Computing with *integrals* in nonlinear algebra

#5

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Plan

#1 Differential equations as a datastructure

#2 High-precision numerical evaluation. Application in experimental maths.

#3 Formal integration: diagonals, constant terms, residues

#4 Periods of algebraic surfaces

#5 The volume of semialgebraic sets

#6 Problem solving.

Make sure that you have installed Sagemath and ore_algebra!

Semialgebraic sets and volume

Let $f_1, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$

Let $S = \{ p \in \mathbb{R}^n \mid \forall i \in \{1, \dots, s\},\ f_i(p) \geq 0 \}$

Problem: compute $\operatorname{vol} S = \int_S 1 \, d\underline{x}$

The volume of convex polytopes

INPUT: $f_1, \dots, f_s \in \mathbb{Q}[x_1, \dots, x_n]$ of degree 1

OUTPUT: $\mathrm{vol}\{f_i \geqslant 0\}$

Theorem (Dyer, Frieze) This problem is #P-hard

(That is: if you can solve this problem efficiently, you can count efficiently the number of solutions of a SAT problem.)

However...

Theorem (Dyer, Frieze, Kannan) We can approximate the volume at relative precision ε and probability $\frac{3}{4}$ with $\mathrm{poly}(n, s, \varepsilon^{-1})$ operations.

The semiring of volumes (and the ring of periods)

Let S and T be compact sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively

- $\mathrm{vol}\, S \times \mathrm{vol}\, T = \mathrm{vol}(S \times T)$
- $\mathrm{vol}\, S + \mathrm{vol}\, T = \mathrm{vol}\left([0,1] \times S \ \cup \ [2,3] \times [0,1]^{n-m} \times T \right)$ (when $n \leq m$)
- The set $\mathcal{V} \subseteq \mathbb{R}$ of all volumes of compact semialgebraic sets defined over $\mathbb{Q}$ is a semiring.
- Furthermore, (Viu-Sos) $\mathcal{V} - \mathcal{V}$ is the ring of $\underbrace{\text{Kontsevich-Zagier}}$ periods.

= values of absolutely convergent integrals of rat. functions over semialg. sets.

Chance optimization

A general problem

maximize Prob[success]

over design parameter

A specialized form

maximize $\mathrm{vol}\{\omega \in \mathbb{R}^n \mid f_i(x,\omega) \geq 0, \ 1 \leq i \leq s\}$

subject to $g_i(x) \geq 0, \ 1 \leq i \leq t$

Monte-Carlo method

Let $S \subseteq [0,1]^n$.

approximate_volume(S, ε) :

$N \leftarrow \exp\left(\frac{1}{\varepsilon^2}\right)$ $\quad k \leftarrow 0$

repeat N times :

$p \leftarrow$ random$([0,1]^n)$

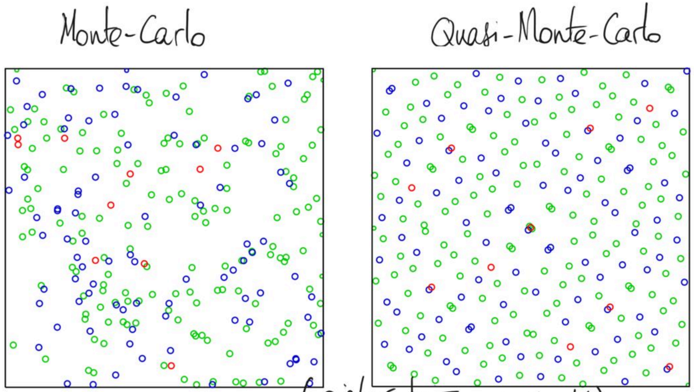
if $p \in S$: $k \leftarrow k+1$

return $\frac{k}{N}$.

