Computing with *integrals* in nonlinear algebra

by PIERRE LAIREZ

Discrete sums

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3} \qquad \text{[Dixon]}$$

$$\sum_{i,j\in\mathbb{Z}} \binom{2n}{n+i}\binom{2n}{n+j} \left| i^3 - j^3 \right| = \frac{2n^2(5n-2)}{4n-1} \binom{4n}{2n} \qquad \text{[Brent, Ohtsuka, Osborn, Prodinger]}$$

$$\begin{tikzcd}
\text{sum} \arrow[r, "\text{generating function}"] & \text{integral} \arrow[r, "\text{symbolic integration}"] & \text{differential equation} \arrow[d] \\
& \text{closed form} & \text{recurrence relation} \arrow[l, "\text{(not always)}"']
\end{tikzcd}$$

MOTIVATING EXAMPLE

Picard group

Find $a_0, \dots, a_4 \in \mathbb{C}^4$ s.t.
$\forall t,\ f\left(\sum_{i=0}^{4} a_i t^i\right) = 0$
+ smoothness cond.

$$X = \{x^4 + y^3z + xyzw + z^3w + yw^3 = 0\} \subset \mathbb{P}^3(\mathbb{C})$$

This surface contains: 0 lines, 0 conics, 0 twisted cubics
but 133 056 smooth rational curves of degree 4.

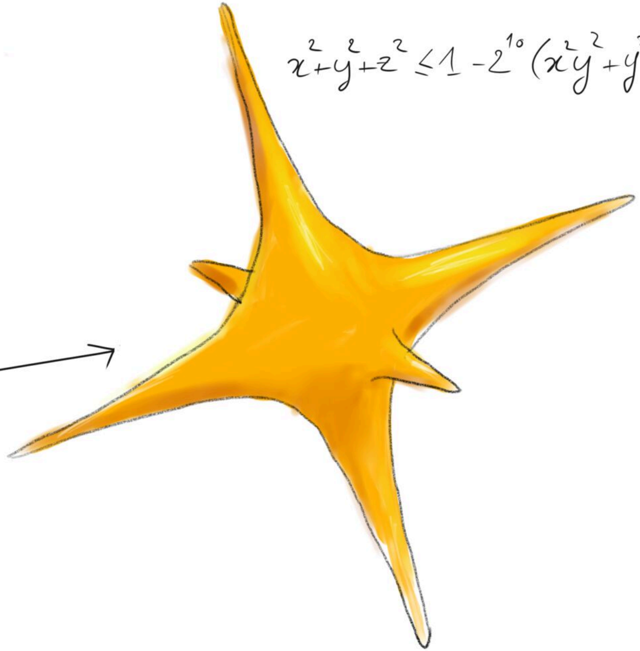
Obtained by computing integrals at high precision (~ 3000 digits),
and finding integer relations between them.

Challenge: compute all these curves! (or just one)

