Computing with *integrals* in nonlinear algebra

#2

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Motzkin words (second try)

$$\mathcal{M} \overset{\text{def}}{=} \{\text{Motzkin words}\} = \{\emptyset, \alpha, \alpha\alpha, (), (\alpha), \alpha(), ()\alpha, \ldots\}$$

$$= \text{`('}\, \mathcal{M}\, \text{`)'} \cup \mathcal{M}\mathcal{M} \cup \{\alpha\} \cup \{\emptyset\}$$

⚠ not disjoint

$$\mathcal{K} = \{ w \in \mathcal{M} \mid w \text{ not in the form } AB, \text{ with } A \text{ and } B \in \mathcal{M} \setminus \{\emptyset\} \}$$

$$= \{\alpha\} \underset{\text{disjoint}}{\sqcup} \text{`('}\, \mathcal{M}\, \text{`)'} \Rightarrow \#\mathcal{K}_n = \begin{cases} 1 & \text{if } n = 1 \\ \#\mathcal{M}_{n-2} & \text{otherwise} \end{cases}$$

$$M(z) = \sum_n \#\mathcal{M}_n z^n$$

$$K(z) = \sum_n \#\mathcal{K}_n z^n$$

$$\mathcal{M} = \{\emptyset\} \underset{\text{disjoint}}{\sqcup} \mathcal{K}\mathcal{M}$$

decomposition is unique

$$\Downarrow$$

$$\#\mathcal{M}_n = \begin{cases} 1 & \text{if } n = 0 \\ \sum_{k=1}^{n} \#\mathcal{K}_k \#\mathcal{M}_{n-k} & \text{otherwise} \end{cases}$$

$$\Downarrow$$

$$K = z + z^2 M$$

$$\Downarrow$$

$$M = 1 + KM$$

$$z^2 M^2 + (z-1) M + 1 = 0$$

$$\Downarrow$$

$$(3t^3 + 2t^2 - t) M'' + (12t^2 + 7t - 3) M' + (6t + 3) M = 0$$

$$\Downarrow$$

$$(n+2)\, \#\mathcal{M}_n = (2n+1)\, \#\mathcal{M}_{n-1} + (3n-3)\, \#\mathcal{M}_{n-2}$$

Linear differential equations in the complex plane

Let $L \in \mathcal{D}$ $\quad L = a_r(t)\partial^r + \cdots + a_1(t)\partial + a_0(t)$, $\quad a_r \neq 0$, $a_i(t) \in \mathbb{C}[t]$

Let $\Sigma = \{ \text{singularities of } L \} = \{ \alpha \in \mathbb{C} \mid a_r(\alpha) = 0 \}$

Let $U \subseteq \mathbb{C} \setminus \Sigma$ be a simply connected open set.

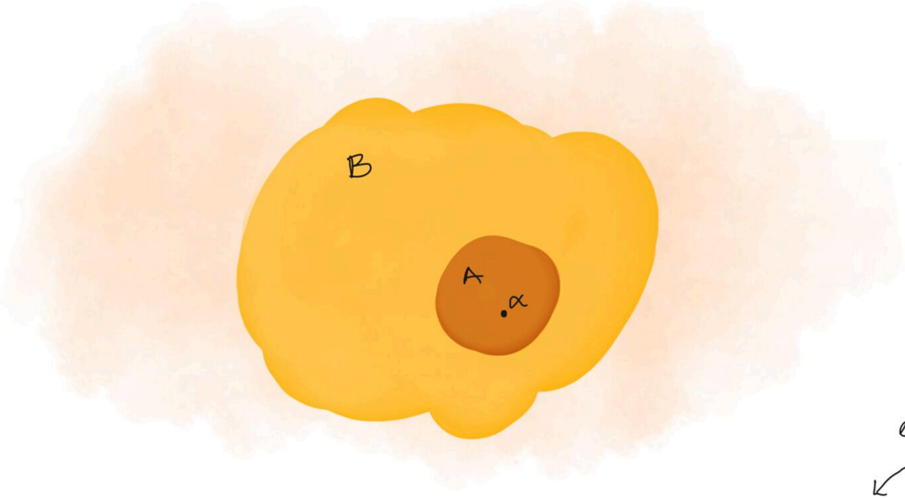
Let $V_U = \{ \text{holomorphic functions } f : U \to \mathbb{C} \text{ s.t. } L \cdot f = 0 \}$

