Two algorithms for computing with Feynman integrals

Pierre Lairez
Université Paris–Saclay, Inria, France

based on joint work with Pierre Vanhove (IPhT)

December 8, 2022

European Research Council
Established by the European Commission

Outline

1. Introduction

2. Computation of Picard–Fuchs operators

3. Minimality of the Picard–Fuchs operator

4. Order of the PF operator from GKZ systems

The meta-question

What is the the motive of a Feynman integral?
(and also, what is a motive?)

That is, *explain* the nature of a Feynman integral in terms of *basic* varieties.

The three-loops sunset graph: an example

(Bloch et al., 2015)

$$I(t) = \iiint_0^\infty \frac{1}{\left(1 + \sum_{i=1}^3 x_i\right)\left(1 + \sum_{i=1}^3 x_i^{-1}\right) - t} \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\frac{\mathrm{d}x_3}{x_3}.$$

Theorem

$$\underbrace{\left(t^2(t-4)(t-16)\frac{\mathrm{d}^3}{\mathrm{d}t^3} + \cdots\right)}_{\text{Picard-Fuchs operator}} \cdot I(t) = -24.$$

$$I(t) = (\text{period of a K3 family}) \cdot (\text{elliptic trilogarithms})$$

The Picard-Fuchs operator

- $x_1, \dots, x_n$, integration variables
- t, parameter
- $R(t, x_1, \dots, x_n)$, a rational function
- γ, a n-cycle in $\mathbb{C}^n$ on which R is continuous
- $I(t) = \oint_\gamma R(t, x_1, \dots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n$

Problem

Find $p_0(t), \dots, p_r(t) \in \mathbb{C}[t]$ such that

$$p_r(t) I^{(r)}(t) + \cdots + p_1(t) I'(t) + p_0(t) I(t) = 0.$$

The Picard-Fuchs operator

- $x_0, x_1, \ldots, x_n$, integration variables
- t, parameter
- $R(t, x_0, x_1, \ldots, x_n)$, a rational function
- γ, a $n+1$-cycle in $\mathbb{C}^{n+1}$ on which R is continuous
- $I(t) = \oint_\gamma R(t, x_0, x_1, \ldots, x_n) \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_n$
- **homogeneity** :
  $R(t, \lambda x_0, \ldots, \lambda x_n) \mathrm{d}(\lambda x_0) \cdots \mathrm{d}(\lambda x_n) = R(t, x_0, \ldots, x_n) \mathrm{d}x_0 \cdots \mathrm{d}x_n$

Problem

Find $p_0(t), \ldots, p_r(t) \in \mathbb{C}[t]$ such that

$$p_r(t) I^{(r)}(t) + \cdots + p_1(t) I'(t) + p_0(t) I(t) = 0.$$

The order of the PF operator

Let $\gamma \in H_n(\mathbb{P}^n \setminus \mathrm{pole}(R))$ generic and $I(t) = \int_\gamma R(t,\mathbf{x})\mathrm{d}\mathbf{x}$.

$$\underbrace{\dim_{\mathbb{C}(t)} \mathrm{Vect}_{\mathbb{C}(t)} \left\{ I^{(k)}(t) \right\}_{k\geq 0}}_{\text{order of the PF operator}} = \dim_{\mathbb{C}} \mathrm{Vect}_{\mathbb{C}} \left\{ \int_\eta R(t,\mathbf{x})\mathrm{d}\mathbf{x} \right\}_{\eta \in H_n(\mathbb{P}^n \setminus \mathrm{pole}(R))}.$$

$\rightsquigarrow$ the order of the PF operator reflects an intrinsic geometry.

See also (Agostini et al., 2022)

Outline

Fundamental relations

Integral of derivatives

$$\oint \sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i} \mathrm{d}\mathbf{x} = 0$$

Integration by part

$$\oint F \frac{\partial G}{\partial x_i} \mathrm{d}\mathbf{x} = -\oint \frac{\partial F}{\partial x_i} G \mathrm{d}\mathbf{x}$$

Derivation under $\int$

$$\frac{\partial}{\partial t} \oint F \mathrm{d}\mathbf{x} = \oint \frac{\partial F}{\partial t} \mathrm{d}\mathbf{x}$$

Griffiths–Dwork reduction

Let $R = a/f^q$ an homogeneous rational function, $q > 1$

- If $a = \sum_{i=0}^{n} b_i \partial_i f$, then

$$\oint \frac{a}{\boxed{f^q}} \mathrm{d}\mathbf{x} = \oint \sum_{i=0}^{n} b_i \frac{\partial_i f}{f^q} \mathrm{d}\mathbf{x} = \frac{1}{q-1} \oint \sum_{i=0}^{n} \frac{\partial_i b_i}{\boxed{f^{q-1}}}.$$

Rewriting rule $\quad \dfrac{\sum_i b_i \partial_i f}{f^q} \longrightarrow \dfrac{1}{q-1} \dfrac{\sum_i \partial_i b_i}{f^{q-1}}$

Proposition $\quad$ If $R \longrightarrow^* R'$, then $\oint R \mathrm{d}\mathbf{x} = \oint R' \mathrm{d}\mathbf{x}$.

