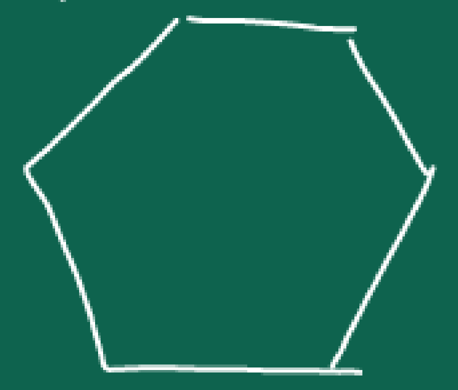
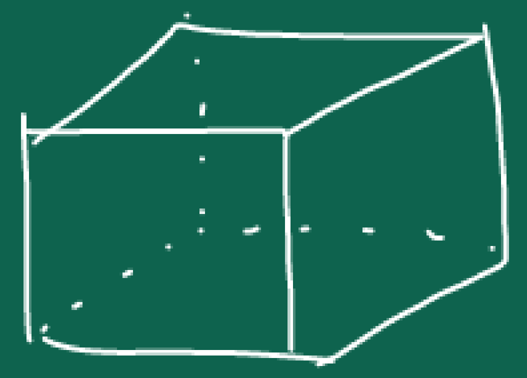
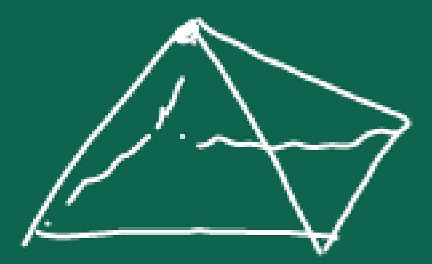
Toric amplitudes & universal adjoints (2504.00897)

1 Definitions

P – simple polytope in $\mathbb{R}^d$, full-dim'l

examples:  

non-examples: 

