Calculations with Poisson brackets on $\mathbb{R}^d$: some bases (of graphs) are more equal than others.

INRIA (26/Apr/2024)

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?: Deform Poisson bracket such that $\{\cdot,\cdot\}(\varepsilon)$ stays Poisson.

Refs.: • M. Kontsevich (Ascona'96); • T. Willwacher [1009.1654]

• Bouisaghouane, Buring, AVK: [arXiv: 1608.01710]

• Buring, AVK: [arXiv: 1811.07878, 2112.03897, 2212.08063]

§1 Nambu-Poisson bracket on $\mathbb{R}^d \ni \underline{x}=(x^1,\ldots,x^d)$; $f,g\in C^\infty(\mathbb{R}^d)$

$$\{f,g\}_{P(\varrho,[\underline{a}])}(\underline{x}) = \varrho(\underline{x})\cdot\det\left(\frac{\partial(a_1,\ldots,a_{d-2},f,g)}{\partial(x^1,\ldots,x^d)}\right) = \varrho\cdot\left|\frac{\partial(f,g,a_1,\ldots,a_{d-2})}{\partial(x^1,\ldots,x^d)}\right|$$

• Casimirs $a_i\in C^\infty(\mathbb{R}^d)$; $\qquad x(x')\rightleftarrows x'(x)$:

$$\varrho(x^1,\ldots,x^d)\,\partial_{x^1}\wedge\ldots\wedge\partial_{x^d} = \varrho(x^{1'},\ldots,x^{d'})\cdot\left|\frac{\partial(x')}{\partial(x)}\right|\partial_{x^{1'}}\wedge\ldots\wedge\partial_{x^{d'}}.$$

Ex. $\{f,g\}_{2D}(x,y) = \varrho(x,y)\cdot\begin{vmatrix} f_x & g_x \\ f_y & g_y\end{vmatrix}$;

$$\{f,g\}_{3D}(x,y,z) = \varrho(x,y,z)\cdot\begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z\end{vmatrix};$$

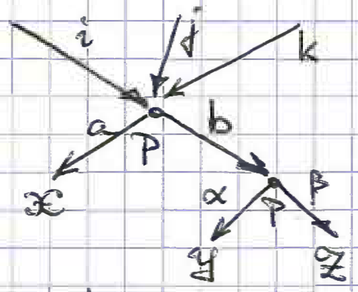
$$\{f,g\}_{4D}(x,y,z,w) = \varrho(x,y,z,w)\cdot\begin{vmatrix} f_x & g_x & a^1_x & a^2_x \\ f_y & g_y & a^1_y & a^2_y \\ f_z & g_z & a^1_z & a^2_z \\ f_w & g_w & a^1_w & a^2_w\end{vmatrix};$$

Ex. (Euler): $a(x,y,z)=\frac{1}{2}(x^2+y^2+z^2)$;

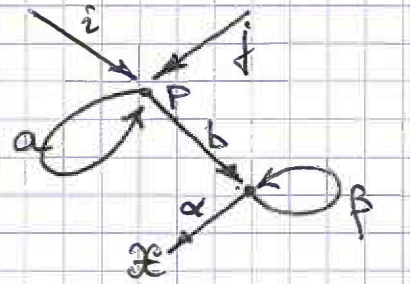
$\{x^i, x^j\} = \varepsilon^{ijk}\cdot x^k$;

§2 KONTSEVICH'S GRAPH CALCULUS on $(M^d_{aff} := \mathbb{R}^d;\ \forall P)$.

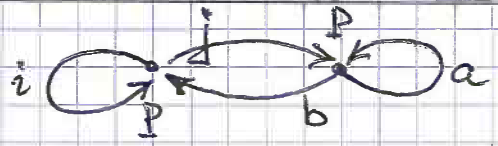
$\partial/\partial x^i \rightsquigarrow \xrightarrow{\ i\ }$ $\qquad P = (P^{ij}) \rightsquigarrow$ 

Ex. 

