0} \, \mathrm{d}x =$$



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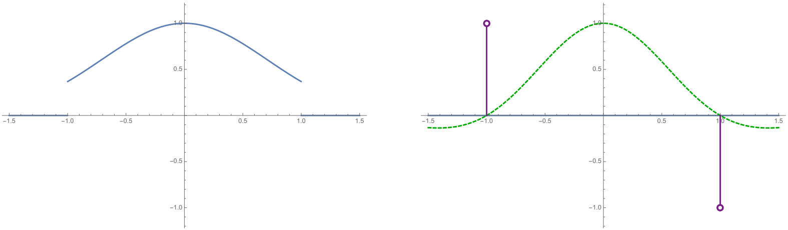

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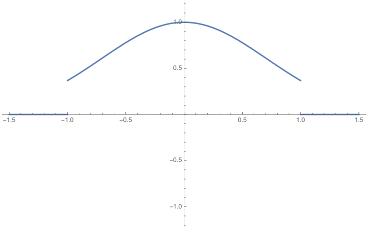
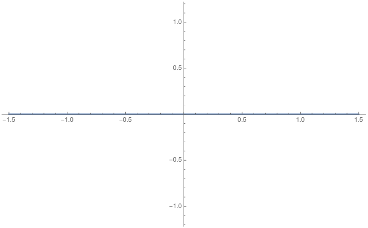
$\Rightarrow (1-x^2)(\partial_x - 2x) \in \mathfrak{Ann}(\mu)$





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– replace $\varphi = x^k$ to obtain a recurrence

$$\int_{-1}^{1} (\partial_x + 2x)(x^2-1) x^k \, f(x) \mathrm{d}x = 0$$

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$\Rightarrow$ Recurrence for the m_k:

$$2m_{k+3} + km_{k+1} - km_{k-1} = 0$$

$\mu = f\mathbf{1}_G, \quad L \in \mathbb{K}[x]\langle\partial_x\rangle$ of order r,

- Use **Lagrange identity**:

$$\varphi\ (L\,f)\ -\ (L^*\,\varphi)\ f\ =\ \partial_x\ \mathcal{L}_L(f,\varphi)$$

→ $\mathcal{L}_L$ bilinear concomitant in f, φ with derivatives of order $\leqslant r-1$

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The General Case

$\mu = f\mathbf{1}_G, \quad L \in \mathbb{K}[x]\langle\partial_x\rangle$ of order $r, \qquad x = (x_1, \ldots, x_n)$

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$$-\int_G \varphi\,(L\,f)\,\mathrm{d}x \;-\; \overbrace{\int_G (L^*\,\varphi)\,f\,\mathrm{d}x}^{\langle L\,\mu,\varphi\rangle} \;=\; \int_G \nabla\cdot\mathcal{L}_L(f,\varphi)\,\mathrm{d}x \;=\; \int_{\partial G}\mathcal{L}_L(f,\varphi)\cdot\vec{n}\,\mathrm{d}S$$

→ use **Stokes' theorem**

$\mu = f\mathbf{1}_G, \quad L \in \mathbb{K}[x]\langle\partial_x\rangle$ of order $r, \qquad x = (x_1, \ldots, x_n)$

– Use **Lagrange identity**:

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$$-\int_G \mathbf{g}^{\mathbf{r}}\varphi\,(L f)\,\mathrm{d}x - \overbrace{\int_G (L^*\,\mathbf{g}^{\mathbf{r}}\varphi)\,f\,\mathrm{d}x}^{\langle \mathbf{g}^{\mathbf{r}} L\,\mu,\varphi\rangle} = \int_G \nabla\cdot\mathcal{L}_L(f,\mathbf{g}^{\mathbf{r}}\varphi)\,\mathrm{d}x = \overbrace{\int_{\partial G}\mathcal{L}_L(f,\mathbf{g}^{\mathbf{r}}\varphi)\cdot\vec{n}\,\mathrm{d}S}^{=\,\mathbf{0}}$$

→ where $\mathbf{g} = 0$ on ∂G → use **Stokes' theorem**

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→ if $L \in \mathfrak{Ann}(f)$ → where $\mathbf{g} = 0$ on ∂G → use **Stokes' theorem**

⇒ $\overline{L} = \mathbf{g}^r L \in \mathfrak{Ann}(\mu)$

– Translate $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$ into a recurrence on (m_α):

$$x_i \quad \to \quad S_{\alpha_i} \qquad\qquad \partial_{x_i} \quad \to \quad -\alpha_i S_{\alpha_i}^{-1}$$

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Direct Problem

1. $\{L_1, \ldots, L_k\} \subseteq \mathfrak{Ann}(f)$ D-finite
2. $\{\overline{L}_1, \ldots, \overline{L}_k\} \subseteq \mathfrak{Ann}(\mu)$
3. Translate into $\{R_1, \ldots, R_k\} \subseteq \mathfrak{Ann}(m_\alpha)$
4. Gröbner basis algo on $\{R_1, \ldots, R_k\}$

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Theorem

If $f(x) = \exp(p(x))$ and $g = 0$ on ∂_G s.t. $\{x \in \mathbb{C}^n \mid g(x) = 0 \text{ and } \nabla g(x) = 0\} = \varnothing$, then the recurrences system is holonomic.