Let $u_i = \begin{cases} 1 & \text{if the } i\text{th sample is in } S \\ 0 & \text{otherwise} \end{cases}$

Lemma.

$$E\left[\left(\mathrm{vol}\, S - \frac{1}{N}\sum_i u_i\right)^2\right]^{1/2} \simeq \frac{1}{\sqrt{N}}$$



(picture by James Heald)

Main result

Theorem (Lairez, Safey El Din, Mezzarobba)

Given $f_1, \dots, f_s \in \mathbb{Q}[x_1, \dots, x_n]$ and $p \geq 0$,

we can compute $V \in \mathbb{R}$ s.t. $|V - \mathrm{vol}\{f_i \geq 0\}| \leq 2^{-p}$

with $\tilde{O}(p)$ bit operations ($f_1, \dots, f_s$ being fixed)

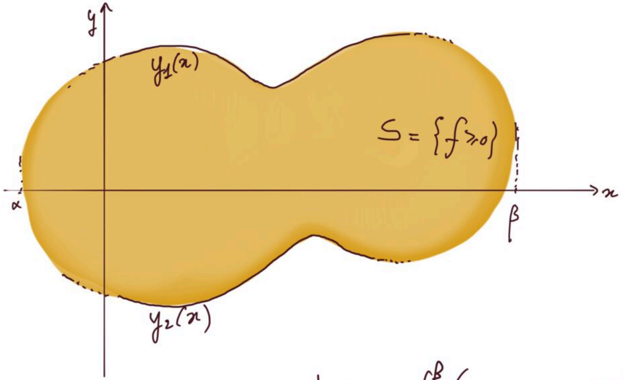
(NB: $(sD)^{O(n^2)}$ operations for the "algebraic" part.)

Low dimensional cases, one inequality

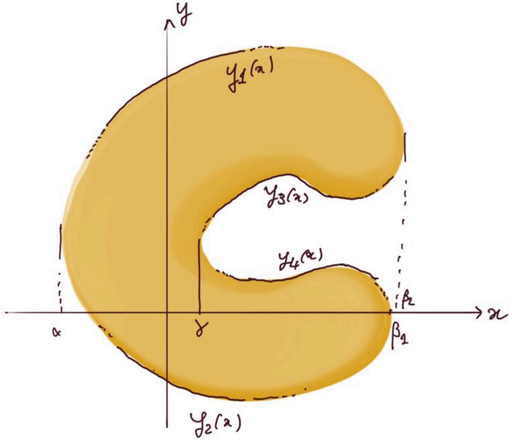
$f \in \mathbb{Q}[x_1, \dots, x_n]$

$\underline{n=1}$ Compute the roots of f. Conclude easily.

$\underline{n=2}$ $f \in \mathbb{Q}[x,y]$



$$\mathrm{vol}\, S = \int_\alpha^\beta \left(y_1(x) - y_2(x)\right) dx$$



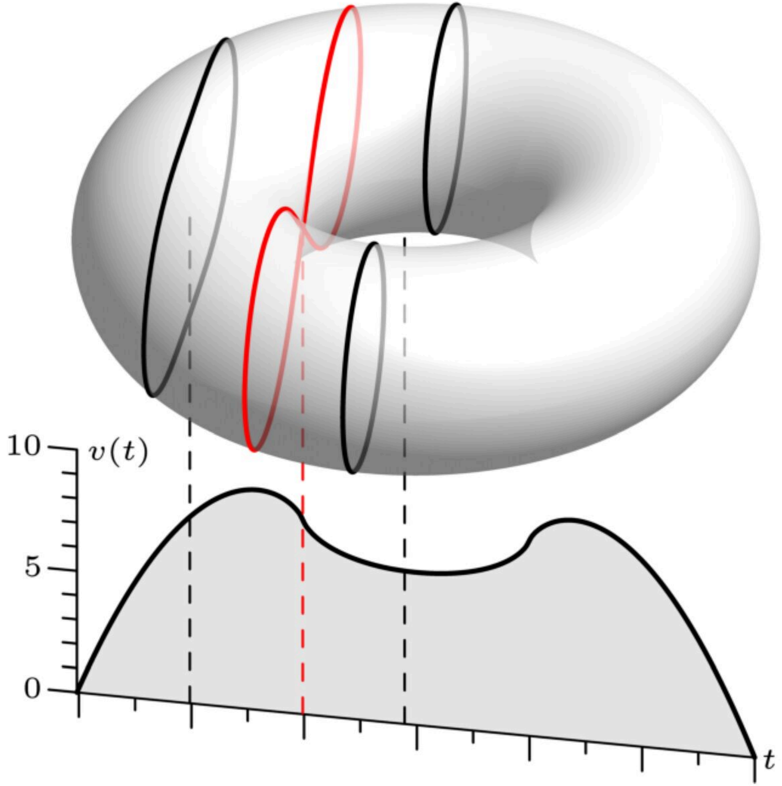
$$\mathrm{vol}\, S = \int_\alpha^\gamma \left(y_1(x) - y_2(x)\right) dx + \int_\gamma^{\beta_1} \left(y_1 - y_3 + y_4 - y_2\right) dx + \int_{\beta_1}^{\beta_2} \left(y_1 - y_3\right) dx$$

Volume of sections

<u>Theorem</u> Let $S \subseteq \mathbb{R}^n$ be a compact semialgebraic set $\{f \geq 0\}$ such that $\{f = 0\}$ is smooth.