NB.
Σ : singular points
$\mathbb{C}\setminus\Sigma$: ordinary points

Theorem

- $\dim_{\mathbb{C}} V_U = r$
- For any $\alpha \in U$, $f \in V_U \longmapsto (f(\alpha), f'(\alpha), \ldots, f^{(r-1)}(\alpha)) \in \mathbb{C}^r$ is an isomorphism.
- For any $\alpha \in U$, there is a basis $f_0, \ldots, f_{r-1}$ of V_U s.t. $f_i(z+\alpha) = z^i + O(z^r)$ (canonical basis)

Restriction and extension



simply connected

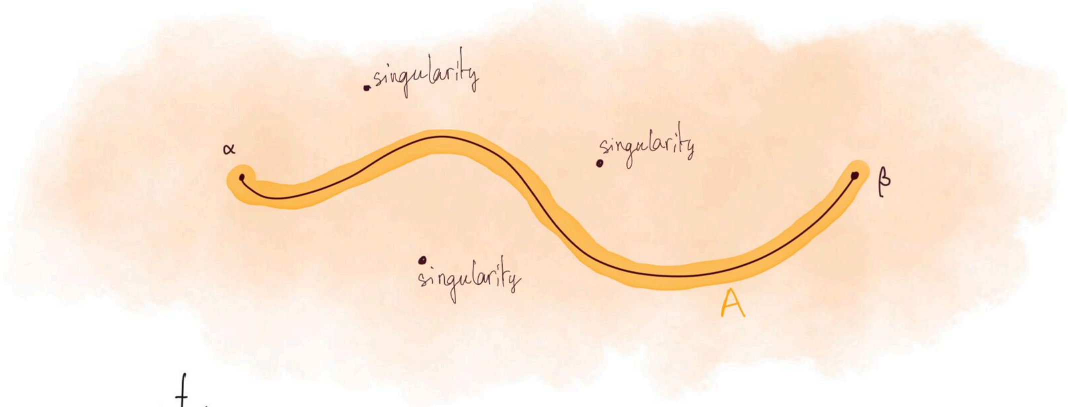
$\alpha \in A \subseteq B \subseteq \mathbb{C}\setminus\Sigma$

Restriction and extension are isomorphisms.

$$\begin{tikzcd}
V_B \arrow[rr, "\text{restriction}"'] \arrow[drr] & & V_A \arrow[ll, bend right, dashed, "\text{extension}"'] \arrow[d] \\
& & \mathbb{C}^r
\end{tikzcd}$$

