Transcendental methods in numerical algebraic geometry

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today’s menu: elliptic curves and quartic surfaces

What is numerical algebraic geometry?

How to do effective complex algebraic geometry?

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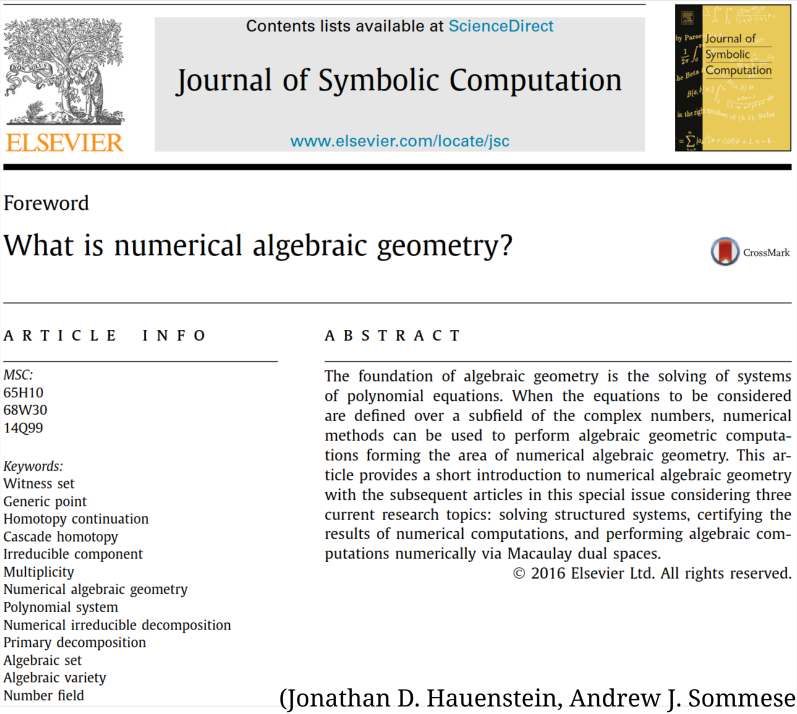
How to do effective complex algebraic geometry?

algebraic side	polynomial rings, polynomial ideals, symbolic algorithms (Gröbner bases, regular chains)
arithmetic side	reduction modulo *p*, *p*-adic numbers, Frobenius structures
geometric side	complex points, numerical approximations, numerical algorithms (path tracking)



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Foreword

What is numerical algebraic geometry?

ARTICLE INFO

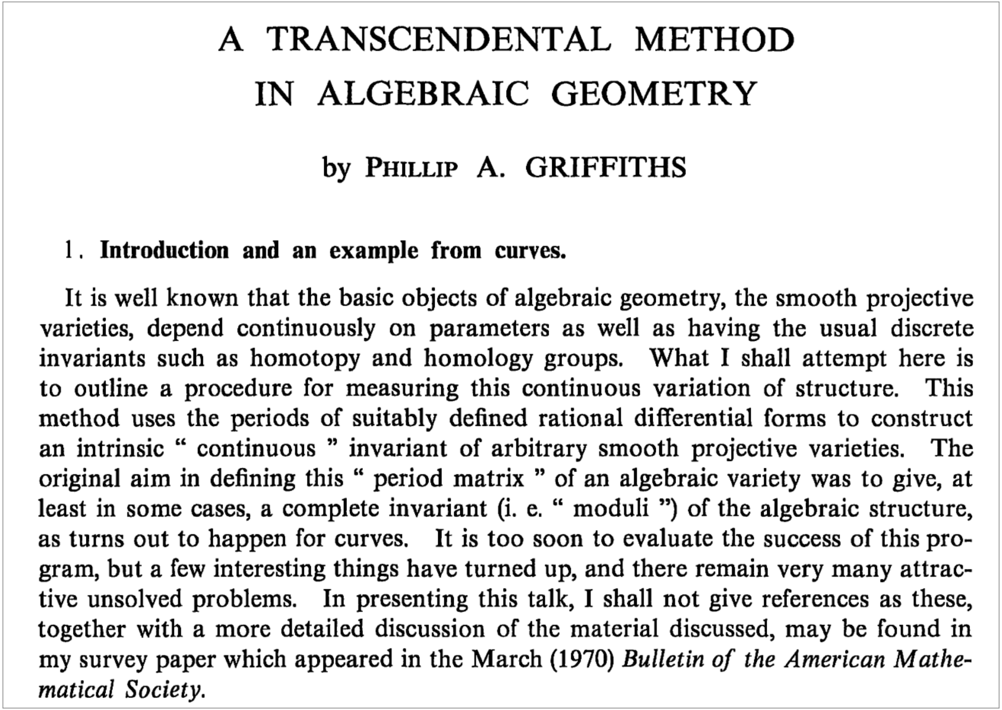
MSC:
65H10
68W30
14Q99

Keywords:
Witness set
Generic point
Homotopy continuation
Cascade homotopy
Irreducible component
Multiplicity
Numerical algebraic geometry
Polynomial system
Numerical irreducible decomposition
Primary decomposition
Algebraic set
Algebraic variety
Number field

ABSTRACT

The foundation of algebraic geometry is the solving of systems of polynomial equations. When the equations to be considered are defined over a subfield of the complex numbers, numerical methods can be used to perform algebraic geometric computations forming the area of numerical algebraic geometry. This article provides a short introduction to numerical algebraic geometry with the subsequent articles in this special issue considering three current research topics: solving structured systems, certifying the results of numerical computations, and performing algebraic computations numerically via Macaulay dual spaces.