Let $v(t) = vol(S \cap \{x_n = t\})$

There is a finite set $\Sigma \subset \mathbb{R}^n$ and a differential operator $L(t, \partial_t)$ s.t. $L \cdot v = 0$ on $\mathbb{R} \setminus \Sigma$.



The exceptional set Σ

$X = \{ f = 0 \} \subseteq \mathbb{R}^n \qquad pr : x \in \mathbb{R}^n \longmapsto x_n \in \mathbb{R}$

$\Sigma \overset{\text{def}}{=} \{ pr(x) \mid f(x) = 0 \text{ and } \ker d_x f \subseteq \{x_n = 0\} \} =$ critical values

$= \{ t \in \mathbb{R} \mid \{x_n = t\} \text{ is tangent to } X \}$

Sard: If X is smooth then Σ is finite

Ehresman: $pr_{|X} : X \longrightarrow \mathbb{R}$ is a locally trivial fibration over $\mathbb{R} \setminus \Sigma$.

That is, for any connected $I \subseteq \mathbb{R}\setminus\Sigma$, and $t \in I$

$$\begin{tikzcd} X \cap pr^{-1}(I) \arrow[rr, dashed, "\sim"] \arrow[dr] & & (X \cap pr^{-1}(t)) \times I \arrow[dl] \\ & I & \end{tikzcd}$$

The differential equation

$S = \{ f \geq 0 \}$ compact $\qquad X = \{ f = 0 \}$ smooth.

$$\mathrm{vol}\, S = \int_S dx_1 \cdots dx_n = \oint_X x_1\, dx_2 \cdots dx_n \qquad \text{(Stokes)}$$

$$= \int_X \left(\oint \frac{u\, \partial_1 f(u, x_2, \dots, x_n)}{f(u, x_2, \dots, x_n)}\, du \right) dx_2 \cdots dx_n$$

$$= \int_{\mathrm{Tube}(X)} \frac{x_1\, \partial_1 f}{f}\, dx_1 \cdots dx_n \qquad \text{(Leray's residue formula)}$$

This a period !

If f depends on a parameter t, then $\mathrm{vol}\, S$ satisfies a Picard-Fuchs equation !

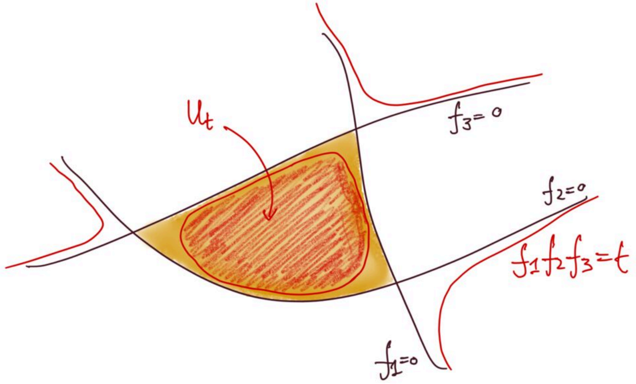
Algorithm (under some assumptions)

INPUT: $f \in \mathbb{Q}[x_1, \dots, x_n]$ s.t. $\{f = 0\}$ is smooth and $\{f \geq 0\}$ compact

OUTPUT: $\mathrm{vol}\,\{f \geq 0\}$

1. Compute the critical values Σ of the projection $x \in \mathbb{R}^n \mapsto x_n$ on $\{f = 0\}$
2. Compute the Picard-Fuchs equation of $t \mapsto \oint \frac{x_1 \partial_1 f(x_1, \dots, x_{n-1}, t)}{f(x_1, \dots, x_{n-1}, t)} dx_1 \cdots dx_{n-1}$
3. For each connected component I of $\mathbb{R} \setminus \Sigma$:
   a. Compute $v_I : t \mapsto \mathrm{vol}(\{f \geq 0\} \cap \{x_n = t\})$ How?
   b. Compute $\int_I v_I(t)\,dt$
4. Return $\sum_I \int_I v_I(t)\,dt$