⇒ Conjecture for the general case?

From Differential Equations to Recurrences

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⇒ Conjecture for the general case?

Inverse Problem

∘ Reconstruct $\overline{L}_i$, then g and L_i from the given moments m_α

⇒ Translation $\overline{L}_i \leftrightarrow R_i$ is linear

⇒ Holonomicity **not** needed

Outline

– To reconstruct g vanishing on ∂G and $L \in \mathfrak{Ann}(f)$ of order r:

1. Make an **ansatz** $\widetilde{L}$ for $\overline{L} = g^r L \in \mathfrak{Ann}(\mu)$
2. Find the coefficients of $\widetilde{L}$ by solving the **linear system**:

$$\langle \widetilde{L}\mu, x^\alpha \rangle = \langle \mu, \widetilde{L}^* x^\alpha \rangle = \int_G (\widetilde{L}^* x^\alpha) f(x) \mathrm{d}x = 0, \qquad |\alpha| \leqslant \boldsymbol{N} \qquad (LS_N)$$

   requiring moments m_α for $|\alpha| \leqslant \boldsymbol{N} + \ldots$

3. Extract g and L from $\widetilde{L}$ using (numerical) GCDs

Inverse Problem — Roadmap and Issues

– To reconstruct g vanishing on ∂G and $L \in \mathfrak{Ann}(f)$ of order r:

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   requiring moments m_α for $|\alpha| \leqslant \boldsymbol{N} + \ldots$
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– Issues to be handled:

- **False** solutions in (LS_N): $\widetilde{L} \notin \mathfrak{Ann}(\mu)$?
- How many moments m_α: **a priori bounds** on $\boldsymbol{N}$?
- Can g and L be always extracted from $\widetilde{L} \in \mathfrak{Ann}(\mu)$?

Outline

- $\mu = f\mathbf{1}_G$ with $f(x) = \exp(p(x))$ for $p \in \mathbb{K}[x]_s$ and $g \in \mathbb{K}[x]_d$ vanishing on ∂G

$$\overline{L}_i \quad = \quad g(\partial_{x_i} - p'_{x_i}) \quad \in \quad \mathfrak{Ann}(\mu)$$

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Algorithm Reconstructexppoly

Input: Moments m_α of μ for $|\alpha| \leqslant \boldsymbol{N} + d + s - 1$
Output: Polynomials $\widetilde{g}$ and $\widetilde{p}$

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Theorem — Correctness of ReconstructExpPoly

If $\boldsymbol{N} \geqslant 3d + s - 1$, then ReconstructExpPoly computes:

- $\widetilde{g} = \lambda g$ with $\lambda \neq 0$
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If $N \geqslant 3d + s - 1$, then ReconstructExpPoly computes:

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Reconstruction of Exp-Poly Densities — Proof

Theorem — Correctness of ReconstructExpPoly

If $N \geqslant 3d + s - 1$, then ReconstructExpPoly computes:

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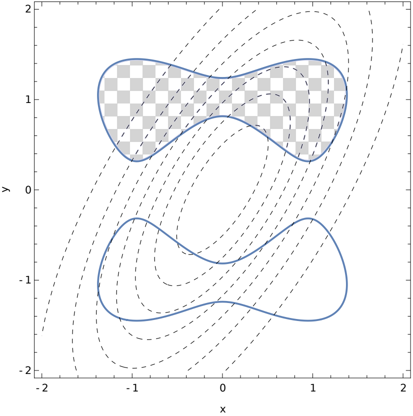
→ Take $\varphi = \widetilde{g} g'_{x_i}$ of degree $2d - 1 \quad \Rightarrow \quad \widetilde{g}^2 {g'_{x_i}}^2 \frac{f}{\|\nabla g\|} = 0$ on $\partial G \quad \Rightarrow \quad \widetilde{g} = 0$ on ∂G

→ Reconstruction of:

$$f(x,y) = \exp(-x^2 + xy - y^2/2) \qquad \text{and} \qquad g(x,y) = \left(x^2 - 9/10\right)^2 + \left(y^2 - 11/10\right)^2 - 1$$