$\partial_i\partial_j\partial_k(P^{ab})\cdot\partial_b(P^{\alpha\beta})\cdot$
$\cdot\partial_a(x)\cdot\partial_\alpha(y)\cdot\partial_\beta(z)$

Ex. 

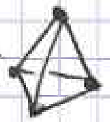
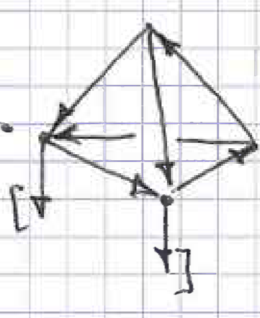
$\partial_i\partial_j\partial_a(P^{ab})\cdot\partial_b\partial_\beta(P^{\alpha\beta})\cdot\partial_\alpha(x)$

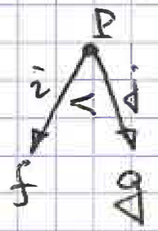
Ex. 

$\partial_b\partial_i(P^{ij})\cdot\partial_j\partial_a(P^{ab})$.

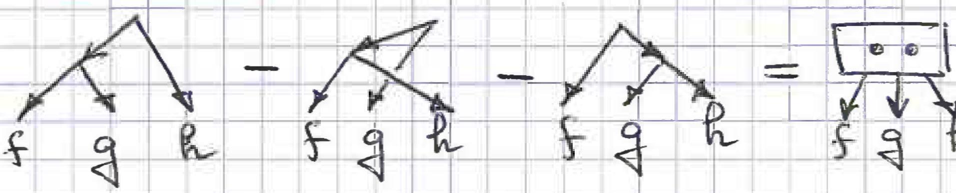
AVK

M. KONTSEVICH (1996):
GRAPH COMPLEX ACTION ON POISSON STRUCTURES.

Ex. $\gamma_3 =$  $\xmapsto{\ \text{OR}\ } \dot{P} = Q^{\gamma_3}(P) =$  $+3\cdot$ 

NB:  $\Leftrightarrow \partial_i(f)\cdot P^{ij}\cdot\partial_j(g)$

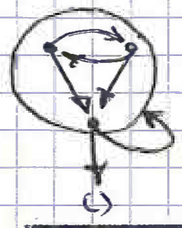
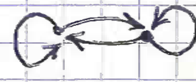
POISSON iff $\mathrm{Jac}(P) = \frac{1}{2}[\![P,P]\!] = 0$;

$\cdot\ \mathrm{Jac}(P)(f,g,h) =$ 

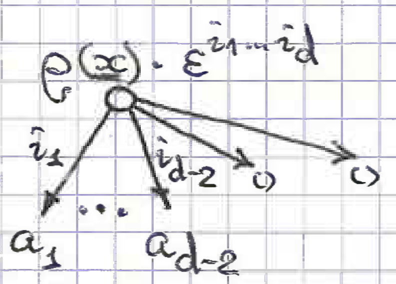
Th. (1996/2017; 2016).
$[\![P, Q^{\gamma_3}(P\otimes P\otimes P\otimes P)]\!] \overset{\checkmark}{=} 0$ whenever
$\mathrm{Jac}(P) = 0$.

$\cdot$ ($\cong$): $P \mapsto P + \varepsilon\cdot Q^{\gamma_3}(P) + \bar{o}(\varepsilon)$ is POISSON mod $\bar{o}(\varepsilon)$.

