Truncated Moment Cone and Connections to the Coalescence Manifold

Georgy Scholten

Joint work with Cynthia Vinzant and Zvi Rosen.

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- We give the minimal number of steps needed to generate the moment cones of monotone functions.

Parametrized Step functions

Let f be a nonnegative step function of at most k steps:

$$f = y_1 \mathbf{1}_{[0,s_1]} + \sum_{i=2}^{k+1} y_i \mathbf{1}_{(s_{i-1},s_i]} \in S_k.$$

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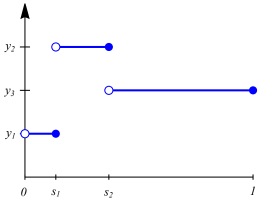

Figure: A step function in S_2.

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- $M(A)$ is convex.

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Main Question: Given $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{Z}_{\geq 0}$, what is the smallest $k \in \mathbb{N}$ such that the cone

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Theorem (Rosen, S., Vinzant, 2020)

$$M_k(A) = M(A) \quad \Leftrightarrow \quad k \geq |A| - 1.$$

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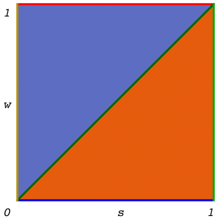

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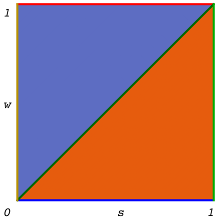

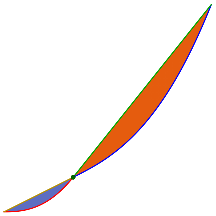

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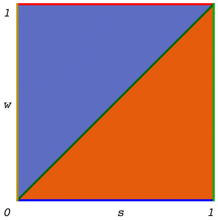

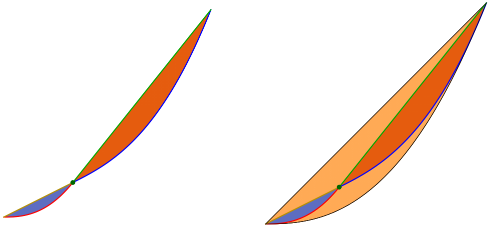

Figure: $M_1(A)$ and $M_2(A)$.

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Reference: Schmüdgen, *The Moment Problem*, Chapter 10.

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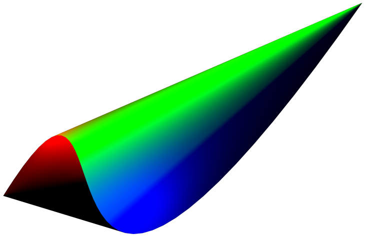

Figure: Affine slice of the conical hull of $v_3 : [0,1] \to \mathbb{R}^4$, for $v_3(t) = (1, t, t^2, t^3)$.

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- $M(A)$ as a projected spectrahedron.

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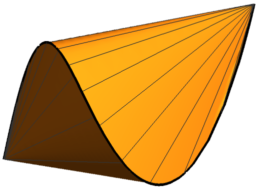

Figure: Affine slice of $M(A)$ in $\mathbb{R}^3$.

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- Denote $\mathcal{C}_{n,k}$ the collection of coalescence vectors arising from step functions of at most k steps.

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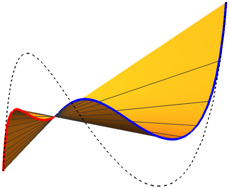

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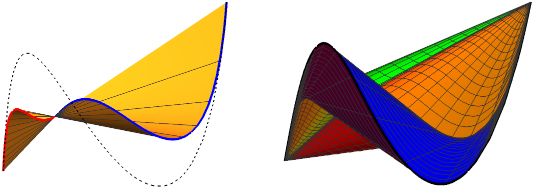

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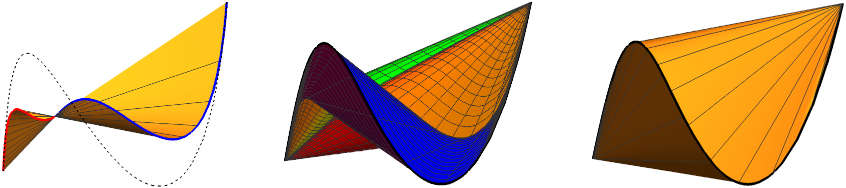

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$$M^{\downarrow}(A) = \overline{\left\{ \left(\int_0^1 x^a f(x) dx \right)_{a \in A} : f \geq 0 \text{ is } \textit{decreasing} \text{ on } [0,1] \right\}}.$$

Theorem (Rosen, S., Vinzant, '20)

Every A-moment vector of a monotone density function is the limit of A-moments of monotone step functions with $\leq k$ breakpoints if and only if $k \geq \lfloor n/2 \rfloor$.

Towards Open Questions

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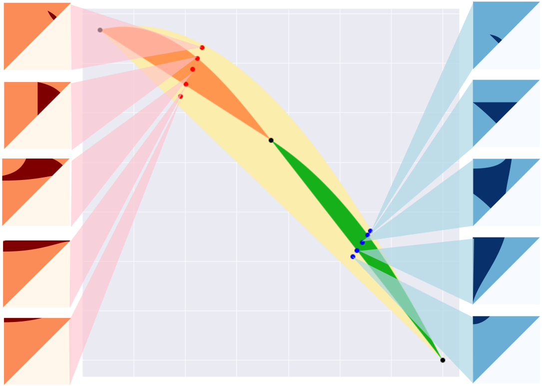

Figure: $\mathcal{C}_4 = M(\{0, 2, 5\})$ and fibers of select points, as subsets of the (s_1, s_2)-simplex.