Moments $(m_{ij})_{i+j\leqslant 18}$ with **10** digits of accuracy

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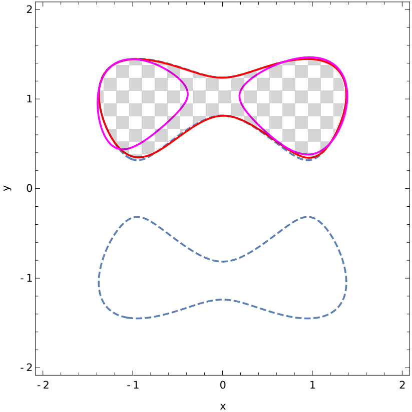

Moments $(m_{ij})_{i+j\leqslant 18}$ with **4**, **6**, **8** digits of accuracy

Example — Algebraic Support, Gaussian Measure

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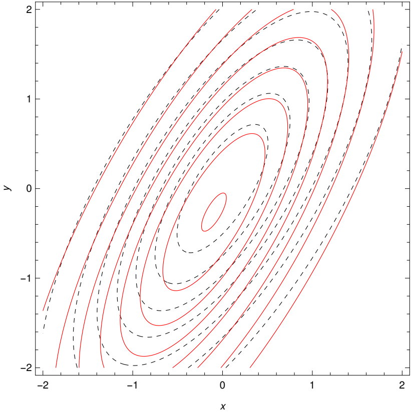

Moments $(m_{ij})_{i+j\leqslant 18}$ with **8** digits of accuracy

Outline

- $\mu = f\mathbf{1}_G$ with $g \in \mathbb{K}[x]_d$ vanishing on ∂G, and $\{L_1, \ldots, L_n\}$ **rectangular** system for f:

$$L_i = q_{ir_i}\partial_{x_i}^{r_i} + \cdots + q_{i1}\partial_{x_i} + q_{i0} \in \mathfrak{Ann}(f) \cap \mathbb{K}[x]\langle\partial_{x_i}\rangle$$

Density and Support Reconstruction in the General Case

- $\mu = f\mathbf{1}_G$ with $g \in \mathbb{K}[x]_d$ vanishing on ∂G, and $\{L_1, \ldots, L_n\}$ **rectangular** system for f:

$$\overline{L}_i = g^{r_i}(q_{ir_i}\partial_{x_i}^{r_i} + \cdots + q_{i1}\partial_{x_i} + q_{i0}) \in \mathfrak{Ann}(\mu) \cap \mathbb{K}[x]\langle\partial_{x_i}\rangle \qquad h_{ij} = g^{r_i} q_{ij} \in \mathbb{K}[x]_s$$

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Algorithm ReconstructDensity

Input: Moments m_α of μ for $|\alpha| \leqslant N + s$
Output: A rectangular system $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ for f

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Input: Moments m_α of μ for $|\alpha| \leqslant N + s$
Output: A rectangular system $\{\widetilde{L}_1, \dots, \widetilde{L}_n\}$ for f

1. Build ansatz $\widetilde{L}_i = \widetilde{h}_{ir_i}\partial_{x_i}^{r_i} + \cdots + \widetilde{h}_{i0}$ for $1 \leqslant i \leqslant n$
2. Compute coefficients of $\widetilde{h}_{ij}$ with nontrivial solution of
$$\langle \mu, \widetilde{L}_i^* x^\alpha \rangle = 0, \qquad 1 \leqslant i \leqslant n, \quad |\alpha| \leqslant N$$
3. Extract (numerical) GCD polynomial factor in $\widetilde{L}_i$

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Algorithm ReconstructSupport

Input: Rectangular $\{L_1, \dots, L_n\}$ and m_α for $|\alpha| \leqslant \boldsymbol{N} + dr + \max_{ij}\{\deg(q_{ij}) - j\}$
Output: Polynomial $\widetilde{g} \in \mathbb{K}[x]_d$

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Output: Polynomial $\widetilde{g} \in \mathbb{K}[x]_d$

1. Compute coefficients of ansatz $\widetilde{h} \in \mathbb{K}[x]_{dr}$ with nontrivial solution of
$$\langle \mu, (\widetilde{h}L_i)^* x^\alpha \rangle = 0, \qquad 1 \leqslant i \leqslant n, \quad |\alpha| \leqslant \boldsymbol{N}$$
2. $\widetilde{g} \leftarrow$ (numerical) GCD of $\{\widetilde{h}, \widetilde{h}'_{x_1}, \ldots, \widetilde{h}'_{x_n}\}$

Theorem — Correctness of ReconstructDensity

For N large enough, the rectangular system $\{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ computed by ReconstructDensity is in $\mathfrak{Ann}(f)$.