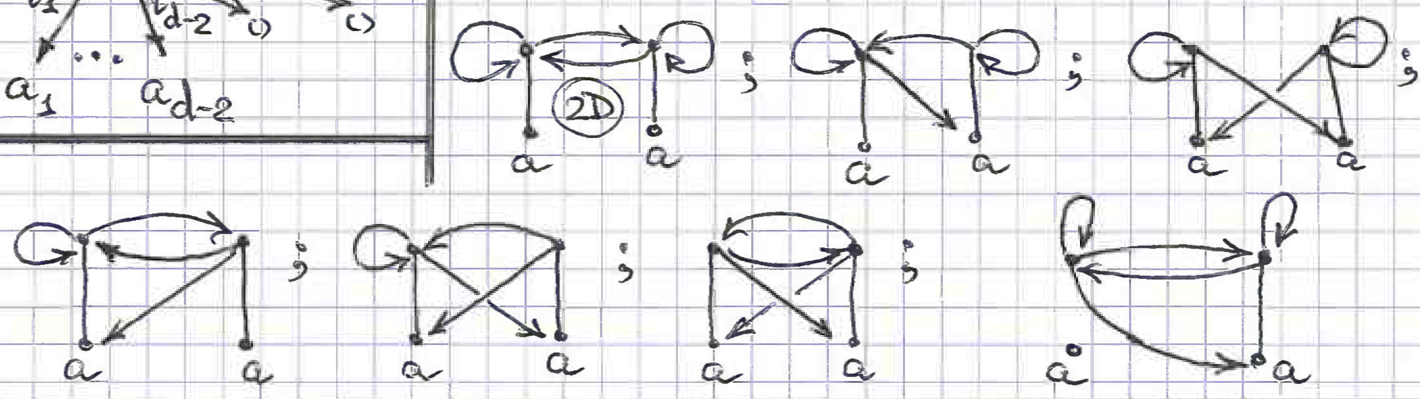
?: $Q^{\gamma_3} \overset{?}{=} [\![P, \vec{X}]\!]$ $\Big\} \leftarrow$ Does $\dot{P} = Q^{\gamma_3}(P)$ amount just to a change of coords. on $\mathbb{R}^d = M$ along trajectories of $\vec{X}$?

Ex. $(d=2)$: $\exists\ \vec{X}^{\gamma_3}_{2D} =$  $+ (\forall \text{const}) \cdot [\![P, \text{Ham} =$  $]\!]$;

§3 NAMBU $\{\cdot,\cdot\}_{P(\rho,[a^1,\ldots,a^{d-2}])}$: MICRO-GRAPH CALCULUS.

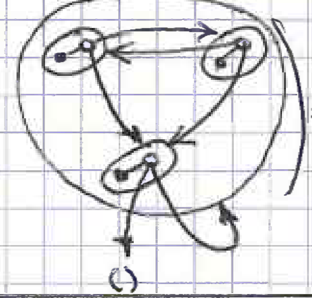


Ex. (3D: HAMILTONIANS = $\emptyset$-vectors)



NB: F-LA $Q^{\gamma_3}_{\dim}$: 2D = 2 lines; 3D = 3 pages; 4D = 3 Gb.

TABLE (1). (dim = 3)

$\#\left(\text{1-vector} \,\middle|\, \begin{array}{l}\#\rho=3\\ \#a=3\end{array}\right) = 366 \supsetneq \#\left(\text{1-vector} \,\middle|\right.$  $\left.\right) = 41$;

$\cup$

$\#(\text{lin. indep. f-las}) = 48$

$\cup$

$\#(\text{lin. indep. f-las}) = 2\emptyset$;

PROP. • R.B. (2020)

$Q^{\gamma_3}_{d=3}(P(\rho,[a])) \overset{\checkmark}{=} [\![P, \vec{X}^{\gamma_3}_{3D}(\rho,[a])]\!] \leftarrow$ #graphs = 11; dim(affine) = 5;

• MJB (2023)

$Q^{\gamma_3}_{d=3}(P(\rho,[a])) \overset{\checkmark}{=} [\![P, \vec{X}^{\gamma_3}_{3D}(P(\rho,[a]))]\!] \leftarrow$ #graphs = 9; dim(affine) = 3;

• FS (2024): • ALL #5 shifts $\Delta\vec{X}^{\gamma_3}_{3D}([\rho],[a]) \overset{\checkmark}{=} [\![P, \text{HAM}([\rho],[a])]\!]$;

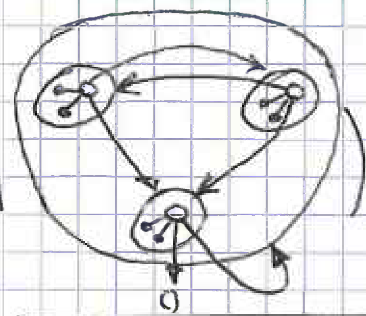
• ALL #3 shifts $\Delta\vec{X}(P(\rho,[a])) \overset{\checkmark}{=} [\![P, \text{Ham}(P(\rho,[a]))]\!]$.

