Computing with integrals in nonlinear algebra
Exercises

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Problem 1. Show that the series $\sum_{n\geq 0}\binom{5n}{n}t^n$ is algebraic.
Show that the series $\sum_{n\geq 0}\frac{(3n)!}{n!^3}t^n$ is not algebraic.

Problem 2. Using the formula $\gamma = -\int_0^\infty e^{-t}\log t\, dt$, compute 1000 digits of the Euler–Mascheroni constant.

Problem 3. Show that

$$\sum_{k=1}^{n}(-4)^{-k}\binom{n-k}{k-1}\sum_{j=1}^{3m}(-2)^{-j}\binom{n+1-2k}{j-1}\binom{m-k}{3m-j} = 0, \quad \forall n, m > 0,$$

and that

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3.$$

Problem 4. Show that

$$\sum_{k=0}^{n}\binom{n}{k}f_k f_{n-k} = \tfrac{1}{5}(2^n g_n - 2),$$

where f_n is defined by $f_{n+2} = f_{n+1} + f_n$ and $f_0 = 0$ and $f_1 = 1$, and g_n is defined by the same recurrence relation but $g_0 = 2$ and $g_1 = 1$.

Problem 5. Inspired by Kontsevich and Odesskii (2020), consider the differential operator $L = \partial z(z-1)(z-\alpha)\partial + z$, where α is a parameter with $|\alpha| \ll 1$.

1. Show that L is Fuchsian.

Let M be the monodromy matrix corresponding to a loop enclosing 0 and α (but not 1).

2. Show that $\det M = 1$.
3. Let $\exp(\pm i2\pi\lambda)$ be the two eigenvalues of M. Check experimentally that λ is a power series in α with rational coefficients. Compute as many coefficients as you can.

Problem 6. Inspired by Koutschan (2013), consider a random walk on a face-centered cubic structure: a point X in $\mathbb{Z}^3$ starts at 0, and at each step the point moves randomly to one of its twelve neighbors in the structure:

$$X + (\pm 1, \pm 1, 0), X + (\pm 1, 0, \pm 1), X + (0, \pm 1, \pm 1).$$

Let X_n be the position after the nth step (this is a random variable). Let p_n be the probability that $X_n = 0$.

1. Let a_n be the probability that $X_n = 0$. Let $A(t) = \sum_{n\geq 0} a_n t^n$. Give a rational function $R(t, x_1, x_2, x_3)$ such that

$$A(t) = \operatorname{res}_{x_1,x_2,x_3} R.$$

2. Let $b_0 = 0$ and, for $n > 0$, let b_n be the probability that $X_n = 0$ and $X_k \neq 0$ for $0 < k < n$. Let $B(t) = \sum_{n\geq 0} b_n t^n$. Show that $B(t) = 1 - A(t)^{-1}$.
3. Evaluate numerically the *return probability*, that is the probability that there is an $n > 0$ such that $X_n = 0$.

References

M. Kontsevich and A. Odesskii (2020). "p-Determinants and monodromy of differential operators". arXiv: 2009.12159.

C. Koutschan (2013). "Lattice Green Functions of the Higher-Dimensional Face-Centered Cubic Lattices". In: *J. Phys. A* 46.12, pp. 125005, 14. DOI: 10.1088/1751-8113/46/12/125005.