Periods

Numerical computation and applications

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Séminaire de lancement ANR « De rerum natura »

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What is a period?

A **period** is the integral on a closed path of a rational function in one or several variables with *rational* coefficients.

"Rational coefficients" may mean

- coefficients in $\mathbb{Q}$
- coefficients in $\mathbb{C}(t)$, **the period is a function of** t.

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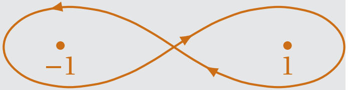

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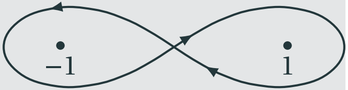
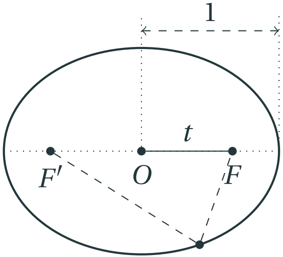


An ellipse

eccentricity t

major radius 1

perimeter $E(t)$



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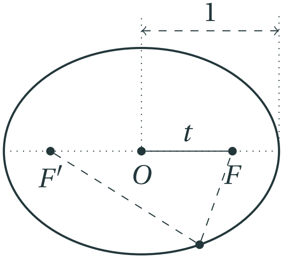
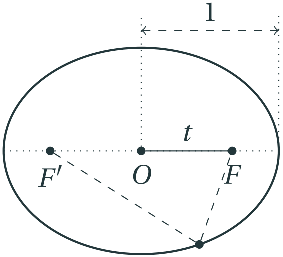


$$E(t) = 2\int_{-1}^{1} \sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x$$



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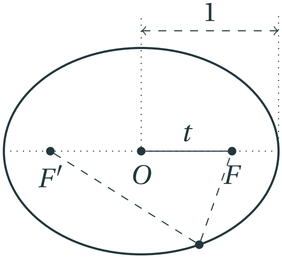

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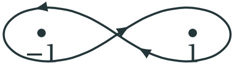

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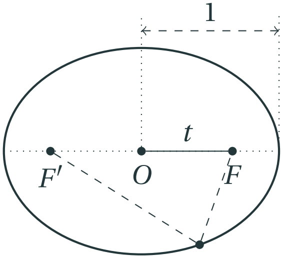

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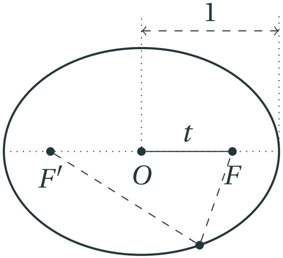

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since then Many applications in algebraic geometry

geometry of the cycles ↔ analytic properties of the periods

Content

Computing periods with a parameter

Differential equations as a data structure

Representation of algebraic numbers

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explicit $\sqrt{5+2\sqrt{6}}$ (also $\sqrt{2}+\sqrt{3}$)

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An example by Bostan, Chyzak, van Hoeij, and Pech (2011)

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explicit $$1+6\cdot\int_0^t \frac{{}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\,\frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\mathrm{d}w$$

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implicit $$t(t-1)(64t-1)(3t-2)(6t+1)y''' + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)y''$$
$$+ 4(576t^3 - 801t^2 - 108t + 74)y' = 0$$ (+ init. cond.)

What can we compute?

Differential equations as a data structure

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- power series expansion
- **equality testing**, given differential equations and initial condtions
- **numerical analytic continuation** with certified precision (D. V. Chudnovsky and G. V. Chudnovsky 1990; van der Hoeven 1999; Mezzarobba 2010)
  More on this later.

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One equation fits all cycles, the **Picard-Fuchs equation**.

recall $$E(t) = \oint \sqrt{\frac{1-t^2x^2}{1-x^2}}\,\mathrm{d}x = \frac{1}{2\pi i}\oint \overbrace{\frac{1}{1-\frac{1-t^2x^2}{(1-x^2)y^2}}}^{R(t,x,y)}\,\mathrm{d}x\mathrm{d}y$$

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Computational proof

$$(t-t^3)\frac{\partial^2 R}{\partial t^2} + (1-t^2)\frac{\partial R}{\partial t} + tR = \frac{\partial}{\partial x}\left(-\frac{t(-1-x+x^2+x^3)y^2(-3+2x+y^2+x^2(-2+3t^2-y^2))}{(-1+y^2+x^2(t^2-y^2))^2}\right) + \frac{\partial}{\partial y}\left(\frac{2t(-1+t^2)x(1+x^3)y^3}{(-1+y^2+x^2(t^2-y^2))^2}\right)$$

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Problem (mostly) solved!

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summation $y(t) = \frac{1}{(2i\pi)^3} \oint \frac{\left(x_1 x_2 x_3 - t \prod_{i=1}^3 (1+x_i)^2\right) dx_1 dx_2 dx_3}{\left(x_1^2 x_2^2 x_3^2 - t \prod_{i=1}^3 (1+x_i)^2\right)\left(1 - t \prod_{i=1}^3 (1+x_i)^2\right)}$

where $y(t)$ is the generating function of the l.h.s.

