```
> restart:
```

Coefficient in a polynomial

```
> with(gfun):
```

```
> P := (1+t)^10000 * (1+t + t^3)^60000;
```

$$P := (1+t)^{10000} \left(t^3+t+1\right)^{60000} \quad \textbf{(1.1)}$$

```
> factor(diff(P, t)/P);
```

$$\frac{10000 \left(19\, t^3+18\, t^2+7\, t+7\right)}{\left(t^3+t+1\right)(1+t)} \quad \textbf{(1.2)}$$

```
> diffeq := {diff(y(t), t) - %*y(t), y(0)=1};
```

$$diffeq := \left\{\frac{\mathrm{d}}{\mathrm{d}t}\, y(t) - \frac{10000 \left(19\, t^3+18\, t^2+7\, t+7\right) y(t)}{\left(t^3+t+1\right)(1+t)},\ y(0)=1\right\} \quad \textbf{(1.3)}$$

```
> rec := diffeqtorec(diffeq, y(t), u(n));
```

$$rec := \{(-190000+n)\, u(n) + (-179999+n)\, u(n+1) + (-69998+n)\, u(n+2) + (-69994+2\, n)\, u(n+3) + (n+4)\, u(n+4),\ u(0)=1,\ u(1)=70000,\ u(2) = 2449965000,\ u(3) = 57164216750000\} \quad \textbf{(1.4)}$$

```
> pr := rectoproc(rec, u(n)):
```

```
> pr(1000);
```

(1.5)

```
28669846942057747534032313545844244597209642473669210074390115\
    39129848268594879721118941698787109993408396294630161199555 0\
    70875856384643733054766271674515940320967978124507431500071 1\
    64390119533434244381775520038518209277957733577585678667785 2\
    97268906065034798765356587222555761705086704347572407743167 6\
    27853427048624417237183631808557909744008094778453124447230 4\
    34630692604244601834635135099468101094798651468871244488641 3\
    76999901989869282635847344874808018139013913904119079167018 0\
    22680834471997693646900125565742429956594875238431790158550 1\
    33501168631383003655802137302035267905175576024887954362781 6\
    02001865140333110670736610338052924797337431236148842693858 2\
    42990928889000068070421106522680370233678912952943038396124 4\
    00331693965944143705182295401507074205788780204215483646723 1\
    81314630617642570952582115221674654872888348452629268331280 1\
    98834499419653369205843147206298569991341255482344843179384 9\
```

```
525176846207285615742722790556768050013031243064920878757570\
036988652963532212313507417166347313204129363919478408712079\
553327838069172082225611340297242087061649177657642228045018\
564861226882696028300087749939559310072872422122335473858843\
018374929655728892339131759453227065657269247708133165665591\
243065576412919020371341512972196629784923290717692140291535\
080820538485210701499512266113018198950536054355551857263686\
118226269896226034174765239298829163747603339560659753104978\
178144854163504329862287022806782910084170938201563646896931\
047571456545594892637928386954914442382484320342946396202914\
724688363383917271328139602193871883652389540440622489452861\
725894248887627428904186676538733387079996144739284202348050\
249910693100537258506431729763475508753444489314265149025653\
650393296292994619271220433266020791051318177354662949062018\
764197282357647662908462718818643689826602061681912719364095\
741780612685670367049592304706494937312148348513237882863018\
243256321303443086219302402134296481738527464554870309357445\
194978340994751255035206526398493828530704504319937376217987\
452029459251949258306916950583237064002052501195989405748720\
884867959240321299467773047371970010454814599990502494587696\
097625287143027150090197853789986626802200241034537182344341\
070168798599421663408798912584151871284118597178245761057456\
4410434883706506559348196640812873004862702305133898
```

```
> # coeff(expand(P), t, 1000); # ARGH!!
```

```
> coeff(series(P, t, 1001),t, 1000);  # OK, but slow
```

```
286698469420577475340323135458442445972096424736692100743901115\   (1.6)
391298482685948797211189416987871099934083962946301611995550\
708758563846437330547662716745159403209679781245074315000711\
643901195334342443817755200385182092779577335775856786677852\
972689060650347987653565872225557617050867043475724077431676\
278534270486244172371836318085579097440080947784531244472304\
346306926042446018346351350994681010947986514688712444886413\
769999019898692826358473448748080181390139139041190791670180\
226808344719976936469001255657424299565948752384317901585501\
335011686313830036558021373020352679051755760248879543627816\
020018651403331106707366103380529247973374312361488426938582\
429909288890000680704211065226803702336789129529430383961244\
```

```
003316939659441437051822954015070742057887802042154836467231\
813146306176425709525821152216746548728883484526292683312801\
988344994196533692058431472062985699913412554823448431793849\
525176846207285615742722790556768050013031243064920878757570\
036988652963532212313507417166347313204129363919478408712079\
553327838069172082225611340297242087061649177657642228045018\
564861226882696028300087749939559310072872422122335473858843\
018374929655728892339131759453227065657269247708133165665591\
243065576412919020371341512972196629784923290717692140291535\
080820538485210701499512266113018198950536054355551857263686\
118226269896226034174765239298829163747603339560659753104978\
178144854163504329862287022806782910084170938201563646896931\
047571456545594892637928386954914442382484320342946396202914\
724688363383917271328139602193871883652389540440622489452861\
725894248887627428904186676538733387079996144739284202348050\
249910693100537258506431729763475508753444489314265149025653\
650393296292994619271220433266020791051318177354662949062018\
764197282357647662908462718818643689826602061681912719364095\
741780612685670367049592304706494937312148348513237882863018\
243256321303443086219302402134296481738527464554870309357445\
194978340994751255035206526398493828530704504319937376217987\
452029459251949258306916950583237064002052501195989405748720\
884867959240321299467773047371970010454814599990502494587696\
097625287143027150090197853789986626802200241034537182344341\
070168798599421663408798912584151871284118597178245761057456\
4410434883706506559348196640812873004862702305133898
```

