Symmetric situations in polynomial optimization

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Tromsø: the ~~Paris~~ Bordeaux of the North

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  $\rightarrow$ Orbit reduction of relative entropy programs.

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How to exploit symmetries using group theory and combinatorics?

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→ Are there any situations in which the set of minimizers contains highly symmetric points?

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$\rightarrow$ From a geometric point of view: the corresponding variety is non empty if and only if it contains a point with at most k distinct coordinates.

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→ Now assume we want to solve the polynomial system

$$P_1 = P_2 = \ldots = P_r = 0,$$

where $\mathcal{S}_n$ permutes the P_i's. Let $I = \langle P_1, \ldots, P_r \rangle$, and $V(I)$ the associated variety.

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→ [M., Riener, Verdure, 2021]: A combinatorial analogue of this result depending on the leading monomials of the P_i's.

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→ [M., Riener, Verdure, 2021]: Which orbit types for the solutions?

Partitions and tableaux

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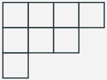

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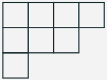

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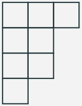

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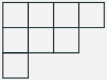

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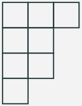

→ λ dominates μ if for every i, $\sum_{j=1}^{i} \lambda_j \geqslant \sum_{j=1}^{i} \mu_j$:

Specht Odyssey

→ A Young tableau of shape $\mu \vdash n$ is a tableau T of shape μ filled-in with numbers from 1 to n.

4	2	6	1
8	7	5	
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→ Note: For $x \in \mathbb{K}^n$, $x \notin V_{\Lambda(x)}$:

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x_3			

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Theorem [M., Riener, Verdure]

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→ The dimension of the variety is at most d.

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→ What about other certificates?

SAGE against the Motzkin

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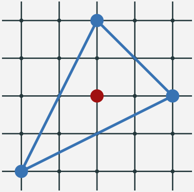

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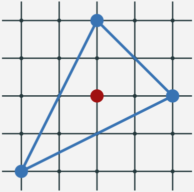

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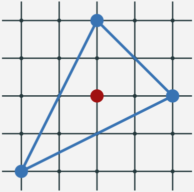

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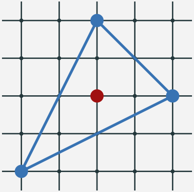

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→ The more general framework of signomials.

→ An AGE signomial is a sum of exponentials of the form

$$f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha e^{\langle \alpha, x \rangle} + c_\beta e^{\langle \beta, x \rangle}$$

such that $\mathcal{A} \cup \{\beta\} \subset \mathbb{R}^n$, $c_\alpha \geq 0$, $c_\beta \in \mathbb{R}$, and $f(x) \geq 0$ on $\mathbb{R}^n$.

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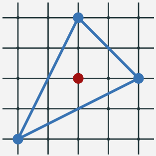

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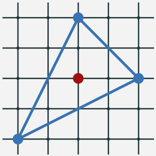

- $D(\nu, e \cdot c) \leqslant c_\beta$,

where $D(\nu, e \cdot c) = \sum_{\alpha \in \mathcal{A}} \nu_\alpha \ln \left(\frac{\nu_\alpha}{e \cdot c_\alpha} \right)$ is the relative entropy function.

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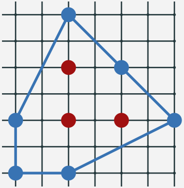
→ Can be solved with relative entropy programming.

Size of the problem



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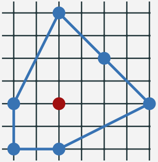

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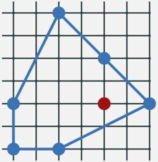

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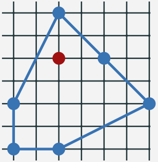