facet representation: $P = \{ y \in \mathbb{R}^d : \overset{n \times d}{U} \cdot y + \overset{\mathbb{R}^n}{Z} \geqslant 0 \}$

$\Sigma'(d) = \{\text{vertices of } P\}$, $\Sigma'(1) = \{\text{facets of } P\}$

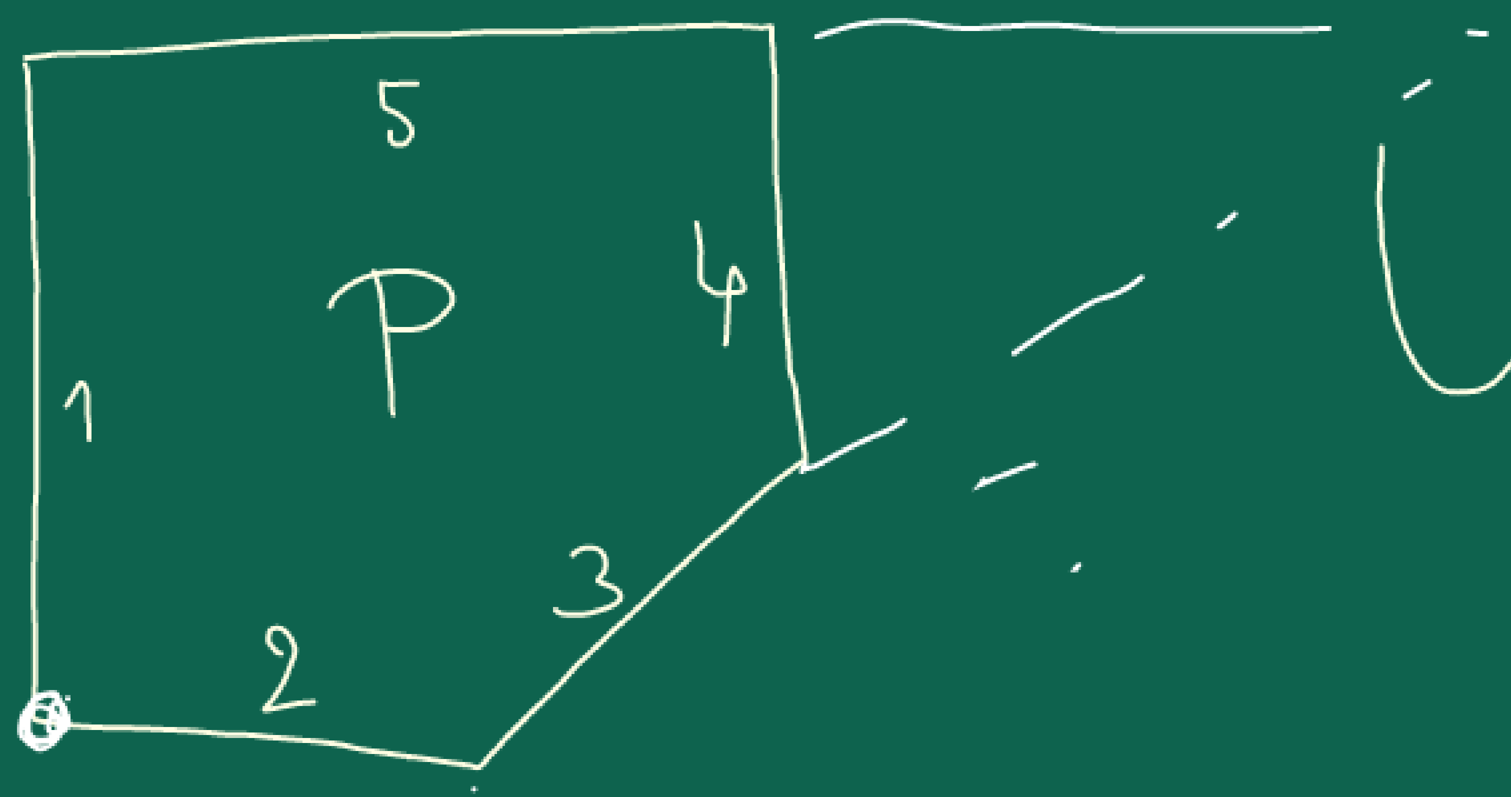
"$\rho \in \sigma(1)$" $\rho \in \Sigma'(1), \sigma \in \Sigma'(d)$