Open Questions

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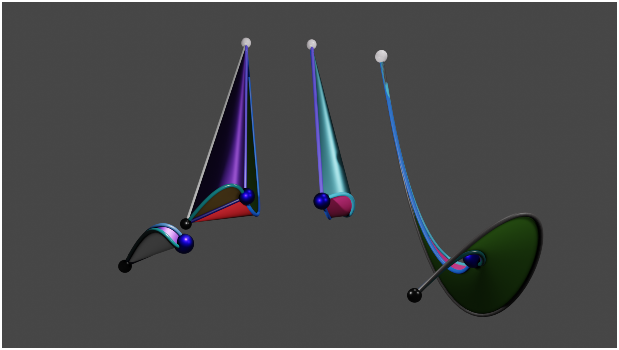

Figure: Algebraic boundaries in $M_2(\{0, 2, 5, 9\})$.

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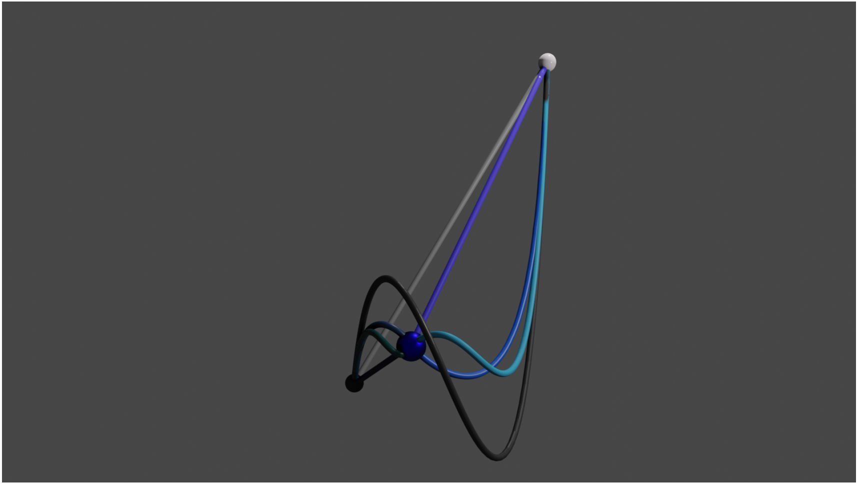
Skeleton of $\mathcal{C}_{5,2}$



Boundary Structure of $M(\{0, 2, 5, 9\})$

Figure: Images of 3-dimensional faces of $\mathcal{C}_{5,2}$.

Boundary Structure of $M(\{0,2,5,9\})$

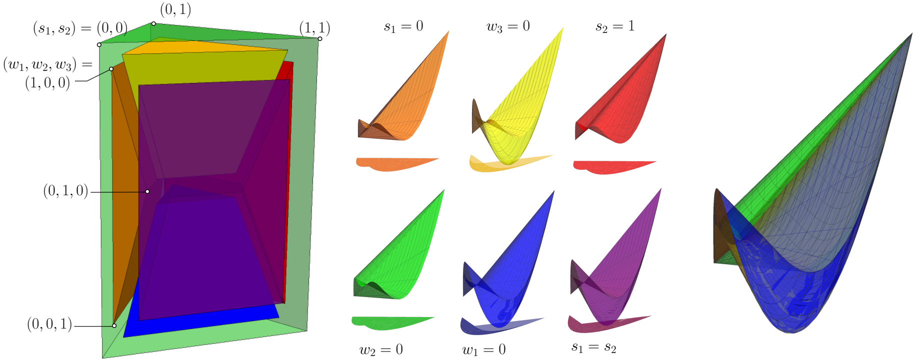


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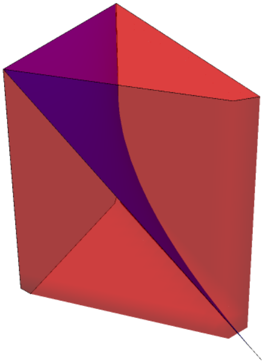
Figure: Image of step functions of a single step and a point mass at 1.

Boundary Structure of $M(\{0, 2, 5, 9\})$



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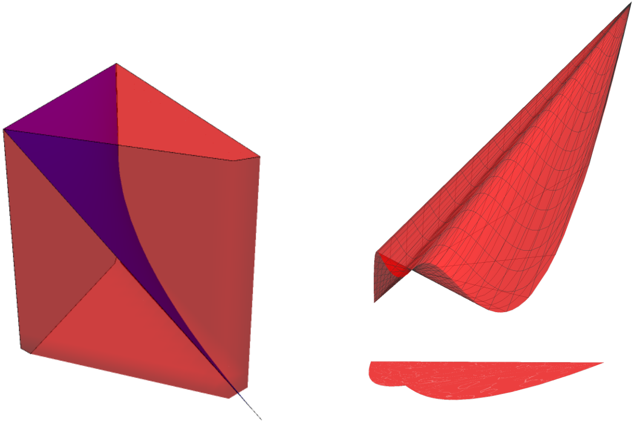

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