$f \mapsto (f(\alpha), f'(\alpha), \ldots, f^{(r-1)}(\alpha))$

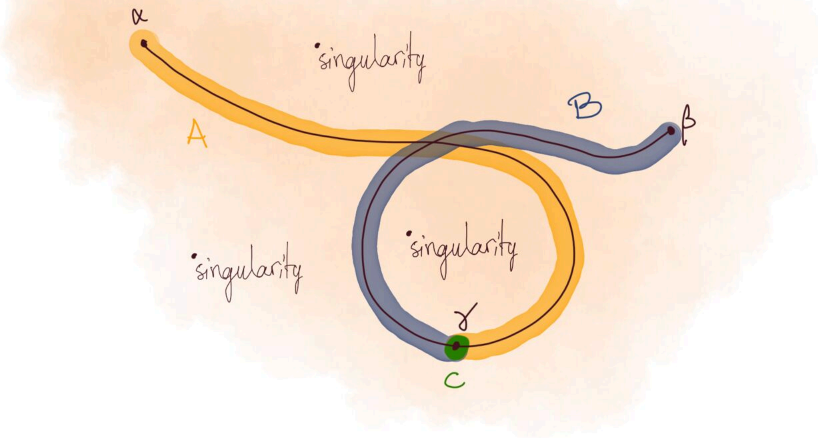
Transition matrices for simple paths



$$
\begin{tikzcd}
 & f \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(f^{(i)}(\alpha))_i & V_A \arrow[dl] \arrow[dr] & (f^{(i)}(\beta))_i \\
\mathbb{C}^r \arrow[rr, "T_{\alpha\to\beta}"'] & & \mathbb{C}^r
\end{tikzcd}
$$

$T_{\alpha\to\beta} \in \mathbb{C}^{r\times r}$ is the transition matrix associated to the path $\alpha \to \beta$.

Transition matrices for general paths



Note for future talks: consider the notation $T_{\beta \leftarrow \alpha}$

$$T_{\alpha\to\beta} \doteq T_{\gamma\to\beta} \circ T_{\alpha\to\beta}$$

$$\begin{tikzcd}
V_A \arrow[r, "\text{restriction}"] \arrow[d, "\alpha"'] & V_C \arrow[r, "\text{extension}"] & V_B \arrow[d, "\beta"] \\
\mathbb{C}^r \arrow[rr, "T_{\alpha\to\beta}"'] & & \mathbb{C}^r
\end{tikzcd}$$

Monodromy matrices : transition matrices associated to loops

Monodromy of fractional powers

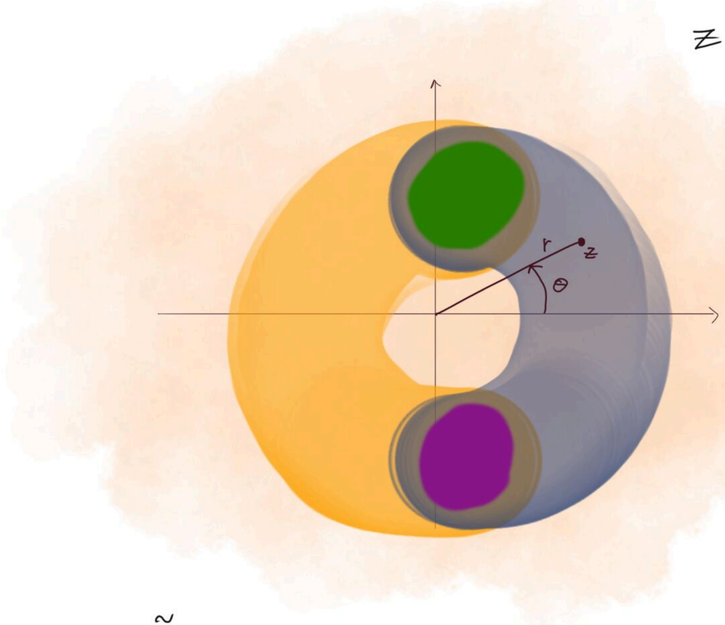
$$\lambda = \frac{p}{q} \in \mathbb{Q}$$

$$L = z\partial - \lambda$$

$$L \cdot z^{\lambda} = 0$$

$$V_{\text{blue}} = \mathbb{C}\ \overbrace{r_z \exp(i\lambda\theta_z)}^{f(z)}$$

$$V_{\text{yellow}} = \mathbb{C}\ \underbrace{r_z \exp(i\lambda\tilde{\theta}_z)}_{\tilde{f}(z)}$$