(§4) SYMMETRY REDUCTIONS IN 4D PROBLEM.

AVK

[P]: $$Q^{\gamma_3 = \triangle}_{d=4}\left(P(\rho,[\underline{a}])^{\otimes \#V=4}\right) \overset{?}{=} [\![P, \vec{X}^{\gamma_3}_{4D}(P(\rho,[\underline{a}]))]\!];$$

TABLE (2): dim = 4.

$\#\left(\text{1-vector} \,\middle|\, \#\rho=3,\ \#a_1=3,\ \#a_2=3\right) = 19{,}957 \gneqq \#\left(\text{1-vector} \,\middle|\, \right.$  $\left.\right) = 324$

$\gneqq$

$\#(\text{lin. indep. as f-las}) = 123.$

PROP. • MJB + R.B. (2023):

(∃) $\vec{X}^{\gamma_3}_{4D}(P) = \langle \#56 \text{ graphs}\rangle$;

$\#\,\Delta\vec{X}^{\gamma_3}_{4D}(P) = 11 \ \} \leftarrow \dim(\text{affine space})$

• F.S. (2024): $\#\,\mathcal{H}am_{4D}(P(\rho,[\underline{a}])) = 21$ | $\#(\text{"}=0\text{" as f-la}) = 1$

$\#(\text{lin. relations}) = 8$

$\#(\text{lin. indep. } \mathcal{H}am) = 11+1.$

(&) All #11 shifts $\Delta\vec{X}^{\gamma_3}_{4D} \overset{\checkmark}{=} [\![P, \mathcal{H}am(P)]\!]$.

[S_{d-2}]: (det) LEMMA. $P(\rho,[a_1],[a_2]) \overset{a_1 \rightleftarrows a_2}{=} (-)\,P(\rho,[a_2],[a_1])$;

(⇒) • $Q^{\triangle}_{d>3}(\underbrace{P\otimes P\otimes P\otimes P}_{(-)^4=(+)}) \overset{?}{=} [\![\underset{(-)}{P}, \vec{X}^{\gamma_3}_{d>3}(\underbrace{P^{\otimes 3}}_{(-)})]\!]$; $\} \leftarrow \vec{X}^{\gamma_3}$ is SKEW.

TABLE (3): dim = 4. • $\Delta\vec{X}^{\gamma_3}_{d>3}(P^{\otimes 3}) \overset{?}{=} [\![P, \mathcal{H}am(P\otimes P)]\!]$: (+)

• #Good (symm) $\mathcal{H}am^{(+)}(P^{\otimes 2}) = 8 = 7 + 1_{(2D)}$.

• #Good (skew) 1-vectors$^{(-)}(P^{\otimes 3}) = 64$.

PROP. • MJB (2024): (∃) $\vec{X}^{\gamma_3}_{4D}(P) = \langle \#27 \text{ skew pairs}\rangle$;

• FS (2024): All #7 shifts $\Delta\vec{X}^{\gamma_3}_{4D,\text{skew}}(P) \overset{\checkmark}{=} [\![P, \mathcal{H}am^{(+)}(P)]\!]$.

[(d+1)↓d]: REQUIRE $\vec{X}^{\gamma}_{d+1}(a_{d-1} := x^{d+1})$ (BE) A SOLUTION $\big|_d$.

(?): DIMENSIONAL STEP $d \mapsto d+1$? $\Longleftrightarrow$ Obstructions?