C^2-finite sequences

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Introduction: C and D-finite sequences

Basic notation

Throughout this talk we consider:

- $\mathbb{K}$: a **computable** field contained in $\mathbb{C}$.
- $\mathbb{K}^{\mathbb{N}}$: ring of sequences over $\mathbb{K}$.
- Termwise addition and product (Hadamard product):

$$(a_n)_n + (b_n)_n = (a_n + b_n)_n,$$

$$(a_n)_n (b_n)_n = (a_n b_n)_n$$

C-finite sequences

Definition

Let $(a_n)_n \in \mathbb{K}^{\mathbb{N}}$. We say that $(a_n)_n$ is C-finite if there exist $d \in \mathbb{N}$ and constants $c_0, \dots, c_d \in \mathbb{K}$ (not all zero) such that:

$$c_d a_{n+d} + \dots + c_0 a_n = 0, \text{ for all } n \in \mathbb{N}.$$

- $\mathcal{C} \subset \mathbb{K}^{\mathbb{N}}$: set of C-finite sequences.

D-finite sequences

Definition

Let $(a_n)_n \in \mathbb{K}^{\mathbb{N}}$. We say that $(a_n)_n$ is D-finite if there exist $d \in \mathbb{N}$ and polynomials $p_0(n), \ldots, p_d(n) \in \mathbb{K}[n]$ (not all zero) such that:

$$p_d(n)a_{n+d} + \ldots + p_0(n)a_n = 0, \text{ for all } n \in \mathbb{N}.$$

- $\mathcal{C} \subset \mathcal{D} \subset \mathbb{K}^{\mathbb{N}}$: set of C-finite sequences.
- $\mathcal{D} \subset \mathbb{K}^{\mathbb{N}}$: set of D-finite sequences.

D-finite sequences

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Finite representation

The elements in $\mathcal{D}$ (and, hence, $\mathcal{C}$) can be represented with finitely many data:

- List of the coefficients $p_0(n), \ldots, p_d(n)$.
- List of initial elements $a_0, \ldots, a_r$ for some $r \in \mathbb{N}$.

Closure properties

The sets $\mathcal{C}$ and $\mathcal{D}$ are subrings of $\mathbb{K}^{\mathbb{N}}$. The operations are computable using *linear algebra*.

Closure properties

The sets $\mathcal{C}$ and $\mathcal{D}$ are subrings of $\mathbb{K}^{\mathbb{N}}$. The operations are computable using *linear algebra*.

Let $(a_n)_n$ and $(b_n)_n$ belong to the same class with oders d_1 and d_2:

Property	**Sequence**	**Order bound**
Addition	$(a_n)_n + (b_n)_n$	$d_1 + d_2$
Product	$(a_n)_n(b_n)_n$	$d_1 d_2$
Shift	$(a_{n+1})_n$	d_1
Inverse shift	$(a_{n-1})_n$	$d_1 + 1$
Arith. subseq.	$(a_{kn+r})_n$	d_1

Also *interlacing* sequences $(a_{1,n})_n, \ldots, (a_{m,n})_n$ is closed in these classes.

Examples

Many sequences are D-finite:

- Fibonacci numbers $[(f_n)_n]$:

$$f_{n+2} - f_{n+1} - f_n = 0.$$

- Catalan numbers $[(c_n)_n]$:

$$(n+2)c_{n+1} - (4n+2)c_n = 0.$$

- Factorial numbers $[(n!)_n]$:

$$(n+1)! - (n+1)n! = 0.$$

- All sequences from D-finite functions.

Non-D-finite examples

There are sequences that **are not** D-finite:

- Bell numbers $[(B_n)_n]$.
- Labelled rooted trees $[(n^{n-1})_n]$.
- Partition sequence $[(p_n)_n]$.
- Fibonorial $[(F_n)_n]$
$$F_n = \prod_{k=1}^{n} f_k.$$
- Sparse D-finite sequences $[(a_{n^2})_n]$.

Extending by iteration?

D-finite functions $\longrightarrow$ DD-finite functions

Idea: use D-finite as coefficients of the recurrence.

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It does not work!