"facet$_\rho$ contains the vertex $\underline{\sigma}$"

"toric amplitude": $\mathrm{Amp}_P = \sum_{\sigma \in \Sigma'(d)} \frac{|\det U_\sigma|}{\prod_{\rho \in \sigma(1)} x_\rho} \in \mathbb{R}(x_\rho : \rho \in \Sigma'(1))$

"universal adjoint": $\mathrm{Adj}_P = \left(\prod_\rho x_\rho\right) \cdot \mathrm{Amp}_P = \sum_{\sigma \in \Sigma'(d)} |\det U_\sigma| \cdot \prod_{\rho \notin \sigma(1)} x_\rho$

Example: pentagon $(d=2,\ n=5)$



$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$U \cdot y + z \geqslant 0$$

$$y_1 + 1 \geqslant 0$$

$$\vdots$$

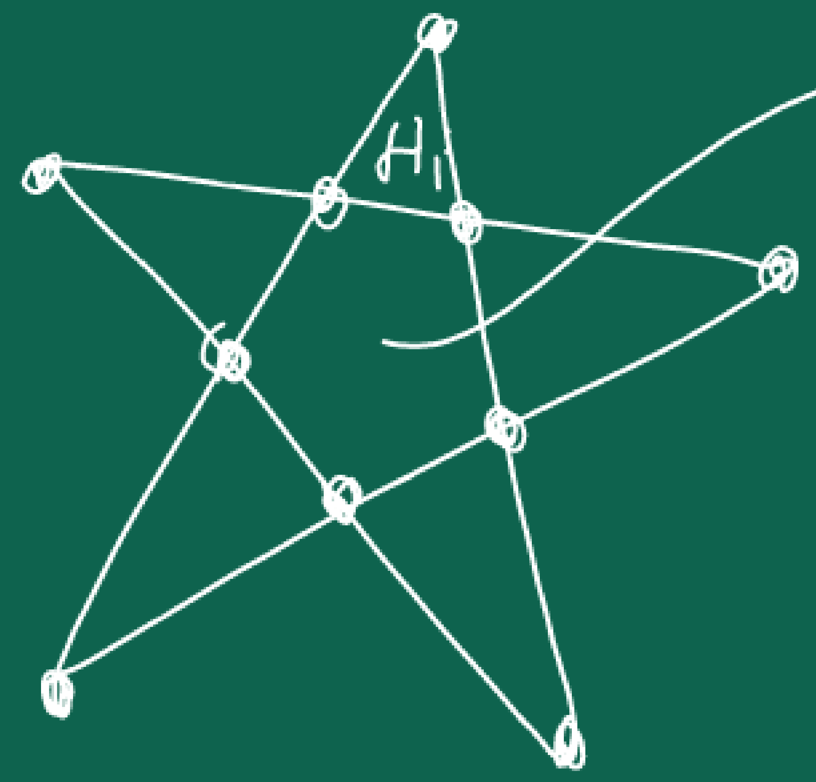
$$-y_2 + 1 \geqslant 0$$

$$\mathrm{Amp}_P = \frac{1}{X_1 X_2} + \frac{1}{X_2 X_3} + \frac{1}{X_3 X_4} + \frac{1}{X_4 X_5} + \frac{1}{X_1 X_5}$$

$$\mathrm{Adj}_P = X_3 X_4 X_5 + X_1 X_4 X_5 + X_1 X_2 X_5 + X_1 X_2 X_3 + X_2 X_3 X_4$$

$\mathcal{A}_P = \{ x \in \mathbb{P}^{n-1} : \mathrm{Adj}_P(x) = 0 \}$ "universal adjoint hypersurface"

for P = pentagon, the cubic threefold $\mathcal{A}_P \subseteq \mathbb{P}^4$ is the Segre cubic. It contains 10 nodes and 15 planes. $(15_4, 10_6)$. Project away from the line $\mathbb{P}(\mathrm{im}\, U)$.



Deformation cone of P.

GOAL: study linear spaces contained in $\mathcal{A}_P$ and $\mathrm{Sing}(\mathcal{A}_P)$.

② Motivation

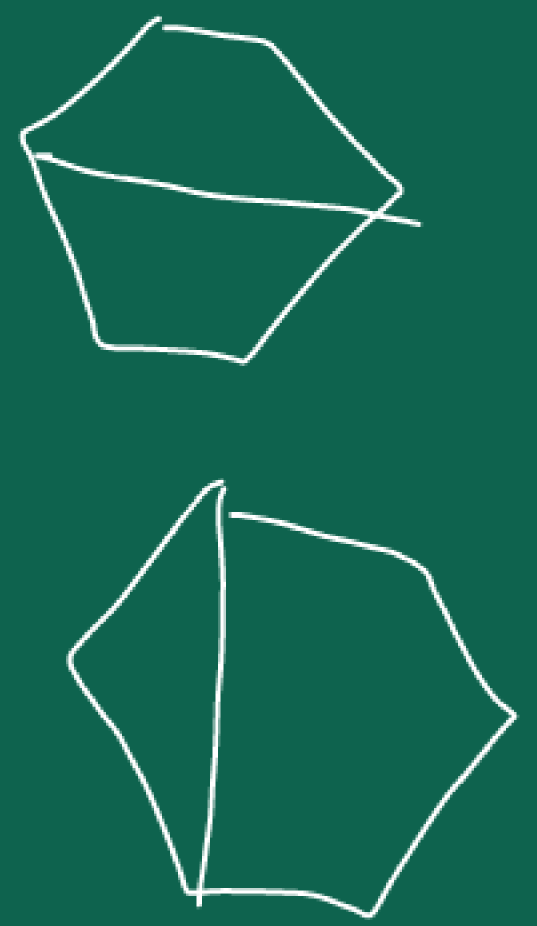
Particle physics: $P \subset \mathbb{R}^d$ is the ABHY associahedron.