(Jonathan D. Hauenstein, Andrew J. Sommese)



A TRANSCENDENTAL METHOD IN ALGEBRAIC GEOMETRY

by PHILLIP A. GRIFFITHS

1. Introduction and an example from curves.

It is well known that the basic objects of algebraic geometry, the smooth projective varieties, depend continuously on parameters as well as having the usual discrete invariants such as homotopy and homology groups. What I shall attempt here is to outline a procedure for measuring this continuous variation of structure. This method uses the periods of suitably defined rational differential forms to construct an intrinsic " continuous " invariant of arbitrary smooth projective varieties. The original aim in defining this " period matrix " of an algebraic variety was to give, at least in some cases, a complete invariant (i. e. " moduli ") of the algebraic structure, as turns out to happen for curves. It is too soon to evaluate the success of this program, but a few interesting things have turned up, and there remain very many attractive unsolved problems. In presenting this talk, I shall not give references as these, together with a more detailed discussion of the material discussed, may be found in my survey paper which appeared in the March (1970) *Bulletin of the American Mathematical Society.*

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high-precision numerical solving, higher-order methods required

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\+ **integer relation algorithm (LLL, PSLQ, HJLS)**

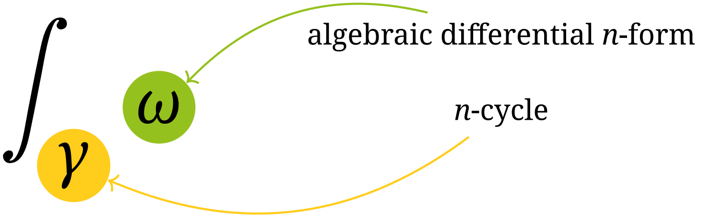
Today’s goal

Explain on two examples:

- ∗ how to compute periods with high precision,
- ∗ how to solve a concrete algebraic problem with them.

Periods

X complex algebraic variety manifold of dimension n



$$\int_{\gamma} \omega$$

algebraic differential n-form

n-cycle

- boils down to a n-fold integral of an algebraic function

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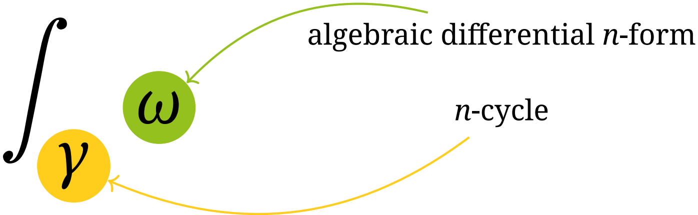

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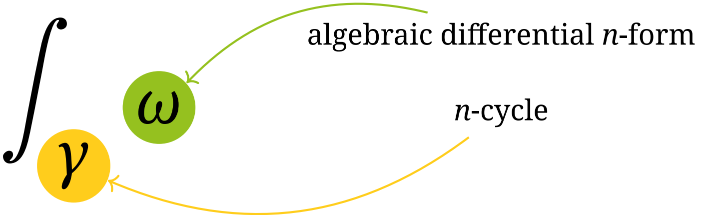

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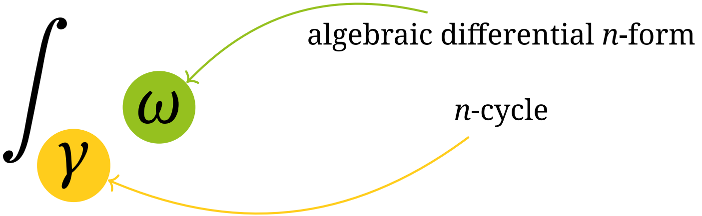

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- ⚠ often not computable exactly, need hundreds or thousands of digits
- ⚠ in this regime, direct numerical recipes do not work well

Why periods are called periods?

$$X = \left\{ (t, s) \in \mathbb{C}^2 \;\middle|\; t^2 + s^2 = 1 \right\}, \qquad s = \pm\sqrt{1 - t^2}$$

$$\sin\left(\int_0^u \frac{\mathrm{d}t}{\sqrt{1 - t^2}} \right) = u$$

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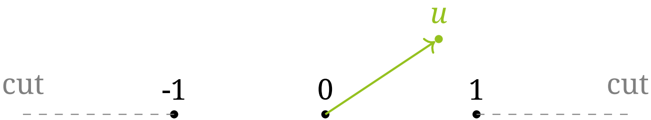

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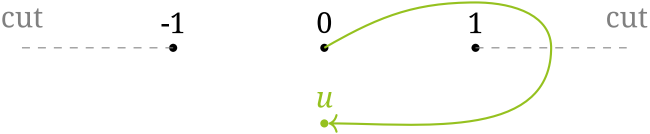

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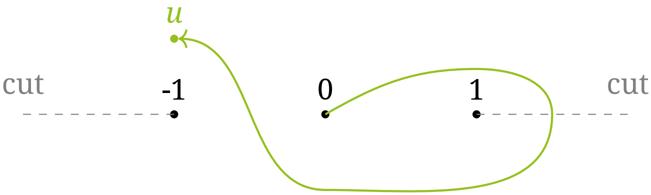