Let $(b_n)_n = (0, 1, 0, 1, 0, 1, \dots)$ and $(a_n)_n$ be defined by the recurrence:

$$b_n a_{n+1} + (b_n + (-1)^n) a_n = 0.$$

To completely define a_n we need an **infinite amount of data**!

Extending by iteration?

D-finite functions $\longrightarrow$ DD-finite functions

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To completely define a_n we need an **infinite amount of data**!

The leading coefficient $(b_n)_n$ has infinitely many zeros!

Extending by iteration? II

We can create this behavior from nice recurrences:

Consider the sequences

$$a_{n+1} + (-1)^n a_n = 0, \quad b_{n+1} + b_n = 0.$$

Getting a recurrence for $(c_n)_n = (a_n + b_n)_n$ by the usual method:

$$z_n c_{n+2} + y_n c_{n+1} + x_n c_n = 0,$$

yields the solution

$$x_n = -(-1)^n + 1, \quad y_n = 2, \quad z_n = (-1)^n + 1,$$

where $(z_n)_n$ has infinitely many zeros.

Extending the class: C^2-finite sequences

Zero divisors

In the ring of sequences: $\mathbb{K}^{\mathbb{N}}$

Any sequence with a zero element is a zero divisor in $\mathbb{K}^{\mathbb{N}}$.

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How worked for C and D-finite?

In the subrings $\mathbb{K}$ and $\mathbb{K}[n]$, zero divisors are very specific:

- For $\mathbb{K}$: only $\mathbf{0} = (0, 0, \ldots)$ is a zero divisor.
- For $\mathbb{K}[n]$: zero divisors has finitely many zeros.

Zero divisors

In the ring of sequences: $\mathbb{K}^{\mathbb{N}}$

Any sequence with a zero element is a zero divisor in $\mathbb{K}^{\mathbb{N}}$.

Lemma (a bit trivial)

There is a recurrence equation for $(a_n)_n$ whose leading coefficient is not a zero-divisor

$$\Updownarrow$$

There is a recurrence equation for $(a_n)_n$ whose leading coefficient has finitely many zeros.

Zero divisors

In the ring of sequences: $\mathbb{K}^{\mathbb{N}}$

Any sequence with a zero element is a zero divisor in $\mathbb{K}^{\mathbb{N}}$.

Lemma (also trivial)

For any ring R, the set of elements that are not zero divisors is multiplicatively closed.
The localization ring over this set is called *total ring of fractions*.

C^2-finite sequences

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$$c_d a_{n+d} + \ldots + c_0 a_n = 0, \text{ for all } n \in \mathbb{N}.$$

C^2-finite sequences

Definition

Let $(a_n)_n \in \mathbb{K}^{\mathbb{N}}$. We say that $(a_n)_n$ is C^2-finite if there exist $d \in \mathbb{N}$ and C-finite sequences $(c_{0,n})_n, \ldots, (c_{d,n})_n \in \mathcal{C}$ (c_d not a zero divisor) such that:

$$c_{d,n}a_{n+d} + \ldots + c_{0,n}a_n = 0, \text{ for all } n \in \mathbb{N}.$$

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$$c_{d,n}a_{n+d} + \ldots + c_{0,n}a_n = 0, \text{ for all } n \in \mathbb{N}.$$

This set includes both C and D-finite sequences.

C^2-finite sequences

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This set is bigger than D-finite sequences

- Fibonorial: $F_{n+1} - f_{n+1}F_n = 0$.
- $(a_n)_n = (c^{n^2})_n$ for $c \in \mathbb{K}$ is C^2-finite:

$$a_{n+1} - c^{2n+1}a_n = 0.$$

C^2-finite sequences

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Let $(a_n)_n \in \mathbb{K}^{\mathbb{N}}$. We say that $(a_n)_n$ is C^2-finite if there exist $d \in \mathbb{N}$ and C-finite sequences $(c_{0,n})_n, \ldots, (c_{d,n})_n \in \mathcal{C}$ (c_d not a zero divisor) such that:

$$c_{d,n}a_{n+d} + \ldots + c_{0,n}a_n = 0, \text{ for all } n \in \mathbb{N}.$$

This set is bigger than D-finite sequences

- Sparse Fibonacci sequence $(f_{n^2})_n$: (Kotek and Makowsky 2014)

$$\begin{aligned} f_{2n+3}(f_{2n+1}f_{2n+3} - f_{2n+2}^2) \quad & f_{n^2} \\ +f_{2n+2}(f_{2n+3} + f_{2n+1}) \quad & f_{(n+1)^2} \\ -f_{2n+1} \quad & f_{(n+2)^2} \quad = 0 \end{aligned}$$

Characterization theorem

Let $\mathcal{R}_{\mathcal{C}}$ be the total ring of fractions of $\mathcal{C}$.