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integration $t(27t+1)y'' + (54t+1)y' + 6y = 0$, i.e. $3(3n+2)(3n+1)u_n + (n+1)^2 u_{n+1} = 0$

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conclusion *Generating functions of binomial sums are periods!*

Computing binomial sums with periods

Theorem + Algorithm (Bostan, Lairez, and Salvy 2016)

One can decide the equality between binomal sums.

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Theorem (Bostan, Lairez, and Salvy 2016)

$(u_n)_{n\geqslant 0}$ is a binomial sum **if and only if** $u_n = a_{n,\dots,n}$, for some rational power series $\sum_I a_I \mathbf{x}^I$.

Volume of semialgebraic sets

joint work with Mezzarobba and Safey El Din

Numerical analytic continuation

input A linear differential equation $L(f) = 0$

Initial conditions at a point $a \in \mathbb{C}$

Another point $b \in \mathbb{C}$

$\varepsilon > 0$

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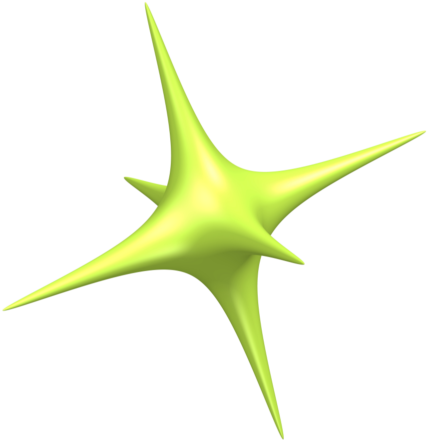
implementation Package *ore_algebra-analytic* by Mezzarobba

```
sage: from ore_algebra import *
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
sage: dop.numerical_solution(ini=[0,1], path=[0,1])
              [0.7853981633974483 +/- 1.08e-18]
sage: dop.numerical_solution(ini=[0,1],
path=[0,i+1,2*i,i-1,0,1])
     [3.926990816987242 +/- 4.81e-17] + [+/- 4.63e-21]*I
```

A numeric integral

$$\{x^2 + y^2 + z^2 \leqslant 1 - 2^{10}\left(x^2 y^2 + y^2 z^2 + z^2 x^2\right)\}$$



What is the volume of this shape?

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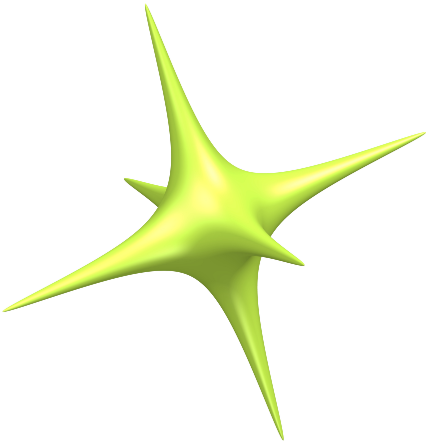

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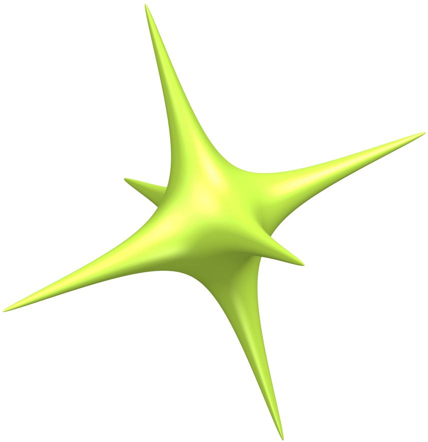

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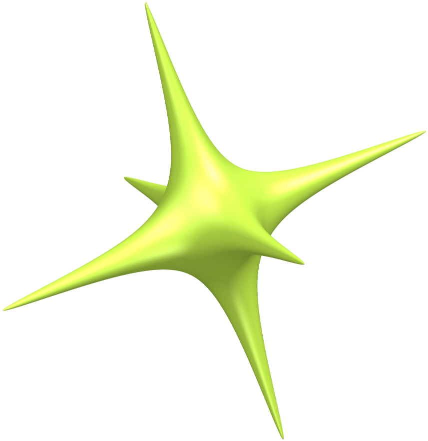

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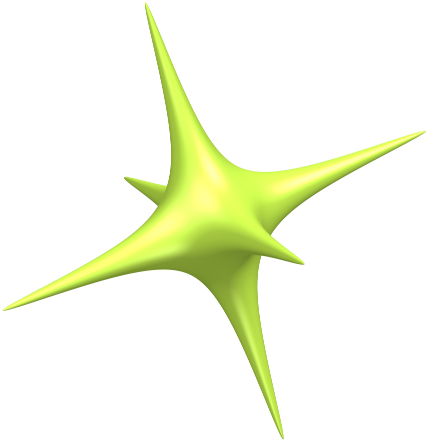

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Volumes are periods

Proposition

For any generic $f \in \mathbb{R}[x_1, \ldots, x_n]$,

$$\operatorname{vol}\{f \leqslant 0\} \triangleq \int_{\{f \leqslant 0\}} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{2\pi i} \oint_{\mathrm{Tube}\{f=0\}} \frac{x_1}{f} \frac{\partial f}{\partial x_1} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