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Theorem — Correctness of ReconstructSupport

ReconstructSupport computes $\widetilde{g} = \lambda g$ with $\lambda \neq 0$ whenever $q_{ir} \neq 0$ on ∂G and $N \geqslant (2r-1)d + (d-1)b + s$ where:

- $r = \max_{1 \leqslant i \leqslant n} r_i$, orders of the L_i
- $b = r \mod 2$
- $s = \max_{1 \leqslant i \leqslant n} \{\deg(q_{ir})\}$ maximal degree of the head coefficients

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Proof.

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- $0 = \langle \widetilde{h} L_i \mu, \varphi \rangle = \int_G \varphi \widetilde{h} (L_i f) \mathrm{d}x - \int_{\partial G} \mathcal{L}_{L_i}(f, \widetilde{h}\varphi) \vec{e}_i \cdot \vec{n} \, \mathrm{d}S \qquad$ for $\varphi \in \mathbb{K}[x]_N$

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- Suppose for contradiction that $\widetilde{h} = g^k h_0$ with $g \nmid h_0$ and $\boldsymbol{k < r}$

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- $0 = \langle \widetilde{h} L_i \mu, \varphi \rangle = \underbrace{\int_G \varphi \widetilde{h}(L_i f) \mathrm{d}x}_{=0} - \int_{\partial G} \mathcal{L}_{L_i}(f, \widetilde{h}\varphi) \vec{e}_i \cdot \vec{n} \, \mathrm{d}S \qquad$ for $\varphi \in \mathbb{K}[x]_N$
- Suppose for contradiction that $\widetilde{h} = g^k h_0$ with $g \nmid h_0$ and $\boldsymbol{k < r}$

$$\begin{aligned}
\mathcal{L}_{L_i}(f, \widetilde{h}\varphi) = f & \left[q_{i1} \widetilde{h}\varphi - \partial_{x_i}(q_{i2}\widetilde{h}\varphi) + \cdots + (-1)^{r-1} \partial_{x_i}^{r-1}(q_{ir}\widetilde{h}\varphi) \right] \\
& + \partial_{x_i}(f) \left[q_{i2} \widetilde{h}\varphi - \partial_{x_i}(q_{i3}\widetilde{h}\varphi) + \cdots + (-1)^{r-2} \partial_{x_i}^{r-2}(q_{ir}\widetilde{h}\varphi) \right] \\
& + \ldots \\
& + \partial_{x_i}^{r-1}(f) \; q_{ir} \widetilde{h}\varphi.
\end{aligned}$$

Theorem — Correctness of ReconstructSupport

ReconstructSupport computes $\widetilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geqslant (2r-1)d + (d-1)b + s$

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Theorem — Correctness of ReconstructSupport

ReconstructSupport computes $\widetilde{g} = \lambda g$ with $\lambda \neq 0$ whenever:

- $N \geqslant (2r-1)d + (d-1)b + s$
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Proof.

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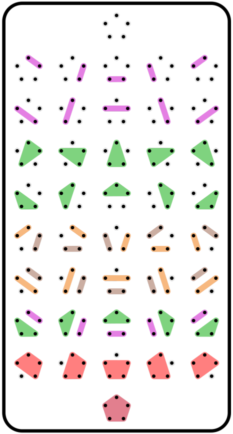
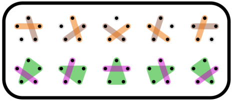
⇒ **Contradiction:** $h_0 = 0$ on ∂_G, hence $g \mid h_0$

→ Express **Catalan numbers** as moments of a measure μ:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \stackrel{?}{=} \int_I x^n f(x)\mathrm{d}x$$





The Singular Case — Example in Combinatorics

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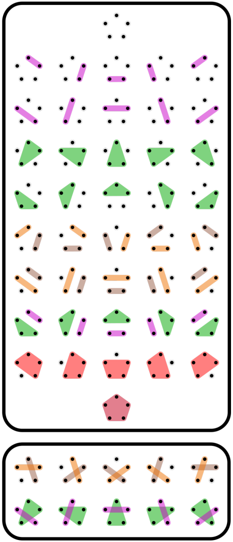

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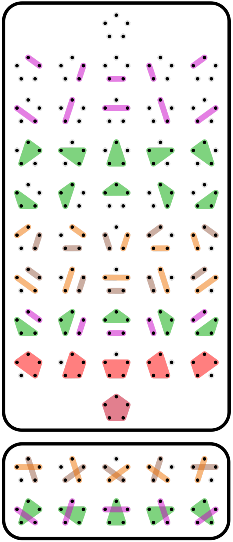
$$(n+2)S_n - (4n+2)$$