Theorem [J.-P., Nuspl and Pillwein 2021]

Let $(a_n)_n \in \mathbb{K}^{\mathbb{N}}$. Then

$$(a_n)_n \text{ is } C^2\text{-finite}$$
$$\Updownarrow$$
The following $\mathcal{R}_{\mathcal{C}}$-module is finitely generated:

$$M((a_n)_n) = \langle (a_n)_n, (a_{n+1})_n, (a_{n+2})_n, \ldots \rangle.$$

Proving addition is closed

Let $(a_n)_n$ and $(b_n)_n$ be C^2-finite sequences. Is $(a_n + b_n)_n$ also C^2-finite?

$$M((a_n + b_n)_n) \subset M((a_n)_n) + M((b_n)_n).$$

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Submodules of finitely generated modules might not be finitely generated. We need a Noetherian module.

Notherianity property

Definition

Let M be a module. We say M is Noetherian if all the submodules of M are finitely generated.

Lemma

For any set of sequences $(a_{1,n})_n, \ldots, (a_{m,n})_n \in \mathcal{C}$, there is a finitely generated $\mathbb{K}$-algebra $T \subset \mathcal{R}_{\mathcal{C}}$ close under shift that contains these sequences.

Proving addition is closed II

Let $(a_n)_n$ and $(b_n)_n$ be C^2-finite sequences. Is $(a_n + b_n)_n$ also C^2-finite?

$$M((a_n + b_n)_n) \subset M((a_n)_n) + M((b_n)_n).$$

Let T be ring provided by the Lemma that contains all the coefficients of the equations for $(a_n)_n$ and $(b_n)_n$. T is a Noetherian ring and the modules

$$\langle (a_n)_n, (a_{n+1})_n, \ldots \rangle_T, \langle (b_n)_n, (b_{n+1})_n, \ldots \rangle_T$$

are Noetherian and finitely generated.

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Let $(a_n)_n$ and $(b_n)_n$ be C^2-finite sequences. Is $(a_n + b_n)_n$ also C^2-finite?

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$$\langle (a_n)_n, (a_{n+1})_n, \ldots \rangle_T, \langle (b_n)_n, (b_{n+1})_n, \ldots \rangle_T$$

are Noetherian and finitely generated.
Hence $\langle (a_n + b_n)_n, (a_{n+1} + b_{n+1})_n, \ldots \rangle_T$ is finitely generated and so it is the module $M((a_n + b_n)_n)$.

C^2-finite sequences form a ring

Theorem [J.-P., Nuspl and Pillwein 2021]

The set $\mathcal{C}^2$ of C^2-finite sequences is a difference subring of $\mathbb{K}^{\mathbb{N}}$.

Proofs for product and shift are similar to the proof for addition.

Iterating the process

The same hold for C^k and D^k-finite sequences.

Other closrue properties

Theorem [J.-P., Nuspl and Pillwein 2021]

Let $(a_n)_n, (a_{0,n})_n, \ldots, (a_{m,n})_n \in \mathcal{C}^2$. Then:

1. Shift:$(a_{n+k})_n \in \mathcal{C}^2$ for all $k \in \mathbb{N}$.
2. Difference:$(a_{n+1} - a_n)_n \in \mathcal{C}^2$.
3. Partial sum:$\left(\sum_{k=0}^n a_k\right)_n \in \mathcal{C}^2$.
4. Subsequence:$(a_{dn})_n \in \mathcal{C}^2$ for all $d \in \mathbb{N}$.
5. $(a_{\lfloor n/d \rfloor})_n \in \mathcal{C}^2$ for all $d \in \mathbb{N} \setminus \{0\}$.
6. Interlacing: $(b_n)_n$ where $b_{mk+r} = a_{r,k}$ is in $\mathcal{C}^2$.