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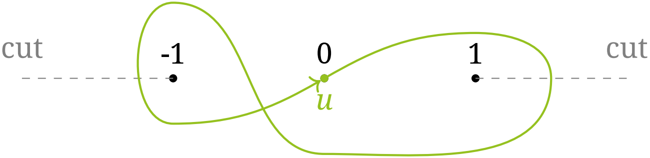

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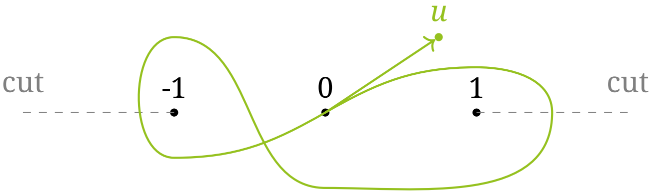

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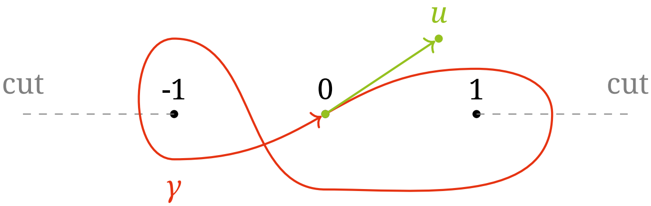

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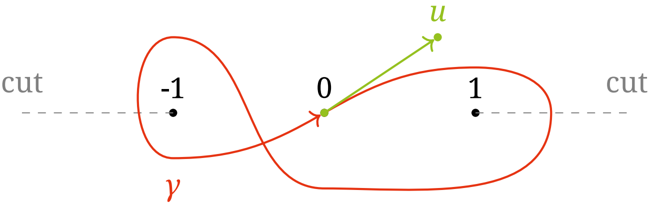

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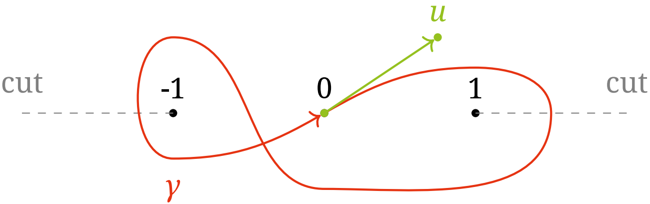
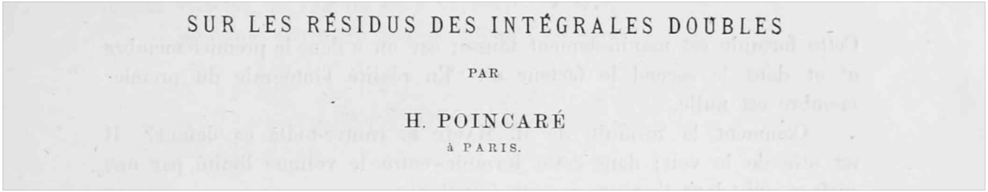
$$\sin\left(\underbrace{\int_{\gamma} \frac{\mathrm{d}t}{\sqrt{1-t^2}}}_{\text{period!}} + z\right) = \sin(z)$$



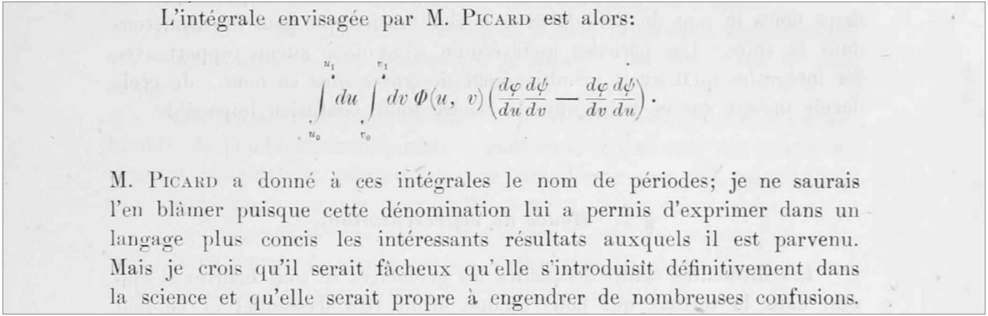


SUR LES RÉSIDUS DES INTÉGRALES DOUBLES

PAR

H. POINCARÉ

à PARIS.



L'intégrale envisagée par M. Picard est alors:

$$\int_{u_0}^{u_1} du \int_{v_0}^{v_1} dv\, \Phi(u,\ v)\left(\frac{d\varphi}{du}\frac{d\psi}{dv} - \frac{d\varphi}{dv}\frac{d\psi}{du}\right).$$

M. Picard a donné à ces intégrales le nom de périodes; je ne saurais l'en blâmer puisque cette dénomination lui a permis d'exprimer dans un langage plus concis les intéressants résultats auxquels il est parvenu. Mais je crois qu'il serait fâcheux qu'elle s'introduisît définitivement dans la science et qu'elle serait propre à engendrer de nombreuses confusions.

"M. Picard gave these integrals the name of periods; I cannot blame him since this name allowed him to express in more concise language the interesting results he achieved. But I believe that it would be unfortunate if it were definitively introduced into science and that it would be likely to generate numerous confusions."