Theorem (Griffiths, 1969) $\quad$ If $V(f)$ is a smooth projective hypersurface, then

$$\oint \frac{a}{f^q} \mathrm{d}\mathbf{x} = 0 \quad \Leftrightarrow \quad \frac{a}{f^q} \rightarrow^* 0.$$

Computation of a PF operator (in the smooth case)

Input An homogeneous rational function $R = a/f^q$, with $V(f)$ smooth
Output The minimal PF operator annihilating $\oint \frac{a}{f^q}\mathrm{d}\mathbf{x}$

for $k = 0, 1, 2, \ldots$:

compute a normal form $\dfrac{\partial^k}{\partial t^k}\dfrac{a}{f^q} \longrightarrow^* \dfrac{b_k}{f^n}$

if rank $\{b_0, \ldots, b_k\} \leq k$:

compute $c_0, \ldots, c_k$ non trivial such that $\sum_i c_i b_i = 0$

return $\sum_{i=0}^k c_i(t)\frac{\mathrm{d}^i}{\mathrm{d}t^i}$

Extended Griffiths–Dwork reduction, principle

Recall the rewriting rule

$$\boxed{\frac{\sum_i b_i \partial_i f}{f^{q+1}} \longrightarrow \frac{1}{q} \frac{\sum_i \partial_i b_i}{f^q}}$$

- There no unicity in the choice of the b_i.
- If $\sum_i b_i \partial_i f = 0$, the *rule*

$$0 \longrightarrow \frac{1}{q} \frac{\sum_i \partial_i b_i}{f^q}$$

  give new relations, maybe unseen by the GD reduction.

Extended Griffiths–Dwork reduction, definition

Extended rank 2 rewrite rules

$$\text{(Griffiths-Dwork)} + \left(\underbrace{\sum_{i=0}^{n} b_i \partial_i f = 0}_{\text{requirement}} \Rightarrow \boxed{\frac{\sum_{i=0}^{n} \partial_i b_i}{f^q} \longrightarrow 0} \right)$$

- The extended rules are still ambiguous
- We may have

$$\begin{tikzcd} & \frac{b}{f^{q-1}} \\ \frac{a}{f^q} \arrow[ur, "\text{rg } 1"] \arrow[dr, "\text{rg } 2"'] & \\ & 0 \end{tikzcd} \quad \text{but not } b/f^{q-1} \xrightarrow{\text{rk } 2} 0.$$

- In this case, we define a new rule $b/f^{q-1} \xrightarrow{\text{rk } 3} 0$.

Extended Griffiths–Dwork reduction, continued

Extended rank 3 rewrite rules

$$(\text{rank 3 rules}) + \left(\begin{array}{ccc} & \xrightarrow{\text{rg } 1} \dfrac{b}{f^{q-1}} & \\ \dfrac{a}{f^q} & & \Rightarrow \boxed{\dfrac{b}{f^q} \longrightarrow 0} \\ & \xrightarrow[\text{rg } 2]{} 0. & \end{array} \right)$$

And so on for higher ranks.

Theorem

$$\boxed{\forall f\ \exists r}\ \forall \frac{a}{f^q},\ \oint \frac{a}{f^q} \mathrm{d}\mathbf{x} = 0 \Rightarrow \frac{a}{f^q} \xrightarrow{\text{rk } r}{}^{*} 0.$$

An example (Beukers' integral for $\zeta(3)$)

Consider

$$f = 2xyz(w-x)(w-y)(w-z) - w^3(w^3 - w^2z + xyz)$$

Let $e(q,r)$ be the number of independent homogeneous rational functions a/f^q that are not reducible with rank r rules.

q	0	1	2	3	4	$q > 4$
without reduction	0	10	165	680	1771	$\sim 36q^3$
$e(q,1)$	0	10	86	102	120	$\sim 18q$
$e(q,2)$	0	10	7	6	6	6
$e(q,3)$	0	9	1	0	0	0

Outline

Asserting the minimality of the PF operator

If the rank r is large enough (?), the PF operator that is computed is minimal. But a specific solution $I(t)$ may satisfy a smaller equation. But we don't know well *the* function in which we are interested.

Problem Given a differential operator $\mathcal{L} = \sum_{i=0}^{m} p_i(t) \frac{\mathrm{d}^i}{\mathrm{d}t^i}$, is there a non zero $I(t) \in \mathrm{Sol}(\mathcal{L})$ such that

$$\dim_{\mathbb{C}(t)} \mathrm{Vect}_{\mathbb{C}(t)} \{I, I', I'', \dots\} < m.$$

Alternative formulation Are there positive order differential operators $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{L} = \mathcal{A}\mathcal{B}$?

Algorithms by van Hoeij (1997), van der Hoeven (2007), and Chyzak et al. (2022).