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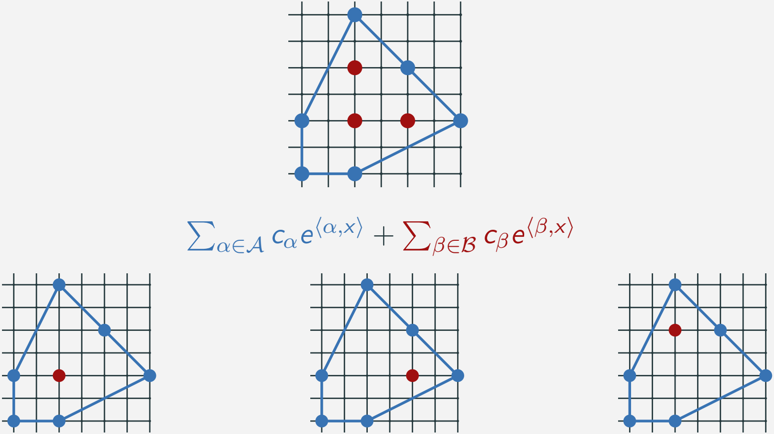

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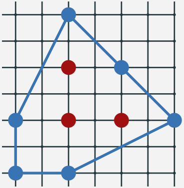

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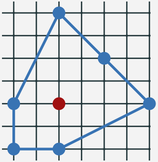

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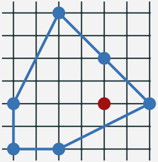

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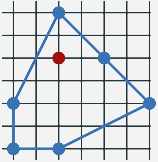

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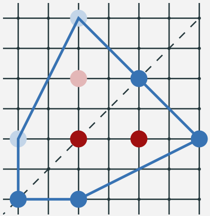

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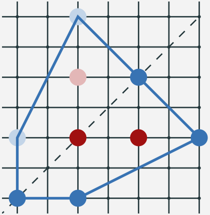

→ Does f have a symmetric decomposition?

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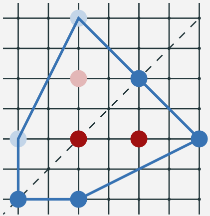

→ Does f have a symmetric decomposition?

→ Can we reduce the size of the relative entropy program?

Orbit decomposition

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Theorem [M., Naumann, Riener, Theobald, Verdure]

The signomial f is a SAGE if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$, there exists an AGE signomial $h_{\hat{\beta}}$ such that

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The functions $h_{\hat{\beta}}$ can be chosen invariant under the action of $\operatorname{Stab}(\hat{\beta})$.

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$\rightarrow$ This already reduces the number of AGE signomials in the decomposition.

$\rightarrow$ Moreover, the invariance under $\mathrm{Stab}(\hat{\beta})$ allows to further reduce the number of variables and constraints.

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(ii) $\sum_{\alpha \in \mathcal{A}/\operatorname{Stab}(\hat{\beta})} \left| \operatorname{Stab}(\hat{\beta}) \cdot \alpha \right| \nu_\alpha^{(\hat{\beta})} \ln \frac{\nu_\alpha^{(\hat{\beta})}}{e c_\alpha^{(\hat{\beta})}} \leqslant c_{\hat{\beta}} \qquad \forall\ \hat{\beta} \in \hat{\mathcal{B}}$,

(iii) $\sum_{\hat{\beta} \in \hat{\mathcal{B}}} \frac{|\operatorname{Stab}(\alpha)|}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\gamma \in (G\cdot\alpha)/\operatorname{Stab}(\hat{\beta})} \left| \operatorname{Stab}(\hat{\beta}) \cdot \gamma \right| c_\gamma^{(\hat{\beta})} \leqslant c_\alpha \qquad \forall\ \alpha \in \hat{\mathcal{A}}$.

Size estimate

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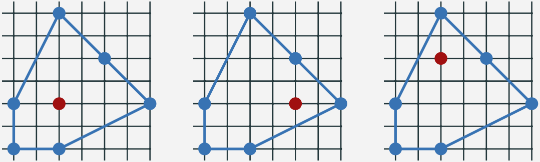
→ Without reduction: $2|\mathcal{B}||\mathcal{A}|$ variables, $n|\mathcal{B}| + |\mathcal{B}| + |\mathcal{A}|$ constraints.