Periods

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$$\alpha(t) = \int_{\gamma_t} \omega_t$$

algebraic differential n-form, rational in t

continuously varying n-cycle

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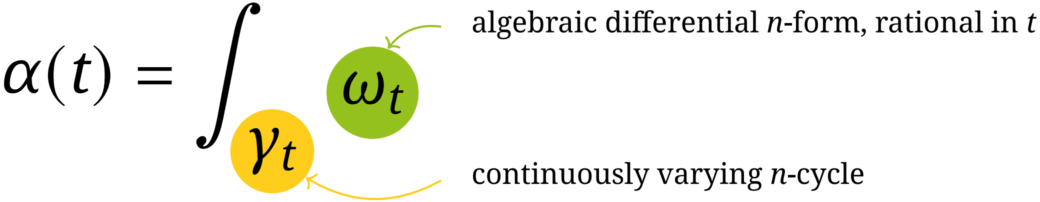

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- **symbolic integration**

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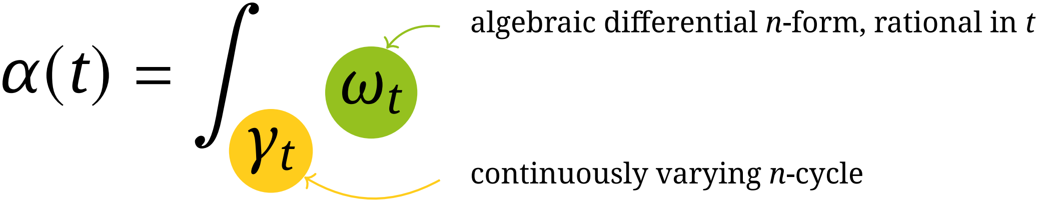

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Picard–Fuchs equations

There are polynomials $p_0(t), \dots, p_r(t) \neq 0$ such that

$$p_r(t)\alpha^{(r)}(t) + \cdots + p_1(t)\alpha'(t) + p_0(t)\alpha(t) = 0.$$

High precision numerical integration of linear ODEs

Theorem (Chudnovsky and Chudnovsky, 1990)

Consider

- *a linear ODE* $\quad (*)\ p_r(t)y^{(r)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0$
- *a path* $\gamma : [0,1] \to \mathbb{C} \setminus \mathit{zeros}(p_r)$
- *initial condition* $u_0, \ldots, u_{r-1} \in \mathbb{C}$

Then we can compute $y(\gamma_1)$, *up to precision* 2^{-p}, *where* y *is the unique solution of* $(*)$ *such that* $y^{(i)}(\gamma_0) = u_i$ $(0 \leq i < r)$, *analytically continued along* γ.

Moreover:

- *The error bound is explicit*
- *As* $p \to \infty$ *(everything else is fixed), the algorithm runs in time* $\tilde{O}(p)$.

See also van der Hoeven (1999) and Mezzarobba (2010).

High precision numerical integration (variant)

Corollary

In the same context, we can compute $\int_\gamma y(z)\mathrm{d}z$, up to precision 2^{-p}.

Moreover:

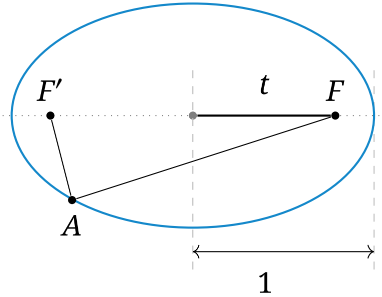
- The error bound is explicit
- As $p \longrightarrow \infty$ (everything else is fixed), the algorithm runs in time $\tilde{O}(p)$.

Proof. Apply the theorem to the differential equation

$$p_r(t)I^{(r+1)}(t) + \cdots + p_1(t)I''(t) + p_0(t)I'(t) = 0$$

of which $I(t) = \int_{\gamma_0}^{t} y(z)\mathrm{d}z$ is solution.

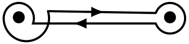
Perimeter of an ellipse



$$E(t) = 2\int_{-1}^{1}\sqrt{1+y'(x)^2}\mathrm{d}x$$

$$= 2\int_{-1}^{1}\sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x$$

$$= \int_{\gamma}\sqrt{\frac{1-t^2x^2}{1-x^2}}\mathrm{d}x$$

Where $\gamma =$ .

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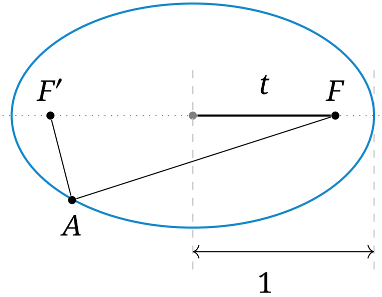

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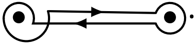
Where $\gamma =$ .

Theorem (Euler, 1733)

$(t-t^3)E'' + (1-t^2)E' + tE = 0$

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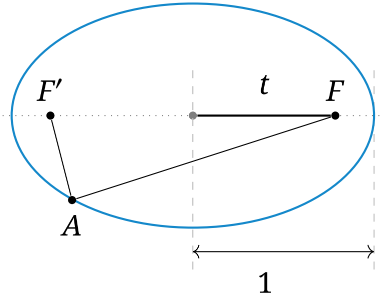

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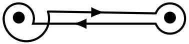
Where $\gamma =$ .

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Theorem (Liouville, 1834)

*$E(t)$ is **transcendental**.*
It is not even expressible in terms of elementary functions.

Proof of Euler’s theorem

Let $F(t, x) = \sqrt{\frac{1-t^2x^2}{1-x^2}}$, so that $E(t) = \int_\gamma F(t, x)\mathrm{d}x$.