Volume of semialgebraic sets

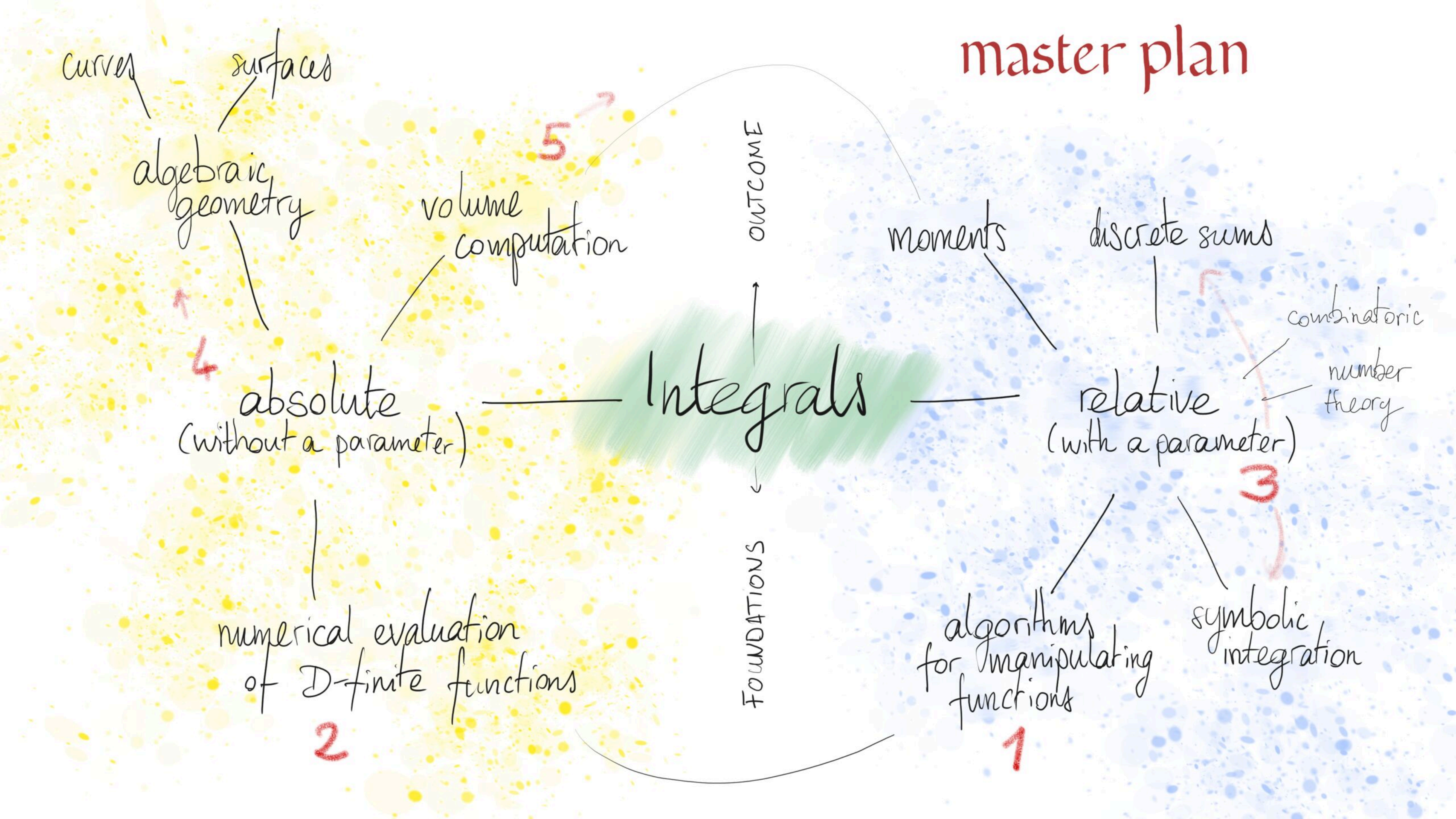
How to compute a thousand digits of the volume of this ?

$$x^2+y^2+z^2 \leq 1 - 2^{10}(x^2y^2+y^2z^2+z^2x^2)$$



Think of possible methods.

How will they scale ?

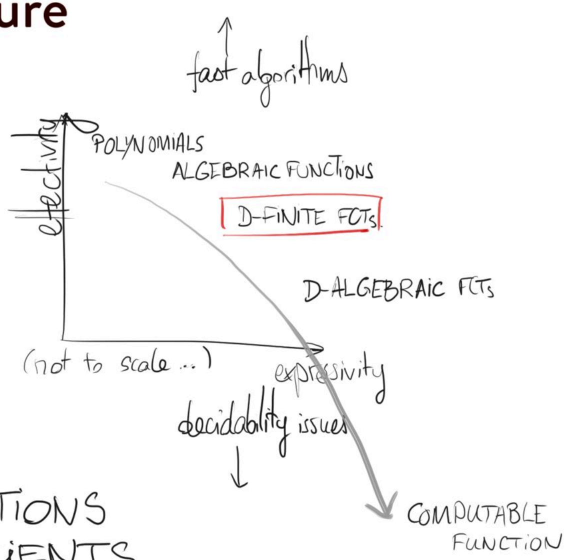

master plan
Integrals
OUTCOME
FOUNDATIONS
absolute
(without a parameter)
relative
(with a parameter)
algebraic geometry
curves
surfaces
volume computation
numerical evaluation of D-finite functions
moments
discrete sums
combinatoric
number theory
algorithms for manipulating functions
symbolic integration
1
2
3
4
5