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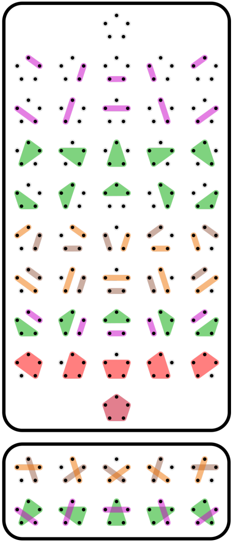

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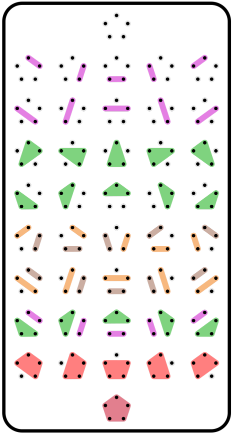

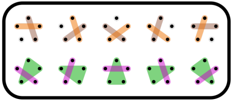

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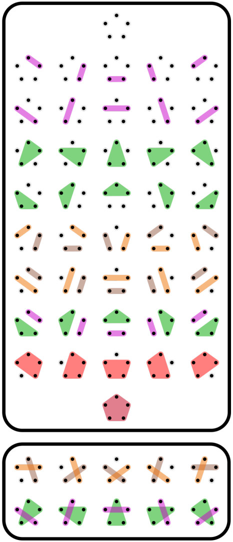

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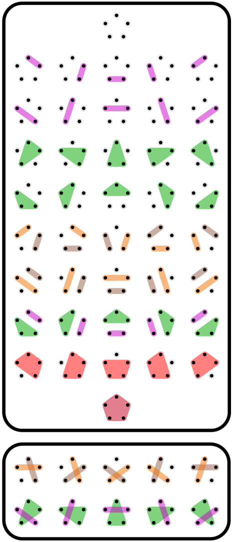
⇒ $(4x - x^2)\partial_x + 2 \in \mathfrak{Ann}(\mu)$



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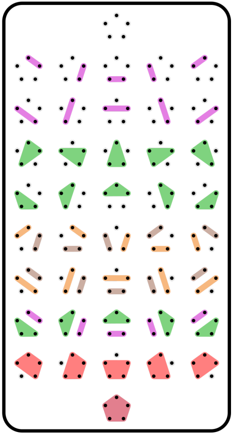
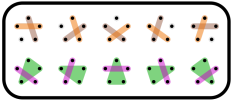
$$\Rightarrow \quad (4x - x^2)\partial_x + 2 \in \mathfrak{Ann}(\mu) \qquad \boldsymbol{g = 1\ ?}$$

$$C_n = \lambda \int_{-\infty}^{+\infty} x^n \sqrt{\frac{4-x}{x}}\mathrm{d}x \quad ?$$





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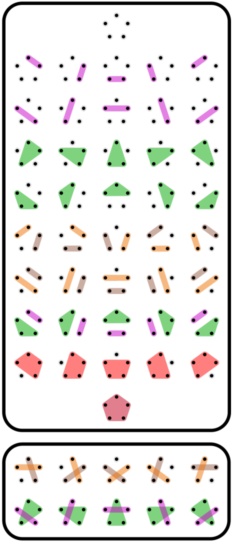
$$C_n = \frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\mathrm{d}x \quad ?$$



Outline

Some Limits and Perspectives

- A priori bounds for $\boldsymbol{N}$ in the general case with unknown D-finite density?
- Full determination of the density, including initial conditions
- Extracting the component of $V(g)$ corresponding to ∂G

– Is there an explicit bound N_0 on N s.t. for ansatz $\widetilde{L}$ of $\overline{L} = g^r L$:

$$\langle \widetilde{L}\mu, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathbb{K}[x]_N \qquad \Rightarrow \qquad \widetilde{L}\mu = 0 \qquad \text{when } N \geqslant N_0 \qquad ?$$

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- The proof of the Exp-Poly density case doesn't generalize:

$$\langle \widetilde{L}\mu, \varphi \rangle = \int_G \varphi (\widetilde{L} f) \, \mathrm{d}x - \int_{\partial G} \mathcal{L}_{\widetilde{L}}(f, \varphi) \cdot \vec{n} \, \mathrm{d}S$$

Bounds for the Number of Moments?