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$*$ Symbolic integration provides algorithms for finding the magical relation $(*)$. Keywords: creative telescoping, D-module integration. (Chyzak, 2000; Koutschan, 2010; Oaku & Takayama, 2001; Lairez, 2016; Chen, van Hoeij, Kauers, & Koutschan, 2018; Bostan, Chyzak, Lairez, & Salvy, 2018)

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- Many implementations

Computing the perimeter, 1st method

Gauss quadrature

Let f be a multivalued analytic function on the complex plane.

$$\int_\gamma f(x)\mathrm{d}x = \sum_{i=1}^{N} w_i f(x_i) + O(C^{-N}),$$

for a suitable choice of w_i and $x_i \in (-1,1)$.

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- $*$ Effective error bounds
- $*$ Complexity $\tilde{O}(N^2)$ for computing the w_i and the x_i
- $*$ Needs evaluation of f at precision C^{-N} at N points
- $\rightsquigarrow$ For k-fold integrals, this leads to a $\tilde{O}(N^{k+1})$ total complexity for computing N digits.

Computing the perimeter, 2nd method

Goal: Compute $E(\frac{1}{2})$

Transcendental continuation, **outer** variant

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Let $R(t) = \sqrt{\frac{1-\frac{1}{4}x^2}{1-x^2}}$, so that $E(\frac{1}{2}) = \int_\gamma R(x)\mathrm{d}x$.

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⚙ Demo!

Wrap up

- Transcendental functions arise from algebraic varieties and $\int$
- We can compute differential equations for integrals with a parameter
- We can compute numerically integrals (without parameter):
  - by the outer method, which introduces a parameter in the integral,
  - by the inner method, which uses the first integration variable as the parameter.
- We can compute to large precision thanks to quasilinear complexity.

The endomorphism ring of an elliptic curve

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- X has the structure of an abelian group.
- $\mathrm{End}(X) = \{\text{regular maps } f : X \to X \text{ with } f(0) = 0\}$ (they are automatically group endomorphisms).
- $\mathrm{End}(X)$ contains at least all the maps $p \in X \mapsto np$ with $n \in \mathbb{Z}$.

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Problem

Is $\mathrm{End}(X)$ nontrivial ($\neq \mathbb{Z}$)?

Nature of the problem

Theorem

The set for all $a, b \in \mathbb{C}^2$ *such that the curve* $X = \left\{y^2 = x^3 + ax + b\right\}$ *has a nontrivial endomorphism is the union of countably many curves in* $\mathbb{C}^2$.

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- The problem does *not* reduce directly to polynomial system solving.
- *Most* elliptic curves does not have a nontrivial endomorphism.
- But elliptic curves with a nontrivial endomorphism are *dense*!
- See Cremona and Sutherland (2023) for a recent progress on the question (algebraic approach).

Analytic approach

- There a meromorphic map $\wp : \mathbb{C} \longrightarrow \mathbb{C}$, *Weierstrass' function*, such that $z \longrightarrow (\wp(z), \wp'(z))$ is a sujective group homomorphism.
- It induces an isomorphism $X \simeq \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2$. α_1 and α_2 are the *periods* of $\wp$.

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- The analytic endomorphisms of $\mathbb{C}$ are the maps $z \mapsto uz$, for $u \in \mathbb{C}$.

The endomorphism ring of a torus

Proposition

$$\mathrm{End}(X) \simeq \{u \in \mathbb{C} \mid u\Lambda \subseteq \Lambda\}\,.$$

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$\mathrm{End}(X)$ is nontrivial if and only if the equation

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has a solution $z \in \mathbb{C}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ not scalar.

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$\mathrm{End}(X)$ is nontrivial if and only if the equation

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$\mathrm{End}(X)$ is nontrivial if and only if the equation

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has a nonzero solution, $a, b, c \in \mathbb{Z}$, where $\tau = \alpha_2/\alpha_1$.

Recover exact data from approximate numbers?

Assume that we have computed τ with large precision.

Can we decide if there are nonzero integers a, b, and c such that $b\,\tau^2 + a\,\tau - c = 0$?

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Yet, we do it every day. Which one of the following numbers is rational?

$$\begin{gathered} 1.6180339887498948482045868343656381177203091798057628 \\ 6213544862270526046281890244970720720418939113 8\ldots \end{gathered}$$

$$\begin{gathered} 1.6153846153846153846153846153846153846153846153846153 \\ 8461538461538461538461538461538461538461538461 5\ldots \end{gathered}$$

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84615384615384615384615384615384615384615384615\ldots
\end{gathered}
$$

Impossible question, but good practical answer: lattice reduction.

Computation of the periods

Recall that $\wp : \mathbb{C} \to \mathbb{C}$ is Weierstrass' functions and $(\wp(z), \wp'(z)) \in X$, that is

$$\wp'(z)^2 = \wp(z)^3 + a\wp(z) + b.$$

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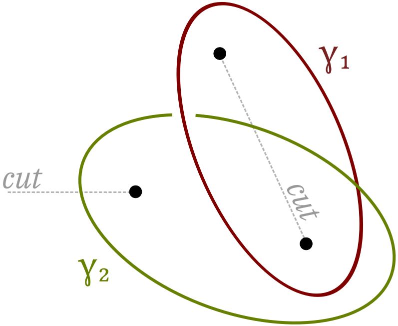
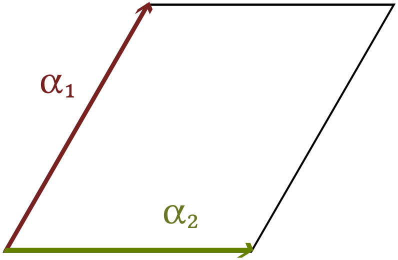
It follows that

$$\wp\left(\int_0^u \frac{\mathrm{d}x}{\sqrt{x^3 + ax + b}}\right) = u.$$

(Does it remind you of something?)