Differential equations as a data structure

To compute, we need representations

How to represent a function?

We want expressivity and effectivity
TRADE-OFF



LINEAR DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS
(+ initial conditions)

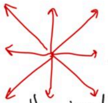
Elementary examples

$f(t)$	differential equation
1	$y' = 0$
t^n	$ty' = ny$
e^t	$y' = y$
$Ai(t)$	$y'' = ty$
$J_\alpha(t)$	$t^2 y'' + ty' + (t^2 - \alpha^2) y = 0$

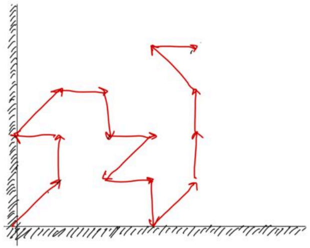
Also many non-examples

$\tan(t)$, $\Gamma(t)$, ...

Explicit vs. implicit



authorized steps



$a_n \stackrel{\text{def}}{=}$ # of paths of length n

$f(t) \stackrel{\text{def}}{=} \sum_{n \geq 0} a_n t^n$

(OEIS A151331)

Theorem (Bostan, Chyzak, van Hoeij, Kauers, Pech)

i. $f(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4u)^3}\ {}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ & 2 \end{matrix} \middle| \frac{16u(1+u)}{(1+4u)^2}\right) du$

ii. $(64t^6+40t^5-30t^4-5t^3+t^2)\, f''' + (\dots) f'' + \cdots = 0$

EXPLICIT: reveals the nature of the function
does not always exist

IMPLICIT: enable computations

The algebra of differential operators

$\mathcal{D} \doteq \mathbb{C}[t]\langle\partial\rangle$ is the algebra generated by t and ∂

($\mathcal{D}_{\mathrm{rat}} \doteq \mathbb{C}(t)\langle\partial\rangle$) subject to the relation $\partial t = t\partial + 1$

(Leibniz rule)

It acts on "functions"

(normal form of an operator)

For $L = \sum_{i=0}^{r} p_i(t)\partial^i \in \mathcal{D}$ and f a "function", $L \cdot f : t \longmapsto \sum_{i=0}^{r} p_i(t) f^{(i)}(t)$

Examples: $\partial \cdot f = f'$

$p(t) \cdot f = p(t) f$

$(\partial - 1) \cdot \exp = 0$

$(t\partial - n) \cdot t^n = 0$

diff. op. $\equiv$ linear diff. eq.

Basic properties of differential operators

For $L \in \mathcal{D}_{rat}$, $\operatorname{ord} L$ = maximum exponent of ∂ in L.

that's actually a right division and there is a similar left division

Lemma. (Euclidean division).

$\forall\ A, B \in \mathcal{D}_{rat},\ A \neq 0 \Rightarrow \exists!\ Q, R \in \mathcal{D}_{rat}\quad A = QB + R$ and $\operatorname{ord} R < \operatorname{ord} A$

Corollary: $\mathcal{D}_{rat}$ is a principal ideal domain.