Monodromy and irreducibility

Let $\mathcal{L}$ be a *Fuchsian* differential operator of order m.
(Fuchsian = all the solutions have at most polynomial growth everywhere.)

Theorem
Let G be the monodromy group acting on $\mathrm{Sol}(\mathcal{L})$. Let $I \in \mathrm{Sol}(\mathcal{L})$.

$$\dim_{\mathbb{C}(t)} \mathrm{Vect}_{\mathbb{C}(t)} \{I, I', I'', \dots\} = \dim_{\mathbb{C}} \mathrm{Vect}_{\mathbb{C}} (G \cdot I) .$$

Factorization algorithm

(van der Hoeven, 2007; Chyzak et al., 2022)

Input A Fuchsian differential operator $\mathcal{L}$

Output A factorization of $\mathcal{L}$, or IRREDUCIBLE

fix a working precision
while true:
  compute numerically generators of the monodromy group G
  if we can find a non trivial subspace invariant under G:
    reconstruct numerically a factorization $\mathcal{L} = \mathcal{A}\mathcal{B}$
    if the factorization is exact:
      return $\mathcal{A}$ and $\mathcal{B}$
  else:
    use the error bounds to certify that there is no such space
    if it worked:
      return IRREDUCIBLE
  increase the precision

A-hypergeometric holonomic systems

Let $A \in \mathbb{Z}^{d\times n}$ such that $(1,\ldots,1)^t \in A(\mathbb{Q}^n)$. Let $\beta \in \mathbb{Q}^n$.

The A-hypergeometric system, with parameter β, is the left ideal of the Weyl algebra in n variable generated by:

- $\partial^u - \partial^v$, for all $u, v \in \mathbb{N}^n$ such that $Au = Av$
- $\sum_{i=1}^n a_{ij}x_j\partial_j - \beta_j$, for $1 \leq i \leq d$

✓ Rich combinatorial structure

✓ Some integrals are solutions of A-hypergeometric systems

Generalized Euler integrals

Let $f_1, \ldots, f_l$ be polynomials where each coefficient is a variable c_i. Let

$$E(\mathbf{c}) = \oint \prod_k f_k^{\beta_k} \mathrm{d}\mathbf{x}$$

Theorem (Gelfand et al., 1990)
$E(c)$ is solution of an A-hypergeometric system.

- ✓ Computation of the integral *for free*
- ✗ Generic coefficients

Specialization of generalized Euler integrals

Let $I(t)$ be some Feynman integral, over a cycle.

Then $I(t) = E(\mathbf{c}(t))$ for some generalized Euler integral E and some rational function $\mathbf{c} : \mathbb{C} \to \mathbb{C}^n$.

Question Does the A-hypergeometric system for E provide any help to determine the order of the minimal differential equation annihilating I?

Remarks

- We may need extra equations for E (Hosono et al., 1996)
- D-module restriction seems useless?
- Power series expansion may help!...
- ... but we need to consider Nilsson rings.

Example

$$I(t) = \oint \frac{\mathrm{d}x\mathrm{d}y}{y^2 + x(x-1)(x-t)}$$

It satisfies a differential equation of order 2, but going through A-hypergeometric systems leads to equations of order 3.

$$J(t) = \oint \frac{\mathrm{d}x\mathrm{d}y}{(\text{random cubic}) + t(\text{random cubic})}$$

$\rightsquigarrow$ differential equation of order 2...
... but a A-hypergeometric system of rank 9.

References I

Agostini, D., Fevola, C., Sattelberger, A.-L., & Telen, S. (2022, August 18). Vector Spaces of Generalized Euler Integrals. https://doi.org/10.48550/arXiv.2208.08967

Bloch, S., Kerr, M., & Vanhove, P. (2015). A Feynman integral via higher normal functions. *Compos. Math.*, *151*(12), 2329–2375. https://doi.org/10/f74v85

Chyzak, F., Goyer, A., & Mezzarobba, M. (2022, February). *Symbolic-numeric factorization of differential operators*.

Gelfand, I. M., Kapranov, M. M., & Zelevinsky, A. V. (1990). Generalized Euler integrals and A-hypergeometric functions. *Advances in Mathematics*, *84*(2), 255–271. https://doi.org/10/bf8dtx

Griffiths, P. A. (1969). On the periods of certain rational integrals. *Ann. Math.*, *90*, 460–541. https://doi.org/10/dq3zc6

Hosono, S., Lian, B. H., & Yau, S.-T. (1996). GKZ-generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces. *Commun. Math. Phys.*, *182*(3), 535–577. https://doi.org/10/b2mcgm

References II

van der Hoeven, J. (2007). Around the numeric–symbolic computation of differential Galois groups. *J. Symb. Comput.*, *42*(1), 236–264. https://doi.org/10/cjxwkq

van Hoeij, M. (1997). Factorization of differential operators with rational functions coefficients. *J. Symb. Comput.*, *24*(5), 537–561. https://doi.org/10/cwbfq7