Motzkin numbers

```
> with(gfun):
```

```
> algeq := eliminate({K = t^2*M+t, M = K*M+1}, K)[2,1];
```

$$algeq := -t^2 M^2 - t M + M - 1 \tag{2.1}$$

```
> diffeq := algeqtodiffeq(algeq, M(t), homogeneous=true);
```

$$diffeq := \left\{(6\,t+3)\,M(t) + \left(12\,t^2+7\,t-3\right)\left(\frac{d}{dt}\,M(t)\right) + \left(3\,t^3+2\,t^2-t\right)\left(\frac{d^2}{dt^2}\,M(t)\right), M(0)=1\right\} \quad \textbf{(2.2)}$$

```
> rec := diffeqtorec(diffeq, M(t), u(n));
```

$$rec := \{(3\,n+3)\,u(n) + (2\,n+5)\,u(n+1) + (-n-4)\,u(n+2), u(0)=1, u(1)=1\} \quad \textbf{(2.3)}$$

Guessing

```
> L := [seq(add(binomial(n,k)^3, k=0..n), n=0..12)];
```

$$L := [1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260, 278415920, 2046924400] \quad \textbf{(3.1)}$$

```
> f := add(L[i+1]*t^i, i=0..nops(L)-1);
```

$$f := 2046924400\,t^{12} + 278415920\,t^{11} + 38165260\,t^{10} + 5280932\,t^9 + 739162\,t^8 + 104960\,t^7 + 15184\,t^6 + 2252\,t^5 + 346\,t^4 + 56\,t^3 + 10\,t^2 + 2\,t + 1 \quad \textbf{(3.2)}$$

```
> add(add(a[i,j]*t^i*`if`(j > 0, t^j*diff(f, t$j), f), i=0..2), j=
  0..2);
```