LEFT ideals

We define the right G.C.D. by $\quad \mathcal{D}_{rat} \cdot \operatorname{rgcd}(A, B) = \mathcal{D}_{rat} A + \mathcal{D}_{rat} B$

LCM by $\quad \mathcal{D}_{rat} \operatorname{rlcm}(A, B) = \mathcal{D}_{rat} A \cap \mathcal{D}_{rat} B$

the minimal eq. of f by $\quad \mathcal{D}_{rat} \operatorname{ann}(f) = \{ A \in \mathcal{D}_{rat} \mid A \cdot f = 0 \}$

Differential operators determine analytic functions

This is where the leading coefficient of L vanishes

Theorem. Let $L \in \mathcal{D}$ and $a_0, \dots, a_{r-1} \in \mathbb{C}$

For all simply connected open set $U \subseteq \mathbb{C} \setminus \operatorname{sing} L$ and all $z \in U$

there is a unique holomorphic function f on U s.t.

$$L \cdot f = 0 \quad \text{and} \quad f^{(i)}(z) = a_i \quad (0 \leq i < \operatorname{ord} L)$$

A function is represented by a differential operator and initial conditions (at an ordinary point)

Representable functions are called **differentially finite** (or D-finite)

Exercises

#0 Give algorithms to compute GCDs and LCMs in $\mathcal{D}_{rat}$

#1 Given f and g, compute $f+g$

#2 ——— compute fg

#3 Given f and $L \in \mathcal{D}_{rat}$, compute $L \cdot f$

#4 Given f and g, decide if $f = g$

#5 Prove that $\sin(z)^2 + \cos(z)^2 = 1$

#6 Show that algebraic functions are D-finite

interactive board

Structure of differentially finite power series

Let $L = \sum_{i=0}^{r} p_i(t)\partial^i \in \mathcal{D}$ and $f = \sum_{n=0}^{+\infty} a_n t^n \in \mathbb{C}[[t]]$ s.t. $L \cdot f = 0$

Let $\theta = t\partial$ (Euler's derivation) and write $L = \sum_{i=0}^{r} \tilde{p}_i(t)\,\theta^i$

(NB: $\theta \cdot t^n = n t^n$)

$$= \sum_{j=0}^{d} t^j q_j(\theta)$$

$$L \cdot f = \sum_{n=0}^{+\infty} \sum_{j=0}^{d} t^j q_j(\theta) \cdot a_n t^n = \sum_n \sum_j a_n q_j(n)\, t^{n+j} = \sum_{n=0}^{+\infty} t^n \left(\sum_{j=0}^{\min(n,d)} q_j(n-j) a_{n-j} \right)$$

indicial polynomial at 0

$$q_0(n)\, a_n = -q_1(n-1)\, a_{n-1} - \cdots - q_d(n-d)\, a_{n-d}$$

The sequence of coefficients is polynomially recursive (P-recursive)

Sequences and generating functions

$(a_n)_{n \geqslant 0}$ $\qquad$ $f = \sum_{n=0}^{+\infty} a_n t^n$

a_{n+1} $\qquad$ $\frac{1}{t} f$

$n\, a_n$ $\qquad$ $t\partial \cdot f$

P-recursive $\qquad$ D-finite

asymptotic behavior $\qquad$ position and nature of the poles of f

Maple

Exercises

#1 Compute the coefficient of t^{1000} in $(1+t)^{10000}(1+t+t^3)^{50000}$

#2 A Motzkin word is a word on the alphabet $\{`(', `)', `\alpha'\}$ where each opening parenthesis is closed and each closing parenthesis closes an opening one. $M_n \overset{\text{def}}{=}$ # of Motzkin word of length n

$M_4 = 9$ (()), $(\alpha\alpha)$, ()(), $()\alpha\alpha$, $(\alpha)\alpha$, $\alpha()\alpha$, $\alpha(\alpha)$, $\alpha\alpha()$, $\alpha\alpha\alpha\alpha$

Show that $(n+4)M_{n+2} = (5+2n)M_{n+1} + (3+3n)M_n$

Guessing differential equations

$\sum_{k=0}^{n} \binom{n}{k}^3$ (Franel numbers)

1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260, 278415920, 2046924400, ???