- Is there an explicit bound $\boldsymbol{N_0}$ on $\boldsymbol{N}$ s.t. for ansatz $\widetilde{L}$ of $\overline{L} = g^r L$:

$$\langle \widetilde{L}\mu, \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathbb{K}[x]_{\boldsymbol{N}} \quad \Rightarrow \quad \widetilde{L}\mu = 0 \quad \text{when } \boldsymbol{N} \geqslant \boldsymbol{N_0} \quad ?$$

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$$\langle \widetilde{L}\mu, \varphi\rangle = \underbrace{\int_G \varphi\,(\widetilde{L} f)\,\mathrm{d}x}_{???} - \int_{\partial G} \mathcal{L}_{\widetilde{L}}(f, \varphi) \cdot \vec{n}\,\mathrm{d}S$$

– Such a bound N_0 depending only on the structure of $\widetilde{L}$ cannot exist:

Example [Batenkov2009] — Legendre Polynomials P_n over $[-1, 1]$

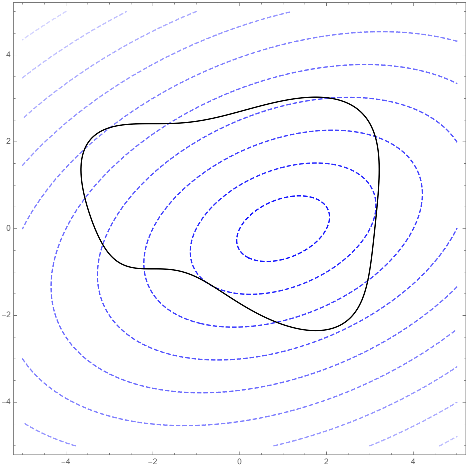
$P_n(x)$ annihilated by $L_n = \partial_x\left((1 - x^2)\partial_x\right) + n(n+1) \quad \Rightarrow \quad$ common ansatz $\widetilde{L}$

but $\quad m_k^{(n)} = \int_{-1}^{1} x^k P_n(x)\mathrm{d}x = 0 \quad$ for $k < n \quad$ and $\quad m_n^{(n)} > 0$

→ Explicit bounds depending on upper bounds on the coefficients of $\widetilde{L}$?

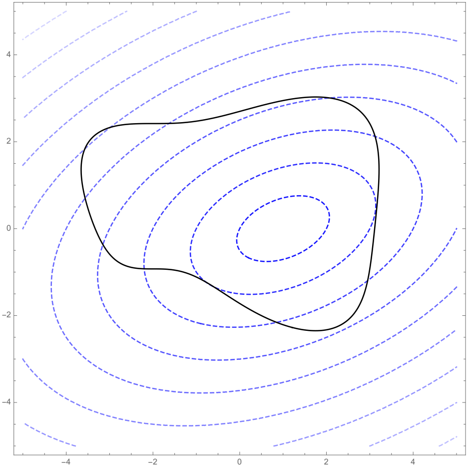
- Algorithm ReconstructDensity only computes a system $\widetilde{\mathfrak{I}} = \{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ but not the initial conditions that fully characterize f

Reconstructing Initial Conditions of the Density



$$f(x,y) = \lambda_1 e^{p_1(x,y)}$$

$$p_1 = -\frac{1}{2}\begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix}^T \Sigma_1^{-1} \begin{pmatrix} x - \mu_{x1} \\ y - \mu_{y1} \end{pmatrix}$$



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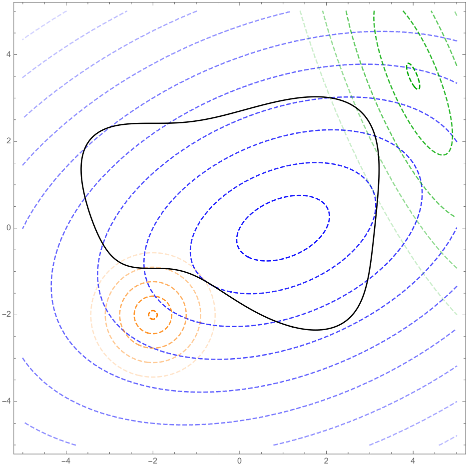
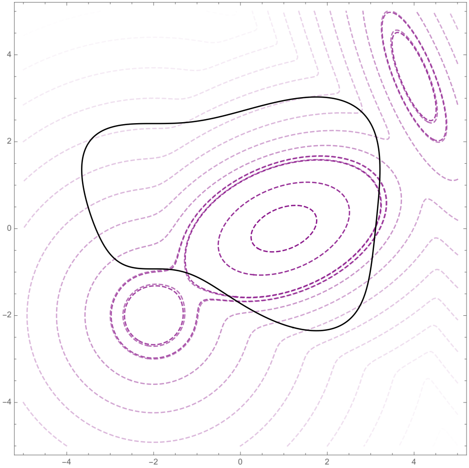


$$f(x,y) = \lambda_1 e^{p_1(x,y)} + \lambda_2 e^{p_2(x,y)} + \lambda_3 e^{p_3(x,y)}$$

$$p_i = -\frac{1}{2} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix}^T \Sigma_i^{-1} \begin{pmatrix} x - \mu_{xi} \\ y - \mu_{yi} \end{pmatrix}$$