$$a_{0,0}\left(2046924400\,t^{12} + 278415920\,t^{11} + 38165260\,t^{10} + 5280932\,t^9 + 739162\,t^8 + 104960\,t^7 + 15184\,t^6 + 2252\,t^5 + 346\,t^4 + 56\,t^3 + 10\,t^2 + 2\,t + 1\right) + a_{1,0}\,t\left(2046924400\,t^{12} + 278415920\,t^{11} + 38165260\,t^{10} + 5280932\,t^9 + 739162\,t^8 + 104960\,t^7 + 15184\,t^6 + 2252\,t^5 + 346\,t^4 + 56\,t^3 + 10\,t^2 + 2\,t + 1\right) + a_{2,0}\,t^2\left(2046924400\,t^{12} + 278415920\,t^{11} + 38165260\,t^{10} + 5280932\,t^9 + 739162\,t^8 + 104960\,t^7 + 15184\,t^6 + 2252\,t^5 + 346\,t^4 + 56\,t^3 + 10\,t^2 + 2\,t + 1\right) + a_{0,1}\,t\left(24563092800\,t^{11} + 3062575120\,t^{10} + 381652600\,t^9 + 47528388\,t^8 + 5913296\,t^7 + 734720\,t^6 + 91104\,t^5 + 11260\,t^4 + 1384\,t^3 + 168\,t^2 + 20\,t + 2\right) + a_{1,1}\,t^2\left(24563092800\,t^{11} + 3062575120\,t^{10} + 381652600\,t^9 + 47528388\,t^8 + 5913296\,t^7 + 734720\,t^6 + 91104\,t^5 + 11260\,t^4 + 1384\,t^3 + 168\,t^2 + 20\,t + 2\right) + a_{2,1}\,t^3\left(24563092800\,t^{11} + 3062575120\,t^{10} \quad \textbf{(3.3)}$$

$$+ 381652600\, t^9 + 47528388\, t^8 + 5913296\, t^7 + 734720\, t^6 + 91104\, t^5$$
$$+ 11260\, t^4 + 1384\, t^3 + 168\, t^2 + 20\, t + 2\big) + a_{0,\,2}\, t^2 \big(270194020800\, t^{10}$$
$$+ 30625751200\, t^9 + 3434873400\, t^8 + 380227104\, t^7 + 41393072\, t^6$$
$$+ 4408320\, t^5 + 455520\, t^4 + 45040\, t^3 + 4152\, t^2 + 336\, t + 20\big)$$
$$+ a_{1,\,2}\, t^3 \big(270194020800\, t^{10} + 30625751200\, t^9 + 3434873400\, t^8$$
$$+ 380227104\, t^7 + 41393072\, t^6 + 4408320\, t^5 + 455520\, t^4 + 45040\, t^3$$
$$+ 4152\, t^2 + 336\, t + 20\big) + a_{2,\,2}\, t^4 \big(270194020800\, t^{10} + 30625751200\, t^9$$
$$+ 3434873400\, t^8 + 380227104\, t^7 + 41393072\, t^6 + 4408320\, t^5 + 455520\, t^4$$
$$+ 45040\, t^3 + 4152\, t^2 + 336\, t + 20\big)$$

```
> series(%, t, 13);
```

$$a_{0,\,0} + \big(2\, a_{0,\,0} + a_{1,\,0} + 2\, a_{0,\,1}\big)\, t + \big(10\, a_{0,\,0} + 2\, a_{1,\,0} + a_{2,\,0} + 20\, a_{0,\,1} + 2\, a_{1,\,1} \tag{3.4}$$
$$+ 20\, a_{0,\,2}\big)\, t^2 + \big(56\, a_{0,\,0} + 10\, a_{1,\,0} + 2\, a_{2,\,0} + 168\, a_{0,\,1} + 20\, a_{1,\,1} + 2\, a_{2,\,1}$$
$$+ 336\, a_{0,\,2} + 20\, a_{1,\,2}\big)\, t^3 + \big(346\, a_{0,\,0} + 56\, a_{1,\,0} + 10\, a_{2,\,0} + 1384\, a_{0,\,1}$$
$$+ 168\, a_{1,\,1} + 20\, a_{2,\,1} + 4152\, a_{0,\,2} + 336\, a_{1,\,2} + 20\, a_{2,\,2}\big)\, t^4 + \big(2252\, a_{0,\,0}$$
$$+ 346\, a_{1,\,0} + 56\, a_{2,\,0} + 11260\, a_{0,\,1} + 1384\, a_{1,\,1} + 168\, a_{2,\,1} + 45040\, a_{0,\,2}$$
$$+ 4152\, a_{1,\,2} + 336\, a_{2,\,2}\big)\, t^5 + \big(15184\, a_{0,\,0} + 2252\, a_{1,\,0} + 346\, a_{2,\,0}$$
$$+ 91104\, a_{0,\,1} + 11260\, a_{1,\,1} + 1384\, a_{2,\,1} + 455520\, a_{0,\,2} + 45040\, a_{1,\,2}$$
$$+ 4152\, a_{2,\,2}\big)\, t^6 + \big(104960\, a_{0,\,0} + 15184\, a_{1,\,0} + 2252\, a_{2,\,0} + 734720\, a_{0,\,1}$$
$$+ 91104\, a_{1,\,1} + 11260\, a_{2,\,1} + 4408320\, a_{0,\,2} + 455520\, a_{1,\,2} + 45040\, a_{2,\,2}\big)\, t^7$$
$$+ \big(739162\, a_{0,\,0} + 104960\, a_{1,\,0} + 15184\, a_{2,\,0} + 5913296\, a_{0,\,1} + 734720\, a_{1,\,1}$$
$$+ 91104\, a_{2,\,1} + 41393072\, a_{0,\,2} + 4408320\, a_{1,\,2} + 455520\, a_{2,\,2}\big)\, t^8$$
$$+ \big(5280932\, a_{0,\,0} + 739162\, a_{1,\,0} + 104960\, a_{2,\,0} + 47528388\, a_{0,\,1}$$
$$+ 5913296\, a_{1,\,1} + 734720\, a_{2,\,1} + 380227104\, a_{0,\,2} + 41393072\, a_{1,\,2}$$
$$+ 4408320\, a_{2,\,2}\big)\, t^9 + \big(38165260\, a_{0,\,0} + 5280932\, a_{1,\,0} + 739162\, a_{2,\,0}$$
$$+ 381652600\, a_{0,\,1} + 47528388\, a_{1,\,1} + 5913296\, a_{2,\,1} + 3434873400\, a_{0,\,2}$$
$$+ 380227104\, a_{1,\,2} + 41393072\, a_{2,\,2}\big)\, t^{10} + \big(278415920\, a_{0,\,0}$$
$$+ 38165260\, a_{1,\,0} + 5280932\, a_{2,\,0} + 3062575120\, a_{0,\,1} + 381652600\, a_{1,\,1}$$
$$+ 47528388\, a_{2,\,1} + 30625751200\, a_{0,\,2} + 3434873400\, a_{1,\,2}$$