$f(t) \doteq 1 + 2t + 10t^2 + 56t^3 + \cdots + O(t^{13})$

Search for polynomials $p_0(t), \ldots, p_r(t)$ such that

$p_0(t) f(t) + p_1(t)\, t f'(t) + \cdots + p_r(t)\, t^r f^{(r)}(t) = O(t^{13})$

$\rightsquigarrow$ linear system of 13 equations in $\sum_i \deg p_i + 1$ unknowns.

We can try $r = 2$ and $\deg p_i = 2$.

$\rightsquigarrow\ t(t+1)(8t-1)\partial^2 + (24t^2 + 14t - 1)\partial + (8t+2)$

Extremely useful

Reliable

Proves nothing

$\rightsquigarrow$ Maple

Diagonals of rational functions

in practice, this will be $\mathbb{Q}$

Let $F(z_1, \dots, z_n) = \dfrac{A}{P} \in \mathbb{C}(z_0, \dots, z_n)$ $\qquad P(0, \dots, 0) \neq 0$

Expand F as $\quad F = \sum_{k_0, \dots, k_n \geq 0} a_{k_0, \dots, k_n} z^{k} \cdots z_n^{k_n} \in \mathbb{C}[[z_0, \dots, z_n]]$

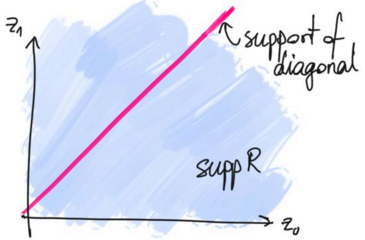
Define $\quad \operatorname{diag} F \stackrel{\text{def}}{=} \sum_{k \geq 0} a_{k, \dots, k}\, t^k \in \mathbb{C}[[t]]$

Theorem (Christol 1985) The power series $\operatorname{diag} F$ is differentially finite

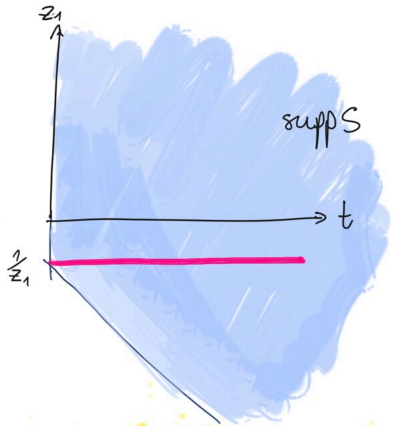
proof on board

A computational handle on diagonals

$R(z_0, \dots, z_n)$



$S(t, z_1, \dots, z_n) \doteq \frac{1}{z_1\cdots z_n} R\left(\frac{t}{z_1\cdots z_n}, z_1, \dots, z_n\right)$



$\rotatebox{90}{$\in$}$

$\mathbb{C}[[z_1,\dots,z_n]]\left[\frac{1}{z_1\cdots z_n}\right][[t]]$

notation for coefficient extraction

$\operatorname{res} S \stackrel{\text{def}}{=} \sum_{k\geq 0} \left(\left[\frac{t^k}{z_1\cdots z_n}\right] S \right) t^k$

Lemma. $\cdot$ $\operatorname{diag} R = \operatorname{res} S$

$\cdot$ $\mathcal{L}(t, \partial_t) \cdot \operatorname{diag} R = \operatorname{res}(\mathcal{L}\cdot S)$

$\cdot$ $\boxed{\operatorname{res}\left(\sum_{i=1}^{n} \frac{\partial S_i}{\partial z_i}\right) = 0}$

$\rightsquigarrow$ Mathematica

Corollary. If there are $A_1, \dots, A_n \in \mathbb{C}(t, z_1, \dots, z_n)$ and $\mathcal{L} \in \mathcal{D}$ s.t. $\mathcal{L}\cdot S = \frac{\partial A_1}{\partial z_1} + \dots + \frac{\partial A_n}{\partial z_n}$, then $\mathcal{L}\cdot \operatorname{diag} R = 0$