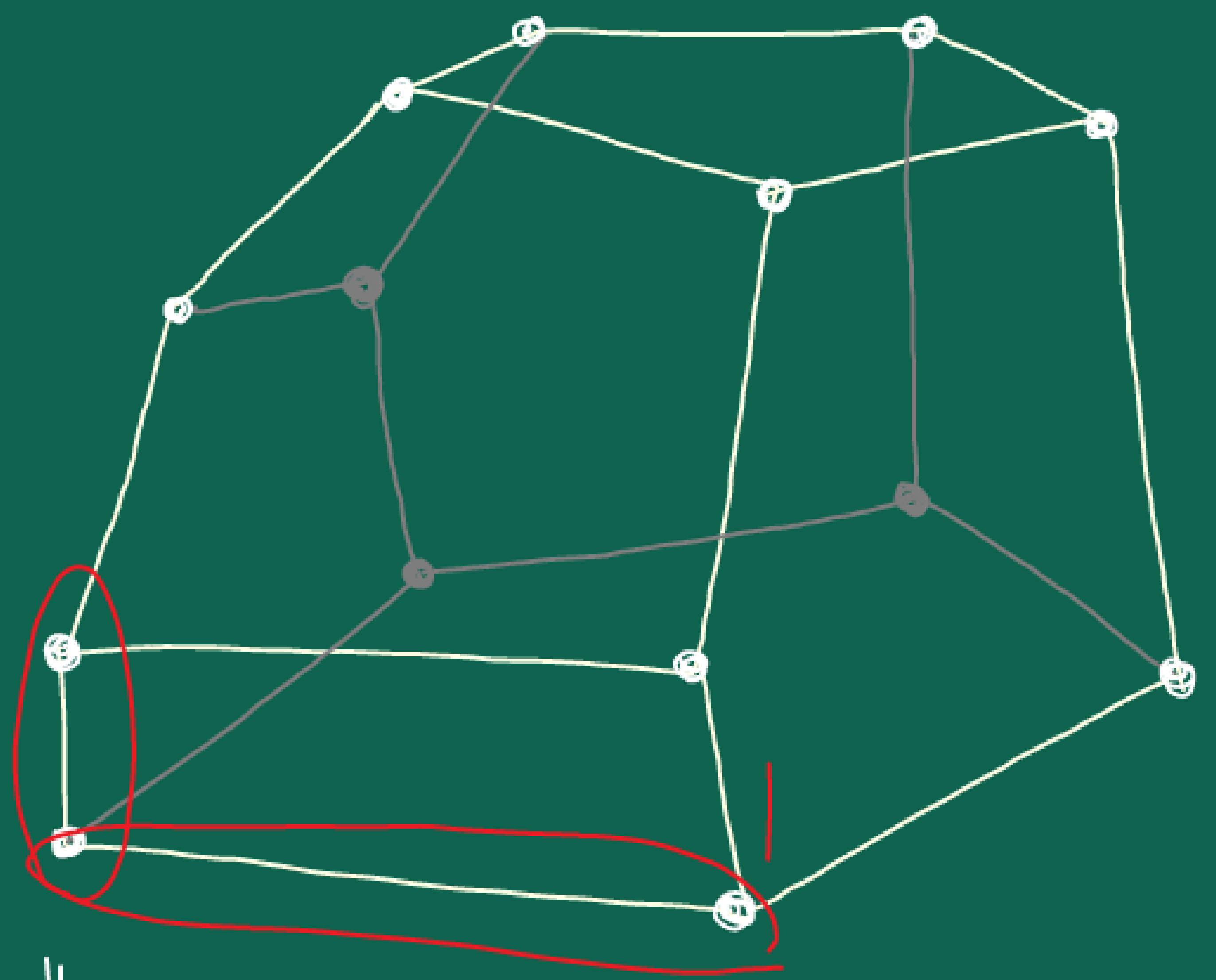
Amp_P is a scattering amplitude

Study "factorization" $\approx$ finding linear spaces in $\mathcal{A}_P$

Example : 3-dim'l ABHY associahedron

$\frac{1}{x} + \frac{1}{y}$

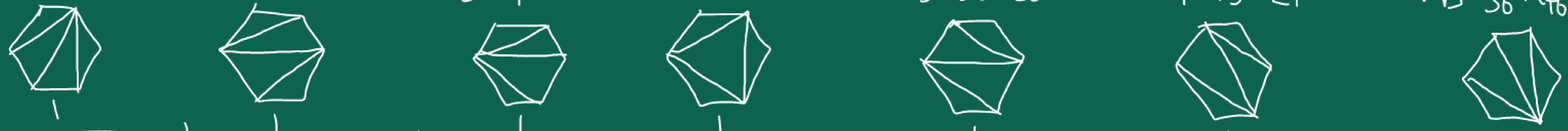




$$U = \begin{array}{c} \begin{array}{ccccccccc} 13 & 14 & 15 & 24 & 25 & 26 & 35 & 36 & 46 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}^t \end{array}$$