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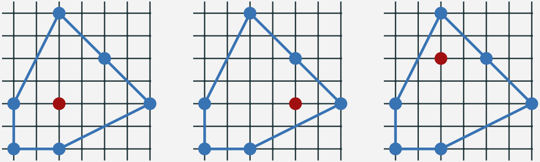

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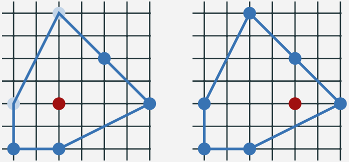

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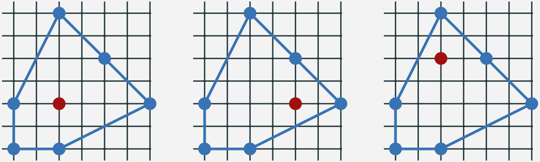

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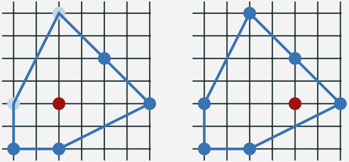
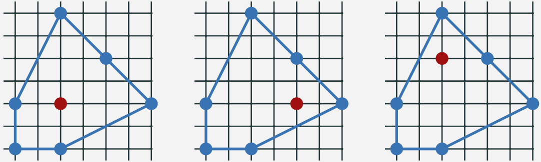


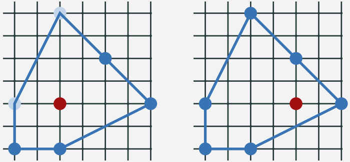
→ $2\sum_{\hat{\beta}\in\hat{\mathcal{B}}}|\mathcal{A}/\operatorname{Stab}(\hat{\beta})|$ variables.

Size estimate

→ Without reduction: $2|\mathcal{B}||\mathcal{A}|$ variables, $n|\mathcal{B}| + |\mathcal{B}| + |\mathcal{A}|$ constraints.



→ With reduction:



→ $2\sum_{\hat{\beta}\in\hat{\mathcal{B}}}|\mathcal{A}/\operatorname{Stab}(\hat{\beta})|$ variables.

→ At most $n|\hat{\mathcal{B}}| + |\hat{\mathcal{B}}| + |\hat{\mathcal{A}}|$ constraints.

A stability result

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→ For $\alpha \in \mathbb{R}^n$, denote by $\mathrm{wt}(\alpha)$ its number of non-zero coordinates.

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Theorem [M., Naumann, Riener, Theobald, Verdure]

Let $k, \ell, w \in \mathbb{N}$ be fixed. Then for every integer $n \geqslant 2w$ and every $\mathcal{S}_n$-invariant signomial such that $|\hat{\mathcal{A}}| \leqslant k$, $|\hat{\mathcal{B}}| \leqslant \ell$, and

$$\max_{\hat{\gamma} \in \hat{\mathcal{A}} \cup \hat{\mathcal{B}}} \mathrm{wt}(\hat{\gamma}) \leqslant w,$$

the number of constraints and the number of variables of the symmetry adapted program are bounded by constants only depending of k, ℓ and w:

$$C_n \leqslant k + \ell + \ell(w+1) \quad \text{and} \quad V_n \leqslant 2\ell k u(w),$$

where $u(w) = \sum_{i=0}^{w} \binom{w}{i}^2 i!\,.$

Concrete size comparisons

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		Standard		**Symmetric**	
$\lvert\mathcal{S}_n \cdot \hat{\beta}\rvert$	$\lvert\mathcal{S}_n \cdot \hat{\alpha}\rvert$	V_n	C_n	V_n	C_n
1	$n!$	$2n! + 3$	$n! + n + 2$	5	4
$n!$	n	$2(n+1)n! + 1$	$(n+1)(n!+1)$	$2n+3$	$n+3$
$n!$	$n!$	$2(n!+1)n! + 1$	$n!(n+2) + 1$	$2n!+3$	$n+3$
n	n	$2n(n+1) + 1$	$(n+1)^2$	7	5

Numerical results (1/4)

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$\rightarrow f_n^{(1)} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sigma \exp(\langle \alpha, x \rangle) - \exp(\langle \beta, x \rangle)$ where $\beta = (1, \ldots, 1)$ and $\alpha = (1, 2, \ldots, n)$.