$$z = r_z \exp(i\theta_z)$$
$$= r_z \exp(i\tilde{\theta}_z)$$

$$\theta_z \in\ ]-\pi, \pi]$$

$$\tilde{\theta}_z \in [0, 2\pi[$$

$$\tilde{\theta}_z = \theta_z \text{ or } \theta_z + 2\pi$$

On green: $\tilde{f} = f$

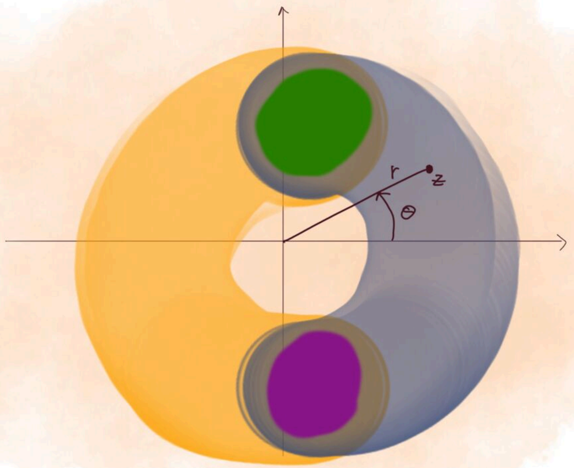
On purple: $\tilde{f} = \exp(2i\pi\lambda) f$

$$M_{\circlearrowleft} = \left(e^{2i\pi\lambda}\right) \in \mathbb{C}^{1\times 1}$$

Monodromy of the logarithm

$$L = \partial z \partial = z\partial^2 + 1$$

$$L \cdot \log(z) = 0$$



$$z = r_z \exp(i\theta_z)$$
$$= r_z \exp(i\tilde{\theta}_z)$$

$$\theta_z \in \,]-\pi, \pi]$$
$$\tilde{\theta}_z \in [0, 2\pi[$$

$$V_{\bullet} = \mathbb{C} + \mathbb{C}\overbrace{\left(z \mapsto \log r_z + i\theta_z\right)}^{\operatorname{Log} z}$$

$$V_{\bullet} = \mathbb{C} + \mathbb{C}\underbrace{\left(z \mapsto \log r_z + i\tilde{\theta}_z\right)}_{\widetilde{\operatorname{Log}}\, z}$$

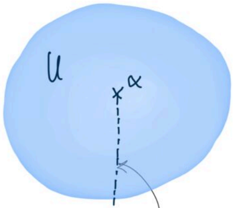
On $\bullet$ $\widetilde{\operatorname{Log}} = \operatorname{Log}$

$\bullet$ $\widetilde{\operatorname{Log}} = \operatorname{Log} + 2i\pi$

$$M_{G_1} = \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

Classification of singularities

Let $\alpha \in \Sigma$ (singular point) and U a slit neighborhood of α



This halfline is excluded from the neighborhood

Apparent singularity

All the solutions on U extend holomorphically to $\overline{U}$

Regular singularity

There is $N > 0$ such that $|f(z)| = O\left((z-\alpha)^{-N}\right)$ as $z \to \alpha$ for any $f \in V_U$.

Irregular singularity

All other cases.

L is **Fuchsian** if it does not have irregular singularities (including at infinity)

Regular singularities

Let $\alpha \in \mathbb{C}$. There is a polynomial $\mathrm{ind}_\alpha(\lambda)$ and an integer c s.t.

$$L \cdot (z-\alpha)^\lambda = \mathrm{ind}_\alpha(\lambda)\,(z-\alpha)^{\lambda+c} + O\left((z-\alpha)^{\lambda+c+1}\right)$$

Easy to compute!

Theorem The following are equivalent:

- α is ordinary, apparent singular or regular singular
- $\mathrm{ind}_\alpha(\lambda)$ has degree r (the order of L)
- L has a basis of solutions in a slit neighborhood of α in the form

$$(z-\alpha)^\lambda \left(y_0(z) + y_1(z)\log(z-\alpha) + \cdots + y_k(z)\log(z-\alpha)^k \right)$$

where $y_0, \dots, y_k$ are holomorphic at α and $\lambda \in \mathbb{C}$.