$$P = \left\{ y \in \mathbb{R}^3 : U \cdot y + \begin{pmatrix} 3 \\ 4 \\ 3 \\ 2 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \geq 0 \right\}$$

$$\frac{1}{X_{15}X_{25}X_{35}} + \frac{1}{X_{13}X_{14}X_{46}} + \frac{1}{X_{13}X_{14}X_{15}} + \frac{1}{X_{13}X_{15}X_{35}} + \frac{1}{X_{13}X_{35}X_{36}} + \frac{1}{X_{14}X_{15}X_{24}} + \frac{1}{X_{13}X_{36}X_{46}}$$



$$\frac{1}{X_{14}X_{24}X_{46}} + \frac{1}{X_{15}X_{24}X_{25}} + \frac{1}{X_{24}X_{25}X_{26}} + \frac{1}{X_{24}X_{26}X_{46}} + \frac{1}{X_{25}X_{26}X_{35}} + \frac{1}{X_{26}X_{35}X_{36}} + \frac{1}{X_{26}X_{36}X_{46}}$$

$$\left(x_{14} \cdot \mathrm{Amp}_P\right)\Big|_{x_{14}=0} = \left(\frac{1}{x_{13}} + \frac{1}{x_{24}}\right)\left(\frac{1}{x_{15}} + \frac{1}{x_{46}}\right)$$

$$\left(\mathrm{Adj}_P\right)\Big|_{x_{14}=0} = x_{25}\, x_{26}\, x_{36}\, x_{35} \left(x_{13} + x_{24}\right)\left(x_{15} + x_{46}\right)$$

$$\Lambda_\tau = \{x_{14} = 0\}$$

Convex geometry

$$P_x = \{ y \in \mathbb{R}^d : Uy + x \geq 0 \}$$

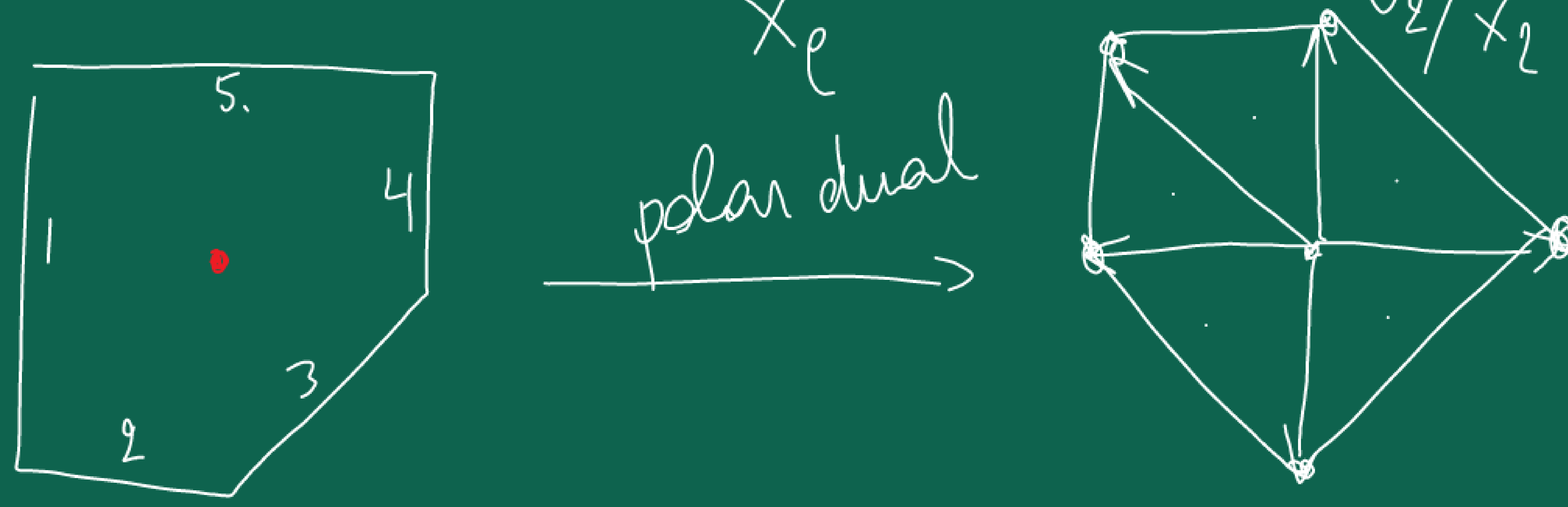
$x \in \mathbb{R}^n_+$ notice: $0 \in \mathrm{int}(P_x)$