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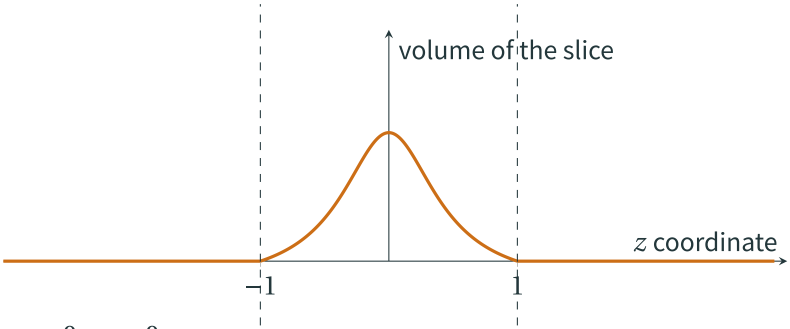
NB. $\operatorname{vol}\{f \leqslant 0\} = \int_{-\infty}^{\infty} \operatorname{vol}\{f \leqslant 0\} \cap \{x_n = t\} \mathrm{d}t$

The “volume of a slice” function

$\{y_1, y_2\}$, basis of the solution space of the Picard-Fuchs equation



$0 \cdot y_1 + 0 \cdot y_2$

$1.0792353\ldots \cdot y_1 - 40.100605\ldots \cdot y_2$

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An algorithm for computing volumes

input $f \in \mathbb{R}[x_1, \ldots, x_n]$ generic

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The complexity is quasi-linear with respect to the precision!

(To get twice as many digits, you need only twice as much time.)

A hundred digits (within a minute)

vol ()

$$= 0.108575421460360937739503\\ 395994207619810917874446\\ 607475444475822993285360\\ 673032928194943474414064\\ 066136624234627959808778\\ 1034932346781568\ldots$$

Computation of the Picard group of quartic surfaces

joint work with Emre Sertöz

The Picard group

quartic surface $X = V(f) \subseteq \mathbb{P}^3$ smooth, where $f \in \mathbb{C}[w, x, y, z]$ is homogeneous of degree 4.

Picard group $\operatorname{Pic} X = \left\{ [\gamma] \mid \gamma \text{ algebraic curve} \right\} \subset H_2(X, \mathbb{Z}) \simeq Z^{22}$

example 1 Pic(very generic quartic surface) $= \mathbb{Z} \cdot$ (hyperplane section)

example 2 $\operatorname{Pic} V(w^4 + x^4 + y^4 + z^4) \simeq \mathbb{Z}^{20}$, generated by the 48 lines

How to compute it? Symbolic approach is difficult because computing elements of $\operatorname{Pic} X$ explicitly involves solving huge polynomial systems. And we do not even have an *a priori* degree bound.

Lefschetz (1,1)-theorem

$X = V(f) \subset \mathbb{P}^3$ smooth quartic surface

periods $\gamma_1, \dots, \gamma_{22}$ basis of $H_2(X, \mathbb{Z})$

$$\eta_i = \oint_{\mathrm{tube}(\gamma_i)} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{f(1, x, y, z)} \in \mathbb{C}$$

Efficiently computable at high precision thanks to Picard-Fuchs equations and numerical analytic continuation!

theorem $\operatorname{Pic} X = \left\{ (a_1, \dots, a_{22}) \in \mathbb{Z}^{22} \;\middle|\; a_1\eta_1 + \cdots + a_{22}\eta_{22} = 0 \right\}$

The Picard group is the lattice of integer relations between the periods of the quartic surface.

algorithm Compute the periods with high precision (typically 1000 digits).
Use LLL to recover $\operatorname{Pic} X$.

How to certify the computation?

goal For $M > 0$, compute $\epsilon_M > 0$ such that for all $a \in \mathbb{Z}^r$,

$$\|a\| \leqslant M \text{ and } \left|\sum_i a_i \eta_i\right| < \epsilon_M \quad \Rightarrow \quad \sum_i a_i \eta_i = 0.$$

For contradiction assume that $0 < \left|\sum_i a_i \eta_i\right| \ll 1$.

perturbation There exists $\tilde{f}$ near f such that the periods $\tilde{\eta}_i$ of $V(\tilde{f})$ satisfy $\sum_i a_i \tilde{\eta}_i = 0$.

Lefschetz Then $V(\tilde{f})$ contains an algebraic curve of a certain type whereas $V(f)$ does not.

algebraic condition There is an explicit polynomial with integer coefficients such that $P(f) \neq 0$ and $P(\tilde{f}) = 0$.

separation If f has integer coefficients, then $|P(f)| \geqslant 1$ so $\tilde{f}$ cannot be too close to f.

Perpectives

Quelques objectifs liés à ces questions

T21 *Calcul des périodes et des volumes*
Plus efficace, plus général

T22 *Calcul symbolique des intégrales à bord*
Elles interviennent dans le calcul des volumes et en arithmétique

T23 *Calcul efficaces de bases de Gröbner différentielles*
Outil important pour l'analyse algébrique