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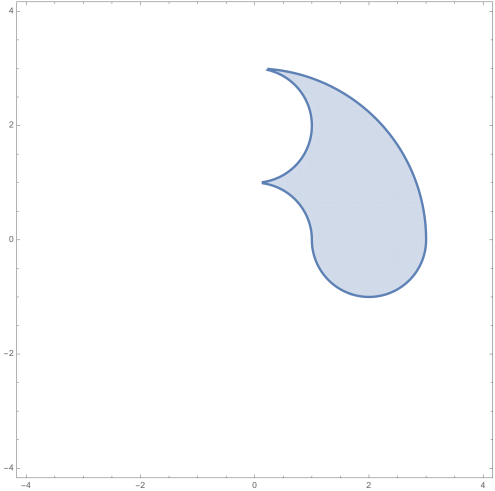
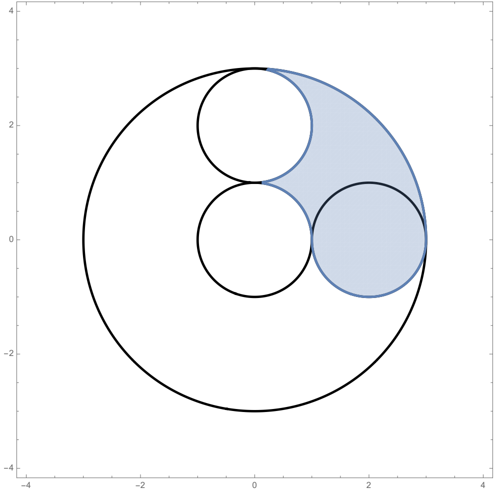
– Algorithm ReconstructDensity only computes a system $\widetilde{\mathfrak{I}} = \{\widetilde{L}_1, \ldots, \widetilde{L}_n\}$ but not the initial conditions that fully characterize f

→ Solution: compute initial moments for a basis of solution densities of $\widetilde{\mathfrak{I}}$

- **Optimization techniques**, e.g., [HenrionLasserreSavorgnan2009]
- **Computer algebra**, e.g., [LairezMezzarobbaElDin2019]

Isolation of the Topological Boundary





$$I(\partial G) = (g) \quad \text{with} \quad g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$$

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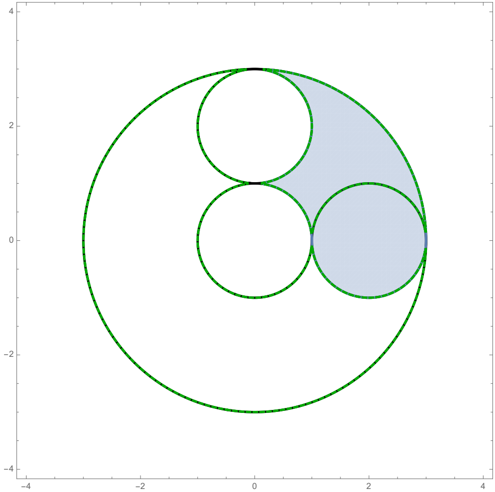

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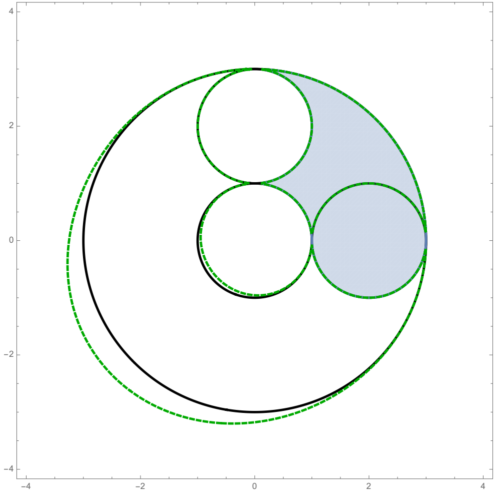
$\widetilde{g}$ reconstructed using **6** digits accuracy for the moments (m_α)

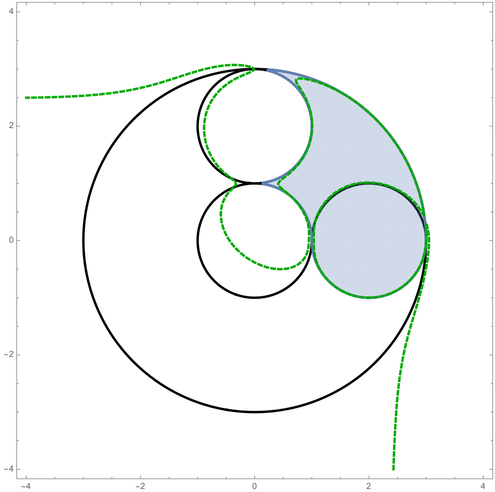
Isolation of the Topological Boundary



$I(\partial G) = (g)$ with $g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$