(bounded) polar dual: $P_x^\circ = \{ v \in (\mathbb{R}^d)^\vee : v \cdot y \geq -1,\ \forall y \in P_x \}$

has vertices $\dfrac{u_\rho}{x_\rho}$, $\rho \in \Sigma(1)$

$$2\,\mathrm{Vol}(P_x^\circ) = \frac{\det U_{12}}{x_1 x_2} + \frac{\det U_{23}}{x_2 x_3} + \cdots$$

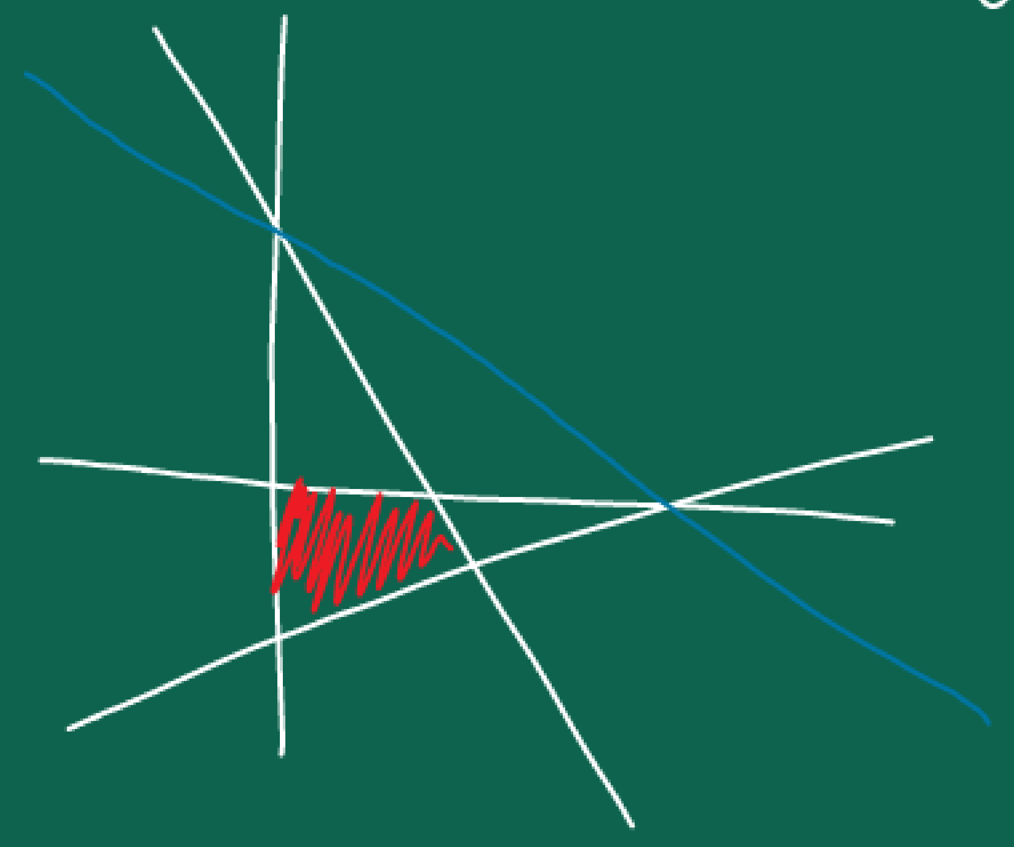
Example

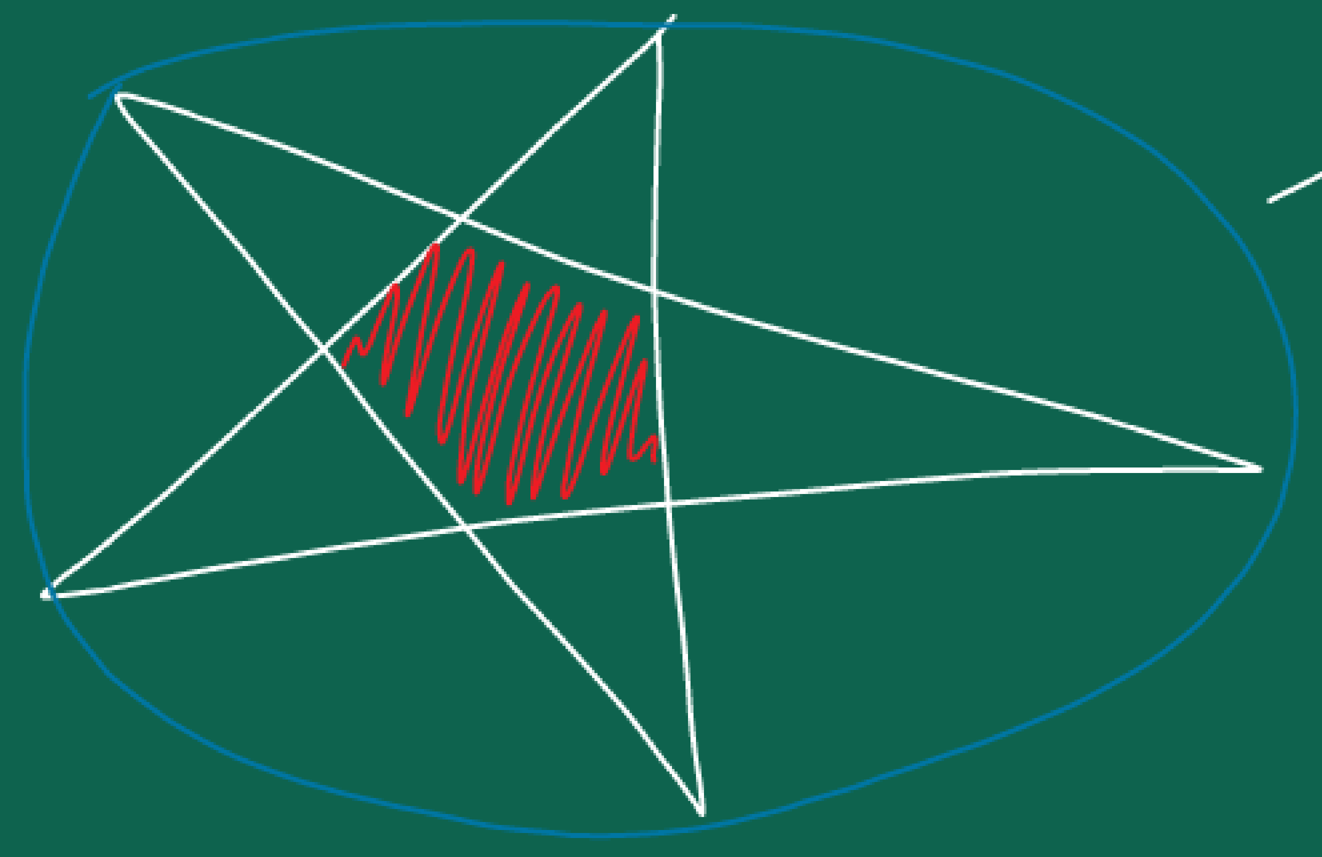


$\mathrm{Amp}_P(x) = d!\,\mathrm{Vol}(P_x^{\circ})$ for x in the deformation cone of P

canonical function of P_z : $d!\,\mathrm{Vol}\left((P_z - y)^{\circ}\right)$, $y \in \mathrm{int}(P_z)$

$$= \frac{\mathrm{adj}_{P_z}(y)}{(u_1 \cdot y + z_1) \cdots (u_n \cdot y + z_n)}$$ → Warren's adjoint





is this curve generally smooth?