The general case



INPUT: $f_1, \dots, f_s \in \mathbb{Q}[x_1, \dots, x_n]$ s.t

$S = \{ f_i \geqslant 0,\ 1 \leqslant i \leqslant n \}$ compact

OUTPUT: $\operatorname{vol} S$

1. $g \leftarrow f_1 \cdots f_n - t$
2. Compute the Picard-Fuchs equation for $\oint \frac{x_1 \partial_1 g}{g} dx_1 \cdots dx_n$
3. Compute initial conditions for $t \mapsto \operatorname{vol} \{ g \geqslant 0,\ \underbrace{f_1 \geqslant 0, \dots, f_s \geqslant 0}_{\text{only useful to select a component, does not affect the geometry}} \}$
4. Compute the limit as $t \to 0$.

Summary

Real algebraic geometry

+

symbolic integration

+

numerical analytic continuation

$\Downarrow$

Volume computation.

Computing probabilities

$X \in \mathbb{R}^n$ a random variable. Typically Gaussian, or uniformly distributed in $[0,1]^n$ or $B(0,1)$.

$f \in \mathbb{R}[x_1, \dots, x_n]$, $a, b \in \mathbb{R}$.

<u>Problem.</u> Compute $\mathbb{P}[a \leq f(X) \leq b]$

<u>NB.</u> If X is uniformly distributed in a semialgebraic set K,

then $\mathbb{P}[a \leq f(X) \leq b] = \operatorname{vol}(K \cap \{a \leq f \leq b\})\,(\operatorname{vol} K)^{-1}$

How people without knowledge of Picard-Fuchs equations handle this?

A method from another world

Henrion, Lasserre, Savorgnan
followed by Jasour, Hofmann, Willi

Fix N. Compute $\mu_k \stackrel{\text{def}}{=} \mathbb{E}[f(x)^k]$ for $0 \leq k < N$

Bottleneck

Naive method = $\Theta((\deg f \cdot N)^n)$ operations

Minimize $\sum_{k<N} a_k \mu_k$

subject to $\sum_{k<N} a_k t^k \geqslant \begin{cases} 1 & \text{if } t \in [a,b] \\ 0 & \text{if } t \in \operatorname{supp} f(x) \\ -\infty & \text{otherwise} \end{cases}$

People in optimization know how to do this

Observe that $\sum_k a_k \mu_k = \mathbb{E}[P(f(x))]$ with $P(t) = \sum_k a_k t^k$

$$\geqslant \mathbb{E}[\mathbb{1}_{[a,b]}(f(x))]$$

$$= \mathbb{P}[f(x) \in [a,b]]$$

Converges as $N \to \infty$ but quite slowly

Efficient computation of moments

Let X be a random uniformly distributed random variable in $[0,1]^n$

Let $f \in \mathbb{R}[x_1, \dots, x_n]$

Not a cycle!

$$\mu(t) := \sum_{k=0}^{+\infty} \mathbb{E}[f(X)^k]\, t^k = \int_{[0,1]^n} \frac{dx_1 \cdots dx_n}{1 - t f(x)}$$

D-finite ? YES!

Compute $L(t, \partial_t)$ s.t. $L \cdot (1 - tf)^{-1} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i$

omission

then $L \cdot \mu = \sum_i \underbrace{\int \left(A_i|_{x_i=1} - A_i|_{x_i=0} \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n}_{\text{induction on dimension}}$

$\Rightarrow$ $\tilde{O}(N^2)$ time complexity bound for computing N moments (but somewhat large const.)

Going further?

Consider $\varphi(t) \doteq \mathbb{E}[\exp(i t f(X))] = \sum_{k=0}^{+\infty} \frac{(it)^k}{k!} \mathbb{E}[f(X)^k]$

(the characteristic function of $f(X)$.)

This is the Fourier transform of the density probability function of $f(X)$.