$\tilde{g}$ reconstructed using **4** digits accuracy for the moments (m_α)

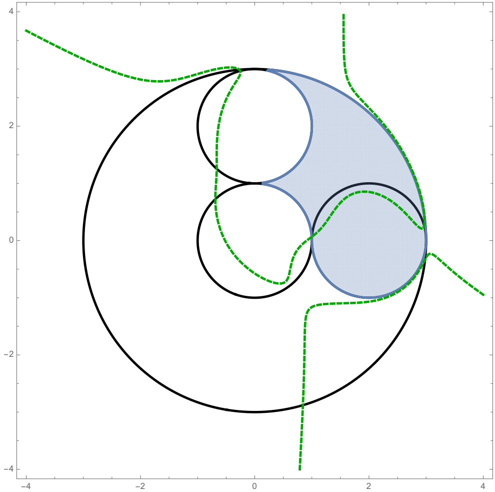
Isolation of the Topological Boundary



$I(\partial G) = (g)$ with $g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$

$\tilde{g}$ reconstructed using **2** digits accuracy for the moments (m_α)

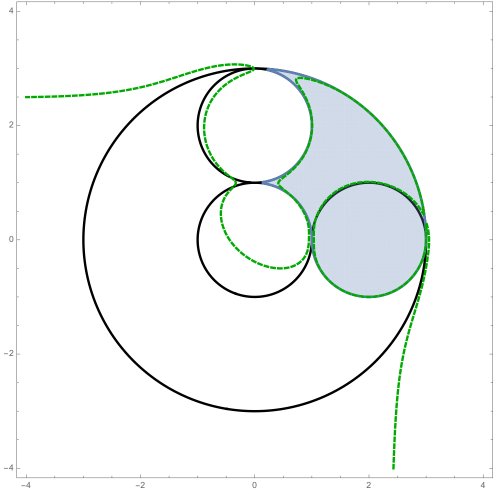
Isolation of the Topological Boundary



$I(\partial G) = (g)$ with $g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$

$\tilde{g}$ reconstructed using **1** digit accuracy for the moments (m_α)

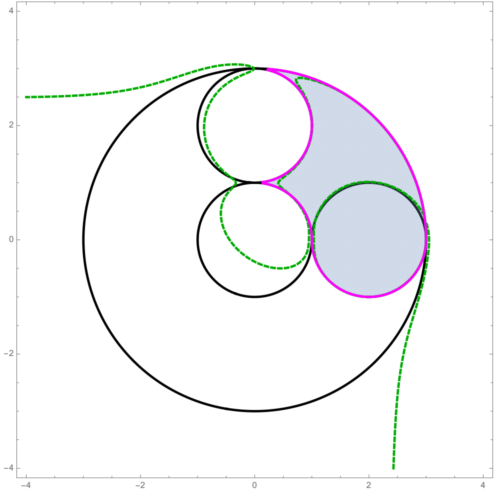
Isolation of the Topological Boundary



$I(\partial G) = (g)$ with $g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$

$\tilde{g}$ reconstructed using 2 digits accuracy for the moments (m_α)

Isolation of the Topological Boundary



$I(\partial G) = (g)$ with $g(x,y) = (x^2+y^2-9)(x^2+y^2-1)((x-2)^2+y^2-1)(x^2+(y-2)^2-1)$

$\partial G \approx \{(x,y) \mid g(x,y) = 0 \text{ and } \mathbb{E}[\widetilde{g}(x,y)^2] \leqslant \varepsilon\}$, $\widetilde{g} \leftarrow$ randomly perturbed $(\widetilde{m}_\alpha)$

Contributions:

- Extension of [LasserrePutinar2015] to reconstruction of unknown **Exp-Poly** density and unknown semi-algebraic support

→ Explicit bound for the number N of required moments

- Reconstruction algorithm for unknown **holonomic** density and unknown semi-algebraic support
- Numerical experiments using **least-squares** approximation when approximate moments are known

Future work:

- Generic bounds for N depending on the magnitude of the coefficients
- **Numerical** aspects: robustness w.r.t. approximate moments, or nonpolynomial boundary
- Isolation of the **topological** boundary via perturbation techniques
- Application to problems involving **combinatorial sequences**

Thanks!