$+ 380227104\, a_{2, 2}) t^{11} + (2046924400\, a_{0, 0} + 278415920\, a_{1, 0}$
$+ 38165260\, a_{2, 0} + 24563092800\, a_{0, 1} + 3062575120\, a_{1, 1}$
$+ 381652600\, a_{2, 1} + 270194020800\, a_{0, 2} + 30625751200\, a_{1, 2}$
$+ 3434873400\, a_{2, 2}) t^{12} + \mathrm{O}(t^{13})$

```
> eqs := [seq(coeff(%, t, i), i=0..12)];
```

$$eqs := [a_{0, 0}, 2\, a_{0, 0} + a_{1, 0} + 2\, a_{0, 1}, 10\, a_{0, 0} + 2\, a_{1, 0} + a_{2, 0} + 20\, a_{0, 1} + 2\, a_{1, 1} + 20\, a_{0, 2}, 56\, a_{0, 0} + 10\, a_{1, 0} + 2\, a_{2, 0} + 168\, a_{0, 1} + 20\, a_{1, 1} + 2\, a_{2, 1} + 336\, a_{0, 2} + 20\, a_{1, 2}, 346\, a_{0, 0} + 56\, a_{1, 0} + 10\, a_{2, 0} + 1384\, a_{0, 1} + 168\, a_{1, 1} + 20\, a_{2, 1} + 4152\, a_{0, 2} + 336\, a_{1, 2} + 20\, a_{2, 2}, 2252\, a_{0, 0} + 346\, a_{1, 0} + 56\, a_{2, 0} + 11260\, a_{0, 1} + 1384\, a_{1, 1} + 168\, a_{2, 1} + 45040\, a_{0, 2} + 4152\, a_{1, 2} + 336\, a_{2, 2}, 15184\, a_{0, 0} + 2252\, a_{1, 0} + 346\, a_{2, 0} + 91104\, a_{0, 1} + 11260\, a_{1, 1} + 1384\, a_{2, 1} + 455520\, a_{0, 2} + 45040\, a_{1, 2} + 4152\, a_{2, 2}, 104960\, a_{0, 0} + 15184\, a_{1, 0} + 2252\, a_{2, 0} + 734720\, a_{0, 1} + 91104\, a_{1, 1} + 11260\, a_{2, 1} + 4408320\, a_{0, 2} + 455520\, a_{1, 2} + 45040\, a_{2, 2}, 739162\, a_{0, 0} + 104960\, a_{1, 0} + 15184\, a_{2, 0} + 5913296\, a_{0, 1} + 734720\, a_{1, 1} + 91104\, a_{2, 1} + 41393072\, a_{0, 2} + 4408320\, a_{1, 2} + 455520\, a_{2, 2}, 5280932\, a_{0, 0} + 739162\, a_{1, 0} + 104960\, a_{2, 0} + 47528388\, a_{0, 1} + 5913296\, a_{1, 1} + 734720\, a_{2, 1} + 380227104\, a_{0, 2} + 41393072\, a_{1, 2} + 4408320\, a_{2, 2}, 38165260\, a_{0, 0} + 5280932\, a_{1, 0} + 739162\, a_{2, 0} + 381652600\, a_{0, 1} + 47528388\, a_{1, 1} + 5913296\, a_{2, 1} + 3434873400\, a_{0, 2} + 380227104\, a_{1, 2} + 41393072\, a_{2, 2}, 278415920\, a_{0, 0} + 38165260\, a_{1, 0} + 5280932\, a_{2, 0} + 3062575120\, a_{0, 1} + 381652600\, a_{1, 1} + 47528388\, a_{2, 1} + 30625751200\, a_{0, 2} + 3434873400\, a_{1, 2} + 380227104\, a_{2, 2}, 2046924400\, a_{0, 0} + 278415920\, a_{1, 0} + 38165260\, a_{2, 0} + 24563092800\, a_{0, 1} + 3062575120\, a_{1, 1} + 381652600\, a_{2, 1} + 270194020800\, a_{0, 2} + 30625751200\, a_{1, 2} + 3434873400\, a_{2, 2}] \quad (3.5)$$

```
> solve(%);
```