Computation of the periods





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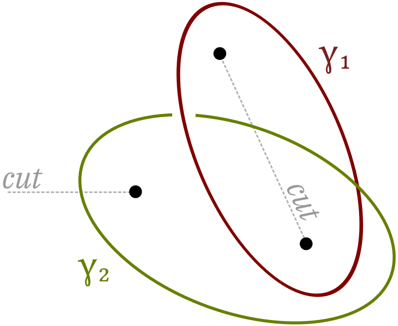

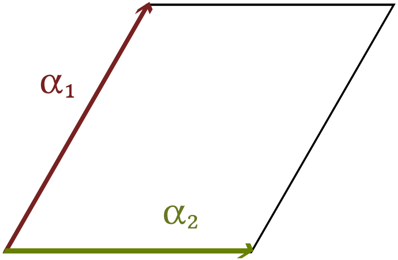

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⚙ Demo!

High precision quadrature uncovers the endomorphism ring of elliptic curves.

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heuristic algorithm, only provides a safe bet.
No known way to trick the heuristic.

- Possibility to certify *a posteriori* (e.g. Costa, Mascot, Sijsling, & Voight, 2019), at the cost of simplicity of course

Curves on a surface

Let $f \in \mathbb{C}[w, x, y, z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

- X contains algebraic curves.
- *Trivial* curves are those obtained by intersecting X with another surface. (Every curve is included in the intersection with another surface, but may not be equal.)

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Does X contain a nontrivial curve?

Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w,x,y,z]_4 \setminus$ (countable union of algebraic hypersurfaces).
Then X_f contains only trivial curves.

Findind hay in a haystack

Theorem (Terasoma, 1985)

There is a smooth $f \in \mathbb{Q}[w, x, y, z]_4$
such that X_f contains only trivial curves.

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Theorem (van Luijk, 2007)

Let $f = 2w^4+w^3z+w^2x^2+2w^2xy+2w^2xz-w^2y^2+w^2z^2+wx^3-wx^2y-wx^2z-wxy^2-wxyz+wxz^2+wy^3+wy^2z+wyz^2-3x^2y^2-xy^2z-4xyz^2-2xz^3-5yz^3-z^4$.
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Theorem (Lairez & Sertöz, 2019)

Let $f = wx^3 + w^3y + xz^3 + y^4 + z^4$. Then X_f contains only trivial curves.

Nature of the problem

Reduction to countably many polynomial systems.

$$\begin{aligned}
\{\text{lines in } X\} &= \left\{(u,v) \in (\mathbb{C}^4)^2 \;\middle|\; u \wedge v \neq 0 \text{ and } \forall t, f(u+tv) = 0\right\} /\!\sim \\
\{\text{conic curves in } X\} &= \Big\{(u,v,w) \in (\mathbb{C}^4)^3 \;\Big| \\
&\qquad\qquad u \wedge v \wedge w \neq 0 \text{ and } \forall t, f(u+tv+t^2w) = 0\Big\}/\!\sim \\
\{\text{twisted cubics in } X\} &= \Big\{(u_0,\dots,u_3) \in (\mathbb{C}^4)^4 \;\Big| \\
&\qquad\qquad u_0 \wedge \cdots \wedge u_3 \neq 0 \text{ and } \forall t, f\left(\sum_{i=0}^{3} u_i t^i\right) = 0\Big\}/\!\sim \\
\{\text{deg. 4 gen. 1 c. in } X\} &= \Big\{(g_1,g_2,h_1,h_2) \in (\mathbb{C}[\mathbf{x}]_2)^4 \;\Big| \\
&\qquad\qquad g_1 \text{ and } g_2 \text{ generic and } f = h_1g_1 + h_2g_2\Big\}/\!\sim
\end{aligned}$$

The structure of curves on a surface

Let X be a smooth quartic complex surface.

Consider the 2nd singular homology group of X:

$$H_2(X,\mathbb{Z}) = \frac{\text{sum of triangles in } X \text{ with no boundary}}{\text{sum of boundaries of 3-simplices in } X} \simeq \mathbb{Z}^{22}.$$

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Noether-Lefschetz theorem (Lefschetz, 1924)

Let $f \in \mathbb{C}[w, x, y, z]_4 \setminus$ (countable union of algebraic hypersurfaces). Then $\mathrm{NS}(X_f) = \mathbb{Z}$.

Periods of a quartic surface

Let $f \in \mathbb{C}[w,x,y,z]_4 \simeq \mathbb{C}^{35}$
such that $X = V(f) \subseteq \mathbb{P}^3$ is smooth.

Let $\gamma_1, \dots, \gamma_{22}$ be a basis of $H_2(X, \mathbb{Z})$,
and let $\omega_X \in \Omega^2(X)$ be the unique holomorphic 2-form on X.

The *periods* of X are the complex numbers $\alpha_1, \dots, \alpha_{22}$ defined – up to scaling and choice of basis – by

$$\alpha_i \overset{\text{def}}{=} \oint_{\gamma_i} \omega_X = \frac{1}{2\pi i} \oint_{\text{Tube}(\gamma_i)} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{f|_{w=1}}$$

Periods determine the Néron-Severi group

The Néron-Severi group of X (a smooth quartic surface) is the sublattice of $H_2(X, \mathbb{Z})$ generated by the classes of algebraic curves on X.