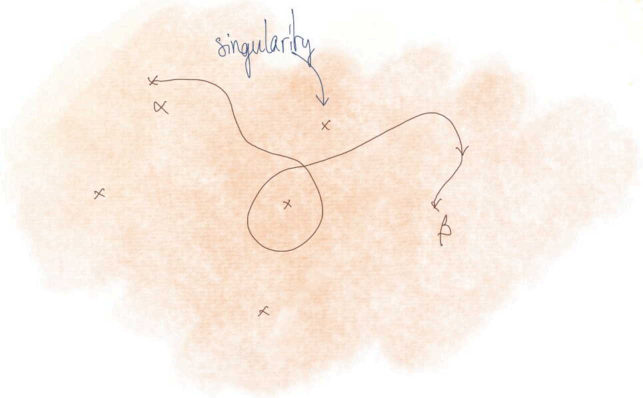
Examples

L	$\mathrm{ind}_0(\lambda)$	V_0 (solutions in a neighborhood of 0)	
$z\partial - u \quad (u \in \mathbb{C})$	$\lambda - u$	$\mathbb{C} z^u$	regular singular (or apparent if $u \in \mathbb{N}$)
∂^r	$\lambda(\lambda-1)\cdots(\lambda-r+1)$	$\mathbb{C}[z]_{<r}$	ordinary
$x^2\partial - 1$	1	$\mathbb{C} e^{-\frac{1}{x}}$	irregular singular
$x^2\partial^2 + x\partial - u$	$\lambda^2 - u$	$\mathbb{C} z^{\sqrt{u}} + \mathbb{C} z^{-\sqrt{u}}$	regular singular
$x\partial^2 + \partial$	λ^2	$\mathbb{C} + \mathbb{C} \log z$	regular singular

Transition matrices

For each $\alpha \in \mathbb{C}$, we have the space V_α of holomorphic solutions of L in a (slit) neighborhood of α. By ordering the terms $(z-\alpha)^\lambda \cdot \log(z-\alpha)^k$ we may also define a canonical basis $\mathcal{B}_\alpha$ of V_α.

Analytic continuation $\rightsquigarrow$ any path from α to β defines an isomorphism $T_{\alpha\to\beta} : V_\alpha \longrightarrow V_\beta$



The transition matrix is the matrix of this isomorphism in the canonical bases.

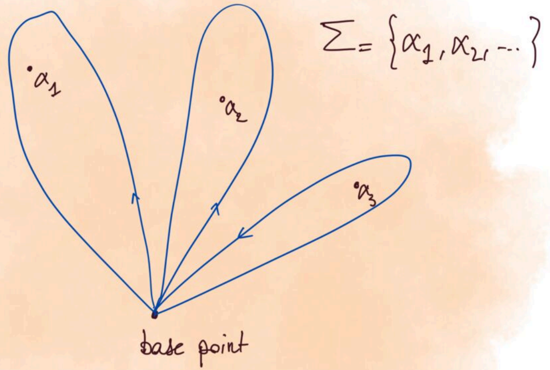
Monodromy group

$M_i = T_{\text{base}\to\text{base}}$ going around α_i
(local monodromy)

In the canonical basis $\mathcal{B}_{\alpha_i}$,

M_i is computable symbolically:

- $z^\lambda \longmapsto e^{2i\pi\lambda} z^\lambda$
- $\log z \longmapsto \log z + 2i\pi$



Monodromy group: $G = \langle M_1, \dots, M_{\#\Sigma} \rangle \subseteq GL(\mathbb{C}^r)$

$f: U \to \mathbb{C}$ is algebraic if $P(z, f(z)) = 0$ for some $P(z,T) \in \mathbb{C}[z,T] \setminus \{0\}$

- Algebraic functions are differentially finite.
- The minimal operator annihilating an algebraic function is Fuchsian with rational exponents
- Its monodromy group is finite.
- Let L be a Fuchsian operator and f a solution. If the orbit of f under the action of monodromy is finite, then f is algebraic.