$$\{a_{0, 0} = 0, a_{0, 1} = a_{0, 2}, a_{0, 2} = a_{0, 2}, a_{1, 0} = -2\, a_{0, 2}, a_{1, 1} = -14\, a_{0, 2}, a_{1, 2} = -7\, a_{0, 2}, a_{2, 0} = -8\, a_{0, 2}, a_{2, 1} = -24\, a_{0, 2}, a_{2, 2} = -8\, a_{0, 2}\} \quad (3.6)$$

```
> subs(%, add(add(a[i,j]*t^(i+j)*D^j, i=0..2), j=0..2));
```

(3.7)

$$-8\,D^2\,t^4\,a_{0,2} - 7\,D^2\,t^3\,a_{0,2} + a_{0,2}\,t^2\,D^2 - 24\,D\,t^3\,a_{0,2} - 14\,D\,t^2\,a_{0,2} + D\,t\,a_{0,2} - 8\,t^2\,a_{0,2} - 2\,t\,a_{0,2} \tag{3.7}$$

```
> collect(%, D, factor);
```

$$-t^2\,a_{0,2}\,(1+t)\,(8\,t-1)\,D^2 - t\,a_{0,2}\,\left(24\,t^2 + 14\,t - 1\right)\,D - 2\,t\,a_{0,2}\,(4\,t+1) \tag{3.8}$$

Impemented in gfun

```
> listtodiffeq(L, y(t))[1];
```

$$\left\{(8\,t+2)\,y(t) + \left(24\,t^2 + 14\,t - 1\right)\left(\frac{d}{dt}\,y(t)\right) + \left(8\,t^3 + 7\,t^2 - t\right)\left(\frac{d^2}{dt^2}\,y(t)\right), y(0) = 1, D(y)(0) = 2\right\} \tag{3.9}$$

```
> rec := SumTools[Hypergeometric][ZeilbergerRecurrence](binomial(n, k)^3, n, k, u, 0..n);
```

$$rec := \left(-8\,n^2 - 16\,n - 8\right)\,u(n) + \left(-7\,n^2 - 21\,n - 16\right)\,u(n+1) + \left(n^2 + 4\,n + 4\right)\,u(n+2) = 0 \tag{3.10}$$

```
> rectodiffeq({rec, u(0)=1, u(1)=2}, u(n), y(t));
```

$$\left\{(-8\,t-2)\,y(t) + \left(-24\,t^2 - 14\,t + 1\right)\left(\frac{d}{dt}\,y(t)\right) + \left(-8\,t^3 - 7\,t^2 + t\right)\left(\frac{d^2}{dt^2}\,y(t)\right), y(0) = 1\right\} \tag{3.11}$$