These results also holds for C^k and D^k-finite sequences.

Computing with C^2-finite sequences: Skolem problem

Why C^2-finite sequences

Theorem [Skolem 1934, Mahler 1935, Lech 1953]

Let $(a_n)_n$ be C-finite over a field of characteristic zero. Then the set

$$\mathcal{Z}_a := \{n \in \mathbb{N} \ : \ a_n = 0\}$$

is comprised of a finite set together with a finite number of arithmetic progressions, i.e.:

$$\mathcal{Z}_a = \{n_0, \ldots, n_m\} \cup \bigcup_{i=1}^{l} \{d_i k + r_i \ : \ k \in \mathbb{N}\}$$

Why C^2-finite sequences

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- Result not know for D-finite sequences.
- We need this for actual computations.

Skolem Problem

Problems related with SML-Theorem

1. Can we decide $|\mathcal{Z}_a|$ is finite?
2. Can we compute the arithmetic progressions?
3. Can we decide the finite set of zeros?

Skolem Problem

Problems related with SML-Theorem

1. Can we decide $|\mathcal{Z}_a|$ is finite? YES!
2. Can we compute the arithmetic progressions? YES!
3. Can we decide the finite set of zeros? not known...
   - We can decide for order ≤ 4 (Ouaknine and Worrell, 2012).
   - We can use CAD to determine sign pattern and zeros (Gerhold and Kauers, 2005).
   - We can use asymptotics to determine growth.
   - If there is a unique dominant root, we can decide.
   - Heuristically, we can check some terms to look for zeros.

Computing: an ansatz method

Ansatz method (addition)

Applying Char. theorem

Let $(a_n)_n, (b_n)_n \in \mathcal{C}^2$. There is $r \in \mathbb{N}$ and $(c_{i,n})_n \in \mathcal{R}_{\mathcal{C}}$ such that

$$(a_{n+r} + b_{n+r}) = c_{0,n}(a_n + b_n) + \ldots + c_{r-1,n}(a_{n+r-1} + b_{n+r-1}).$$

Ansatz method (addition)

Applying Char. theorem

Let $(a_n)_n, (b_n)_n \in \mathcal{C}^2$. There is $r \in \mathbb{N}$ and $(c_{i,n})_n \in \mathcal{R}_{\mathcal{C}}$ such that

$$(a_{n+r} + b_{n+r}) = c_{0,n}(a_n + b_n) + \ldots + c_{r-1,n}(a_{n+r-1} + b_{n+r-1}).$$

Using recurrences for $(a_n)_n$ and $(b_n)_n$

Since $(a_n)_n, (b_n)_n \in \mathcal{C}^2$ of orders d_1 and d_2, we can write:

$$a_{n+j} = \alpha_{j,0,n}a_n + \ldots + \alpha_{j,d_1-1,n}a_{n+d_1-1}$$

$$b_{n+j} = \beta_{j,0,n}b_n + \ldots + \beta_{j,d_2-1,n}b_{n+d_2-1}$$

Ansatz method (addition)

Linear system in $\mathcal{R}_{\mathcal{C}}$

$$\begin{pmatrix} \alpha_{0,0,n} & \alpha_{1,0,n} & \cdots & \alpha_{r-1,0,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0,d_1-1,n} & \alpha_{1,d_1-1,n} & \cdots & \alpha_{r-1,d_1-1,n} \\ \beta_{0,0,n} & \beta_{1,0,n} & \cdots & \beta_{r-1,0,n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{0,d_2-1,n} & \beta_{1,d_2-1,n} & \cdots & \beta_{r-1,d_2-1,n} \end{pmatrix} \begin{pmatrix} c_{0,n} \\ c_{1,n} \\ \vdots \\ c_{r-1,n} \end{pmatrix} = \begin{pmatrix} \alpha_{r,0,n} \\ \vdots \\ \alpha_{r,d_1-1,0} \\ \beta_{r,0,n} \\ \vdots \\ \beta_{r,d_2-1,0} \end{pmatrix}$$