Theorem (Lefschetz, 1924)

$$\mathrm{NS}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \;\middle|\; \int_\gamma \omega_X = 0 \right\}$$

In coordinates, $\mathrm{NS}(X) \simeq \left\{ \mathbf{u} \in \mathbb{Z}^{22} \;\middle|\; u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}$.
This is the lattice of *integer relations between the periods*.

The NS group determine the possible degree and genus of all the algebraic curves lying on X.

The Fermat hypersurface

Let $f = w^4 + x^4 + y^4 + z^4$.
The vector of periods is

$$\begin{pmatrix} 1 & i & i & i & i & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -i & -i & -i & -i & -i & -i$$

$$\operatorname{rank} \mathrm{NS}(X_f) = 22 - \dim \mathrm{Vect}_{\mathbb{Q}} \{\text{periods}\} = 20.$$

Indeed there are 48 lines on X_f spanning a sublattice of $H_2(X, \mathbb{Z})$ of rank 20.

The outer method for computing periods (Sertöz, 2019)

Let $f \in \mathbb{C}[w, x, y, z]_4$
and let $f_t = (1 - t)f + t(w^4 + x^4 + y^4 + z^4) \in \mathbb{C}(t)[w, x, y, z]_4$.

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3. Numerical analytic continuation provides quasilinear-time algorithms for computing the periods.

⚠ Afflicted by the size of the PF equation (generically order 21 and degree $\geq$ 1000), the algorithm does not always terminate in reasonnable time.

Computation of the lattice of integer relations

We have the periods $\alpha_1, \ldots, \alpha_{22}$ with high precision (hundreds of digits); we want a basis of

$$\Lambda = \left\{ \mathbf{u} \in \mathbb{Z}^{22} \,\middle|\, u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0 \right\}.$$

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   this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.

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   this is a rank 22 lattice. Short vectors are expected to come from integer relations between the periods.
3. Compute a LLL-reduced basis of L
4. Output the *short* vectors

What is a short vector?

Let $f = 3x^3z - 2x^2y^2 + xz^3 - 8y^4 - 8w^4$.
With 100 digits of precision on the periods, here is a LLL-reduced basis of the lattice L (last 5 columns omitted).

$$\left[\begin{array}{cccccccccccccc rrr c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1669083212117905913652734 & 0 & 1937019641160560221317687 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1669083212117905913652734 & 1937019641160560221317687 & .. \\
\cline{15-18}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -146511829901195443671789 & 84478429044587822467823 & -365980228690630104919296 & .. \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -337167720252678310258177 & 224110151973403946221421 & -743116955936487279910552 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 357031479253522311483650 & 768066337666351099432748 & 940525994719391079998435 & .. \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -552756671828854153114905 & -126018248279583585486071 & 535095811953165917210863 & .. \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 104335431129908645825133 & -231616284585318363570849 & 502730408585962411025306 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -649159586430203173692632 & 770784867967071100945665 & -2152014469737999315531272 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 277747983934797690835205 & -28625739873061372966384 & -638732179408358479990097 & .. \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 146511829901195443671790 & -84478429044587822467823 & 365980228690630104919296 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 250899146775406645936761 & 575615030011256031395007 & -114830012426104078247291 & .. \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 104335431129908645825133 & -231616284585318363570849 & 502730408585962411025307 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -140644950443454586919439 & -393058206212350140614235 & 429933080833930208291557 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 594933070600140950961561 & 273156103820314126589096 & -671845991848498223316874 & .. \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 337167720252678310258177 & -224110151973403946221421 & 743116955936487279910552 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -824317154838996681984621 & 177119763197465887754938 & -236792300924643740702432 & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 379344119023965108104833 & -76972296432673405118395 & 606366776041154973804541 & .. \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 552756671828854153114905 & 126018248279583585486070 & -535095811953165917210864 & .. \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -467285675585474370500971 & -950623161465256990213520 & -1255629063127217210042702 & .. \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -69025235930677842745100 & 457102914343586863258366 & 660652346877586707848817 & ..
\end{array}\right.$$

A triple alternative

- ↯ Certified error bounds!
- $*$ assume that the periods are known $\pm\beta^{-1}$

Lemma

If the heuristic algorithm succeeds then one of the following holds:

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I do not know how to deal with 2, there are quartic surfaces with NS group minimaly generated by arbitrary large elements (Mori, 1984).

But we can do something about 3.

Separation of periods

Let $f \in \mathbb{Q}[w, x, y, z]_4$
and let $\alpha_1, \ldots, \alpha_{22}$ be the periods.

Theorem (Lairez & Sertöz, 2022)

There exist a computable constant $c > 0$ depending only on f and the choice of the homology basis, such that for any $\mathbf{u} \in \mathbb{Z}^{22}$,

$$|u_1\alpha_1 + \cdots + u_{22}\alpha_{22}| < 2^{-c^{\max_i |u_i|^9}} \Rightarrow u_1\alpha_1 + \cdots + u_{22}\alpha_{22} = 0.$$

An inner method for computing periods?

- Sertöz’ algorithm is very indirect.
- Can we directly compute

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$$\alpha_i = \oint_{\gamma_i} \omega_X?$$

- That’s a *double* integral.
- How do we get γ_i?
  How do we compute a basis of the singular homology group $H_2(X)$?