Wishful thinking: If $d_x f = 0 \Rightarrow \det(\mathrm{Hess}_x f) \neq 0$, then $h \in L^2(\mathbb{R})$

$$\mathbb{P}[a \leq f(X) \leq b] = \int_{\mathbb{R}} h(t) \, \mathbb{1}_{[a,b]}(t) \, dt$$

Plancherel's theorem

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\theta) \, \widehat{\mathbb{1}_{[a,b]}}(-\theta) \, d\theta$$

D-finite function
holomorphic on the complex plane
known initial conditions at 0
but... irregular singularity at ∞

Integrals on semialgebraic sets

Let $F(t, x_1, \ldots, x_n)$ be a function in the form $A(t,x)\exp(B(t,x))$,

with $A, B \in \mathbb{C}(t, x_1, \ldots, x_n)$

Let $U_t = \{ f_i(t, \underline{x}) \geq 0,\ 1 \leq i \leq s \}$ for some polynomials $f_i \in \mathbb{C}[t, \underline{x}]$.

<u>Theorem</u> (Oaku, Shiraki, Takayama)

If $t \mapsto \int_{U_t} F(t, \underline{x})\, d\underline{x}$ is well-defined on an interval $I \subseteq \mathbb{R}^n$,

then it is differentially finite (as a distribution)

Distributions

Schwartz' distributions on $\mathbb{R}^n$ are generalized functions

- any L^1 function is a distribution
- Dirac's δ is a distribution
- any distribution f has well-defined partial derivatives $\dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$
- any distribution can be multiplied by a polynomial (but not divided.)

$$\mathrm{ann}(f) = \{ P \in \underbrace{\mathbb{R}[x_1, \dots, x_n]}_{\text{polynomial coefficients}} \langle \tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n} \rangle \mid P \cdot f = 0 \}$$

Example: $\mathrm{ann}(\mathbb{1}_{[0,+\infty)}) = \langle x \frac{\partial}{\partial x} \rangle$

Holonomic ideals

See Sattelberger & Sturmfels for a gentle introduction

Let $W_n = \mathbb{C}[x_1,\dots,x_n]\langle \frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\rangle$

Let $I \subseteq W_n$ be a left ideal.

Let $F_k W_n$ be the finite dimensional space spanned by all $\underline{x}^{\alpha}\frac{\partial^{|\beta|}}{\partial \underline{x}^{\beta}}$ with $|\alpha|+|\beta|\leq k$

The dimension of I is the unique integer d s.t.

$$\dim_{\mathbb{C}} F_k W_n / F_k W_n \cap I \sim \text{constant}\cdot k^d \quad \text{as } k\to\infty$$

(mirrors the commutative theory of dimension)

Theorem (Bernstein) $\quad n \leq d(I) \leq 2n.$

Definition. I is holonomic if $d(I) = n$.

NB: when $n=1$, $I=0$ or I is holonomic

Integration of holonomic ideals

Let $I \subseteq \mathbb{C}[t, \underline{x}]\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \underline{x}} \rangle$ be a holonomic ideal

the integration ideal $\left(I + \frac{\partial}{\partial x_1} W_{n+1} + \cdots + \frac{\partial}{\partial x_n} W_{n+1} \right) \cap \mathbb{C}[t]\langle \frac{\partial}{\partial t} \rangle$

is holonomic.

Rationale: If $I = \operatorname{ann}(f)$, for some $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

and $A(t, \frac{\partial}{\partial t}) \cdot f = \sum_i \frac{\partial}{\partial x_i} B_i \cdot f$,

then $A(t, \frac{\partial}{\partial t}) \cdot \int f(t, \underline{x})\, d\underline{x} = 0$

in nice cases only...

Back to integrals on semialgebraic sets

(after Oaku, Shiraki & Takayama)
see Oaku 2013 JSC

Let $F(t, x_1, \dots, x_n)$ be a function in the form $A(t,x)\exp(B(t,x))$,
with $A, B \in \mathbb{C}(t, x_1, \dots, x_n)$

Let $U_t = \{ f_i(t, \underline{x}) \geq 0,\ 1 \leq i \leq s \}$ for some polynomials $f_i \in \mathbb{C}[t, \underline{x}]$.

1. $F\,\mathbb{1}_{U_t}$ is a holonomic distribution on $\mathbb{R} \times \mathbb{R}^n$

2. With appropriate convergence hypothesis, the integration ideal of $\operatorname{ann}(F\,\mathbb{1}_{U_t})$ annihilates $t \mapsto \int_{U_t} F(t, \underline{x})\, d\underline{x}$

Conclusion

We can compute volume of semialgebraic using
period of rational integrals

The D-module approach is extremely attractive
but suffers (at the moment) from the lack
of simple algorithms and efficient implementations.