Ansatz method

In general

$M((h_n)_n) \subset \mathcal{M}$ where $\mathcal{M} = \langle \phi_1, \ldots, \phi_k \rangle_{\mathcal{R}_\mathcal{C}}$.
For some $\mathbf{v}_i \in \mathcal{R}_\mathcal{C}^k$:

$$(h_{n+i})_n = (\phi_1, \ldots, \phi_k) \cdot \mathbf{v}_i$$

Leading to the system:

$$\left(\mathbf{v}_0 \middle| \cdots \middle| \mathbf{v}_{r-1}\right) \mathbf{c} = -\mathbf{v}_r.$$

Ansatz method

In general

$$(\mathbf{v}_0 | \cdots | \mathbf{v}_{r-1})\,\mathbf{c} = -\mathbf{v}_r.$$

Ansatz method

1. Choose r *big enough*.
2. Build $A = (\mathbf{v}_0 | \dots | \mathbf{v}_{r-1})$.
3. Build $\mathbf{b} = \mathbf{v}_r$.
4. $\mathbf{y} \leftarrow \texttt{solve}(A, -\mathbf{b})$.
5. Return $\mathbf{y}$.

Ansatz method

In general

$$(\mathbf{v}_0|\cdots|\mathbf{v}_{r-1})\,\mathbf{c} = -\mathbf{v}_r.$$

Ansatz method

1. Choose r *big enough*.
2. Build $A = (\mathbf{v}_0|\ldots|\mathbf{v}_{r-1})$.
3. Build $\mathbf{b} = \mathbf{v}_r$.
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Choosing r

$$A\mathbf{c} = -\mathbf{v}_r.$$

- We need for each n that the rank of A is equal to the rank of $(A|\mathbf{v}_r)$.
- Condition given by zeros of some minors **(Skolem problem)**
- Solve the system in each section (by Moore-Penrose-Inverse)
- Interlace all sections

Example

Consider the sequences

$$a_{n+1} + (-1)^n a_n = 0, \quad b_{n+1} + b_n = 0.$$

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Ansatz of order 2 for the sequence $(a_n + b_n)_n$ yields the linear system:

$$\begin{pmatrix} 1 & -(-1)^n \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This system is not solvable for even n.

Example

Consider the sequences

$$a_{n+1} + (-1)^n a_n = 0, \quad b_{n+1} + b_n = 0.$$

Ansatz of order 3 for the sequence $(a_n + b_n)_n$ yields the linear system:

$$\begin{pmatrix} 1 & -(-1)^n & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} -(-1)^n \\ 1 \end{pmatrix}$$

Here, the system is always solvable using the first and third column, yielding:

$$x_n = \frac{(1-(-1)^n)}{2}, \quad z_n = \frac{1+(-1)^n}{2},$$

Example

Consider the sequences

$$a_{n+1} + (-1)^n a_n = 0, \quad b_{n+1} + b_n = 0.$$

Ansatz of order 3 for the sequence $(a_n + b_n)_n$ yields the linear system:

$$\begin{pmatrix} 1 & -(-1)^n & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} -(-1)^n \\ 1 \end{pmatrix}$$

So the final equation for the addition $(c_n)_n = (a_n + b_n)_n$ is:

$$c_{n+3} - \frac{1 + (-1)^n}{2} c_{n+2} - \frac{(1 - (-1)^n)}{2} c_n = 0.$$

Current Implementation

Implementation of C^2-finite sequences in SageMath.

- Main developer: Philipp Nuspl
- Current features: operations with C^2-finite objects
- Skolem problem: heuristics for finite set of zeros.
- Solving systems: using guessing when possible.

Code available

Contact me or Philipp:

philipp.nuspl@jku.at

Conclusions and future work

Conclusions and Future work

Achievements

- Defined the computable class of C^2-finit sequences.
- Proved closure properties for them.
- Implemented these closure properties on SageMath.

Future work

- Creative telescoping problems on $\mathcal{C}^2$.
- Study the generating functions.
- SML-Theorem for D-finite sequences.
- Better implementation ($\mathcal{D}^2$, improve solving mechanics, ...)

Thank you!

Contact webpage:

- http://www.lix.polytechnique.fr/~jimenezpastor/
- https://www.dk-compmath.jku.at/people/philipp-nuspl