proofs on board

$${}_8F_7\left(\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{5} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{4}{5} & \end{matrix} \middle| \, z \right) =$$

sagemath

$$\sum_{n \geq 0} \frac{(30n)!\, n!}{(15n)!\,(10n)!\,(6n)!} z^n$$

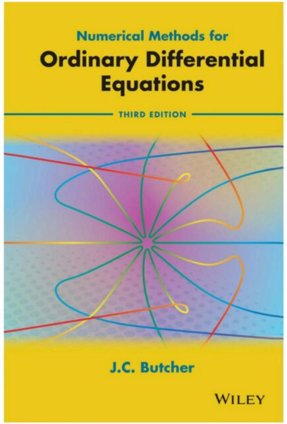
Rodriguez-Villegas : This is an algebraic function of degree 483 840.

+ Beukers & Heckman

Can we check this experimentally ?

Effective analytic continuation

$L \in \mathcal{D}$ of order r
a path $\alpha \to \beta$ in $\mathbb{C}$ $\longrightarrow$ ANALYTIC CONTINUATION $\longrightarrow$ $T_{\alpha\to\beta} \in \mathbb{C}^{r\times r}$



Many methods... but we are doing (experimental) maths:

need very high precision

need rigorous error bounds

differential equations grow big (hundreds of MB)
and they are also very special.

Euler's method

Let f s.t. $L \cdot f = 0$

Let $Y(z) = \begin{pmatrix} f(z) \\ f'(z) \\ \vdots \\ f^{(r-1)}(z) \end{pmatrix}$

$Y'(z) = A(z)\,Y(z)$ where $A(z) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\frac{a_0}{a_r} & -\frac{a_1}{a_r} & \cdots & -\frac{a_{r-1}}{a_r} \end{pmatrix}$

(NB: $L = a_r \partial^r + \cdots + a_1 \partial + a_0$)

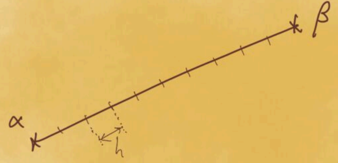
Given $Y(\alpha)$, we want $Y(\beta)$.

<ins>Algorithm</ins>.

$y_0 \leftarrow Y(\alpha)$

$y_{n+1} \leftarrow y_n + h A(\alpha + nh)\, y_n$

$\rightsquigarrow$ sagemath



$h = \frac{\beta - \alpha}{N}$

$Y(z+h) = Y(z) + h \underbrace{Y'(z)}_{= A(z)Y(z)} + O(h^2)$

$\rightsquigarrow \boxed{\| y_N - Y(\beta) \| = O\left(\frac{1}{N}\right)}$

⚠ Exponential complexity w.r.t. precision

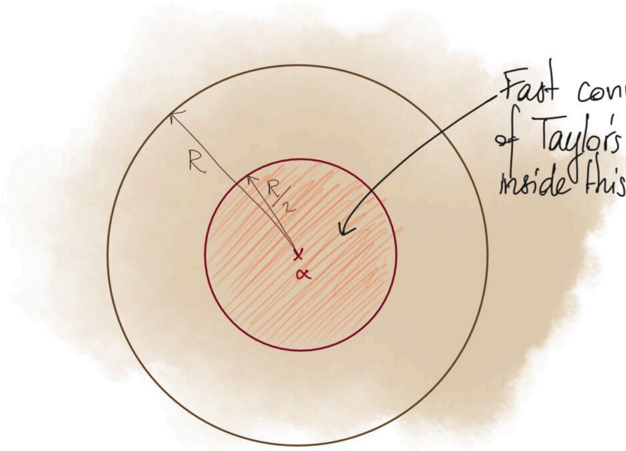
Higher order Taylor series method

Computable!

Principle

If f is holomorphic on $D(\alpha, R)$,

then $\left| f(z) - \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(\alpha) z^k \right| \leq \text{poly}(m) 2^{-m}$ on $D(\alpha, \frac{R}{2})$



Fast convergence of Taylor's series inside this disk

An algorithm for differentially finite functions

INPUT: $L \in \mathcal{D}$, $\alpha, \beta \in \mathbb{C}$ (straight-line path), $m \geq 0$

f s.t. $L \cdot f = 0$, determined by initial conditions at α.

compute efficiently using a recurrence relation.