Double integrals *via* Fubini

- $f \in \mathbb{C}[w,x,y,z]_4$ (generic coordinates)
- $X \triangleq V(f) \subseteq \mathbb{P}^3(\mathbb{C})$
- $X_t \triangleq X \cap \left\{ \frac{w}{x} = t \right\}$ (hyperplane section)
- 💡 Consider the surface as a family of curves

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Main idea

$$\int_\gamma \omega_X = \oint_{\text{loop in } \mathbb{C}} \mathrm{d}t \underbrace{\oint_{\text{cycle in } X_t} \frac{\omega_X}{\mathrm{d}t}}_{} .$$

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⚡ satisfies a Picard–Fuchs equation!

⚙ Requires a concrete description of γ to be implemented.
We need to *compute* $H_2(X, \mathbb{Z})$

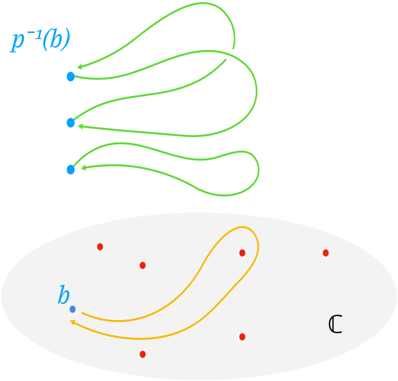
The homology of curves (Tretkoff & Tretkoff, 1984)

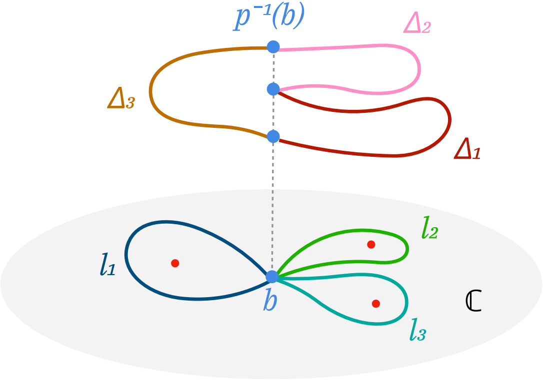
- X a complex algebraic curve
- $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map
- $\Sigma \triangleq \{\text{critical values}\}$

- Given a loop in $\mathbb{P}^1(\mathbb{C}) \setminus \Sigma$, starting from a base point b, and a point in the fiber $p^{-1}(b)$, the loop lifts in X uniquely.
- ⚙ Compute loops in $\mathbb{P}^1(\mathbb{C})$ that lift in a basis of $H_1(X, \mathbb{Z})$

(Deconinck & van Hoeij, 2001; Costa, Mascot, Sijsling, & Voight, 2019)

Principle of the method

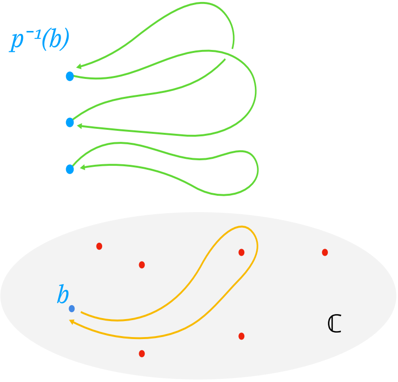


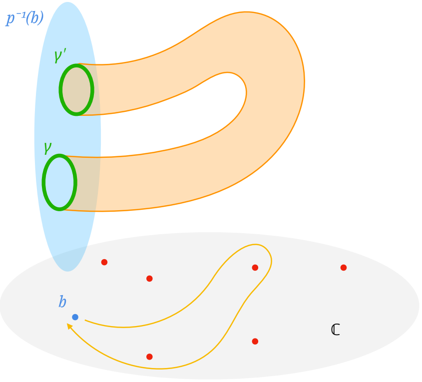


1. compute pieces of paths in X by lifting loops
2. connect them to form loops

Homology of surfaces

	dimension 1	dimension 2
monodromy action	permute the fiber	linear action on $H_1(X)$
lift in X	path	*hosepipe*
computable with	path tracking	numerical ODE solving





Homology of surface from the monodromy

- X a complex algebraic curve
- $p : X \longrightarrow \mathbb{P}^1(\mathbb{C})$ nonconstant map, define $X_t = p^{-1}(t)$
- $\Sigma \triangleq \{\text{critical values}\}$

- Given a loop γ in $\mathbb{P}^1 \setminus \Sigma$ starting from a base point b, and a cycle $c \in H_1(X_b)$, the cycle deforms as t runs along γ.
- This defines the monodromy action $\gamma_* : H_1(X_b) \longrightarrow H_1(X_b)$.
- ⚙ Compute the monodromy action of generators or $\pi_1(\mathbb{P}^1 \setminus \Sigma)$ to construct elements of $H_2(X)$.

(Lefschetz, 1924; Lamotke, 1981; Lairez, Pichon-Pharabod, & Vanhove, 2024; Pichon-Pharabod, 2024)

Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solution of the Picard–Fuchs equation of the periods of X_t.

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⌛⌛⌛

We can compute periods of a quartic surface with hundreds of digits in about 1 hour.

Monodromy computation in higher dimension

De Rham duality

The monodromy action on $H_1(X_t)$ is dual to the monodromy action on the solution of the Picard–Fuchs equation of the periods of X_t.

- We can connect hosepipes by integrating a Picard–Fuchs differential equation.

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We can compute periods of a quartic surface with hundreds of digits in about 1 hour.

Thank you!

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