		Standard method				**Symmetric method**			
dim	bound	V_n	C_n	t_s	t_r	V_n	C_n	t_s	t_r
2	-0.1481	7	6	0.0113	0.0121	5	4	0.0147	0.0158
3	-0.2499	15	11	0.0148	0.0160	5	4	0.0141	0.0149
4	-0.3257	51	30	0.0304	0.0337	5	4	0.0139	0.0147
5	-0.3849	243	127	–	–	5	4	0.0140	0.0147
6	-0.4327	1443	728	–	–	5	4	0.0136	0.0144
7	-0.4724*	10083	5049	–	–	5	4	0.0211	0.0222

Numerical results (2/4)

Numerical results (2/4)

→ $f_n^{(2)} = (n-1)! \sum_{i=1}^{n} \exp(n^2 x_i) - \sum_{\sigma \in \mathcal{S}_n} \sigma \exp(\langle \beta, x \rangle)$, where $\beta = (1, 2, \ldots, n)$, and $\alpha = (n^2, 0, \ldots, 0)$.

		Standard method				**Symmetric method**			
dim	bound	V_n	C_n	t_s	t_r	V_n	C_n	t_s	t_r
2	-0.2109	13	9	0.0173	0.0185	7	5	0.0297	0.0311
3	-0.8888	49	28	0.0427	0.0454	9	6	0.0248	0.0264
4	-4.111	241	125	0.152	0.1701	11	7	0.0296	0.0318
5	-22.30	1441	726	0.7888	0.8433	13	8	0.0356	0.0384
6	-141.0	10081	5047	5.422	5.843	15	9	0.0423	0.0458
7	-1024	80641	40328	57.26	66.67	17	10	0.0491	0.0538
8	-8418	725761	362889	1514	2211	19	11	0.0568	0.0626
9	-77355	7257601	3628810	–	–	21	12	0.0661	0.0835
10		79833601	39916811	–	–	23	13	–	–

Numerical results (3/4)

Numerical results (3/4)

→ $f_n^{(3)} = \frac{1}{n}\sum_{\sigma\in\mathcal{S}_n} \exp(\langle\alpha, x\rangle) - \frac{1}{n}\sum_{\sigma\in\mathcal{S}_n} \sigma\exp(\langle\beta, x\rangle)$, where $\beta = (1, 2, \ldots, n)$ and $\alpha = (2, 8, \ldots, 2n^2)$.

		Standard method				**Symmetric method**			
dim	bound	V_n	C_n	t_s	t_r	V_n	C_n	t_s	t_r
2	-0.4178	13	9	0.0301	0.0323	7	5	0.0431	0.0465
3	-1.0323	85	31	0.0558	0.0603	15	6	0.0531	0.0569
4	-3.494	1201	145	–	–	51	7	0.1212	0.1301
5	-15.13	29041	841	–	–	243	8	0.5750	0.6215
6		1038241	5761	–	–	1443	9	–	–

Numerical results (4/4)

Numerical results (4/4)

$\rightarrow f_n^{(4)} = \frac{1}{n}\sum_{i=1}^{n} \exp(n^2 x_i) - \frac{1}{n}\sum_{i=1}^{n} \exp((n-1)(x_1 + \cdots + x_n) + x_i)$, where $\beta = (n, n-1, n-1, \ldots, n-1)$ and $\alpha = (n^2, 0, \ldots, 0)$.

		Standard method				Symmetric method			
dim	bound	V_n	C_n	t_s	t_r	V_n	C_n	t_s	t_r
2	-0.1054	13	9	0.019 01	0.0204	7	5	0.0213	0.0229
3	-0.092	25	16	0.0268	0.0287	7	5	0.0205	0.0218
4	-0.076	41	25	0.0341	0.0367	7	5	0.0205	0.0218
68	-0.0053	9385	4761	–	–	7	5	0.0475	0.0519
95	-0.0038*	18241	9216	–	–	7	5	0.0267	0.0281

Thank you!