ALGO: Compute intermediate points $\alpha_0, \alpha_1, \dots, \alpha_N = \beta$

For i from 1 to N

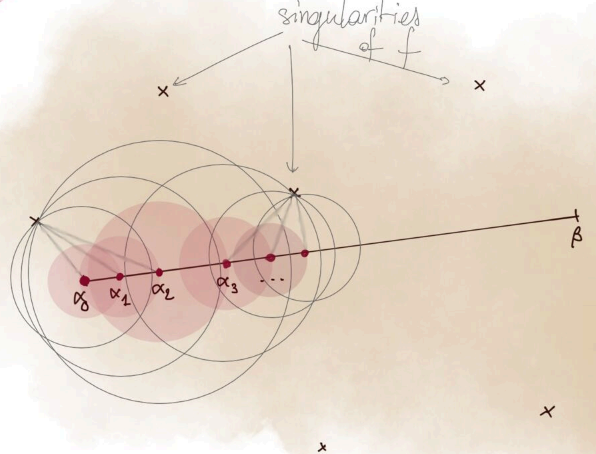
For j from 0 to $r-1$

$$f^{(j)}(\alpha_i) \leftarrow \sum_{k=0}^{m} \frac{1}{k!} f^{(j+k)}(\alpha_{i-1}) \cdot (\alpha_i - \alpha_{i-1})^k$$

OUTPUT: $f(\beta), f'(\beta), \dots, f^{(r-1)}(\beta)$

at precision $\sim 2^{-m}$

COMPLEXITY (w.r.t. m only): $O(m)$ arithmetic operations



The basic operation of analytic continuation

INPUT: • $u_0, \dots, u_{r-1} \in \mathbb{Q}$, $m \in \mathbb{N}$

• a recurrence relation $q_0(n) u_n = q_1(n) u_{n-1} + \dots + q_r(n) u_{n-r}$, $q_i \in \mathbb{Q}[n]$

OUTPUT $\sum_{k=0}^{m} u_k \in \mathbb{Q}$

NAIVE ALGORITHM $O(m)$ arithmetic operations

BUT... define $\mathrm{ht}\left(\frac{p}{q}\right) = \max(\log|p|, \log|q|)$

$\mathrm{ht}(u_k) = O(k \log k)$

$\tilde{O}(m^2)$ time complexity

Binary splitting (a simple case)

$$q(n)\, u_n = p(n)\, u_{n-1} \qquad\qquad u_n = u_0 \prod_{k=1}^{n} \frac{p(k)}{q(k)}$$

$$u_n = u_0 T_{0,n}$$

$$\sum_{k=0}^{n-1} u_n = u_0 S_{0,n}$$

Let $\displaystyle T_{a,b} = \prod_{k=a+1}^{b} \frac{p(k)}{q(k)}$ and $\displaystyle S_{a,b} = \sum_{k=a}^{b-1} T_{a,k}$

Then $T_{a,b} = T_{a,c}\, T_{c,b}$ and $S_{a,b} = S_{a,c} + T_{a,c} S_{c,b}$ $\quad (a \le c \le b)$

$[0, m]$
- $[0, \frac{m}{2}]$
  - $[0, \frac{m}{4}]$
  - $[\frac{m}{4}, \frac{m}{2}]$
- $[\frac{m}{2}, m]$
  - $[\frac{m}{2}, \frac{3m}{4}]$
  - $[\frac{3m}{4}, m]$

ht $\quad \sim m \quad \sim \frac{m}{2} \quad \sim \frac{m}{4}$

2^k arithmetic operations in height $\sim \frac{m}{2^k}$

$\rightsquigarrow$ $\tilde{O}(m)$ time complexity.