3 Results

τ face of P

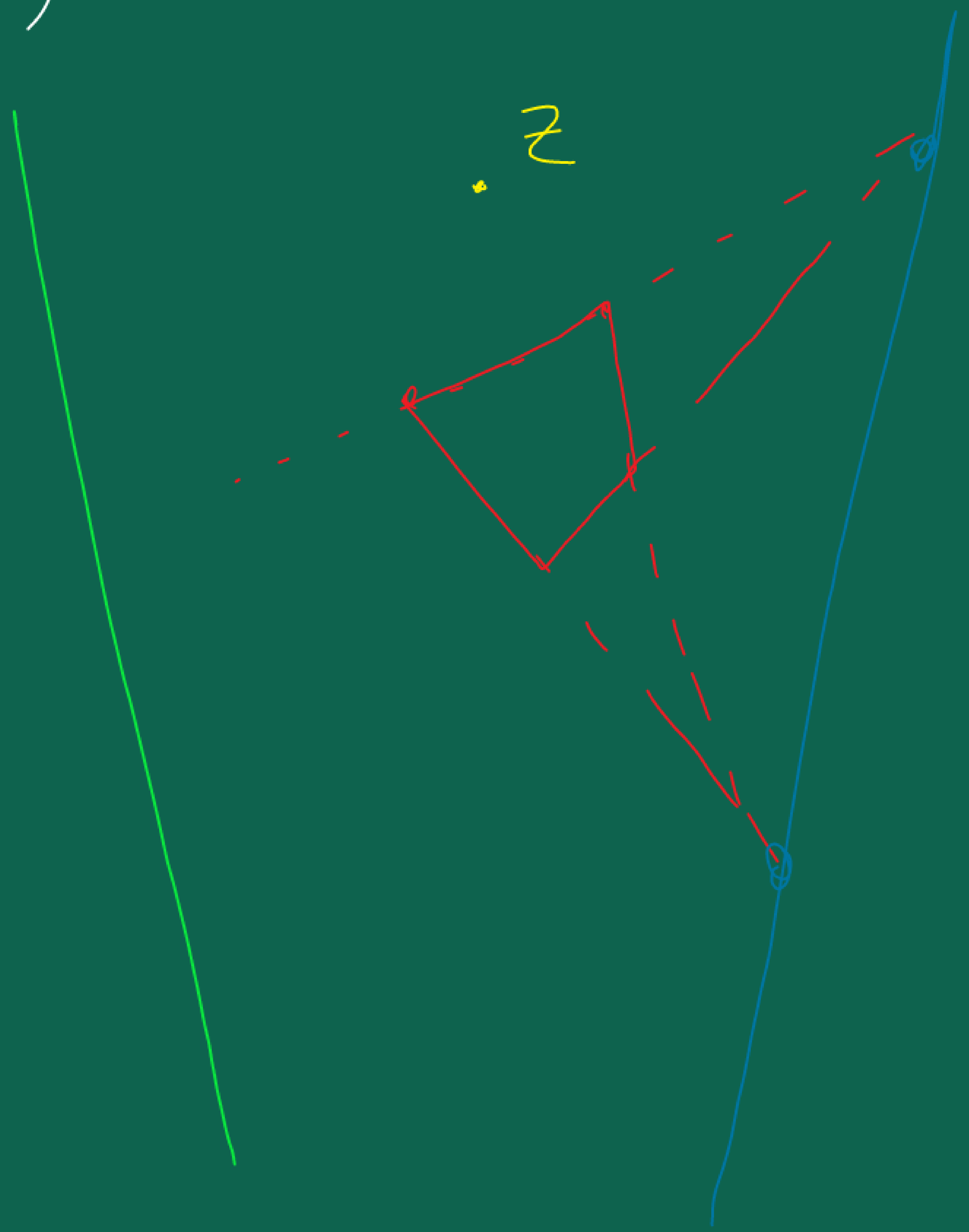
$$\Lambda_\tau = V(x_\ell : \ell \in \tau(1))$$

Prop If τ is a product of simplices of dim. $d-k$ then $A_P \cap \Lambda_\tau$ is a union of $(n-k-2)$-planes

<u>Lemma</u>: $\mathrm{Adj}_{P}\left(\underbrace{U_{y+z}}\right) = \mathrm{adj}_{P_z}(y)$ for $z \in \mathrm{Def}(P)$

<u>Theorem</u> $\mathbb{P}(\operatorname{im} U) \subseteq A_{\cancel{P}\,\Sigma}$

P = quadrilateral $(d=2,\ n=4)$



Theorem For a generic n-gon, $\mathrm{Sing}(\mathcal{A}_P)$ has dimension $\leq n-4$

Corollary: For a generic n-gon, $\mathrm{adj}_{P_z}(y) = 0$ is smooth.